Algebraic expressions 1A

1 a $x^3 \times x^4 = x^{3+4}$
$= x^7$

b $2x^3 \times 3x^2 = 2 \times 3 \times x^{3+2}$
$= 6x^5$

c $\dfrac{k^3}{k^2} = k^{3-2}$
$= k$

d $4p^3 \div 2p = \frac{4}{2} \times p^3 \div p$
$= 2 \times p^{3-1}$
$= 2p^2$

e $\dfrac{3x^3}{3x^2} = \dfrac{3}{3} \times \dfrac{x^3}{x^2}$
$= 1 \times x^{3-2}$
$= x$

f $(y^2)^5 = y^{2 \times 5}$
$= y^{10}$

g $10x^5 \div 2x^3 = 5x^{5-3}$
$= 5x^2$

h $(p^3)^2 \div p^4 = p^6 \div p^4$
$= p^{6-4}$
$= p^2$

i $(2a^3)^2 \div 2a^3 = 2^2 \times a^6 \div 2a^3$
$= 4a^6 \div 2a^3$
$= 2a^{6-3}$
$= 2a^3$

j $8p^4 \div 4p^3 = 2p^{4-3}$
$= 2p^1$
$= 2p$

k $2a^4 \times 3a^5 = 2 \times 3 \times a^4 \times a^5$
$= 6 \times a^{4+5}$
$= 6a^9$

l $\dfrac{21a^3b^7}{7ab^4} = \dfrac{21}{7} \times \dfrac{a^3}{a} \times \dfrac{b^7}{b^4}$
$= 3a^{3-1}b^{7-4}$
$= 3a^2b^3$

m $9x^2 \times 3(x^2)^3 = 9 \times 3 \times x^2 \times x^{2\times 3}$
$= 27x^{2+6}$
$= 27x^8$

n $3x^3 \times 2x^2 \times 4x^6 = 3 \times 2 \times 4 \times x^{3+2+6}$
$= 24x^{11}$

o $7a^4 \times (3a^4)^2 = 7a^4 \times 9a^8$
$= 63a^{12}$

p $(4y^3)^3 \div 2y^3 = 64y^9 \div 2y^3$
$= 32y^6$

q $2a^3 \div 3a^2 \times 6a^5 = 2 \div 3 \times 6 \times a^{3-2+5}$
$= 4a^6$

r $3a^4 \times 2a^5 \times a^3 = 3 \times 2 \times a^{4+5+3}$
$= 6a^{12}$

2 a $9(x-2) = 9x - 18$

b $x(x+9) = x^2 + 9x$

c $-3y(4-3y) = -12y + 9y^2$

d $x(y+5) = xy + 5x$

e $-x(3x+5) = -3x^2 - 5x$

f $-5x(4x+1) = -20x^2 - 5x$

g $(4x+5)x = 4x^2 + 5x$

h $-3y(5-2y^2) = -15y + 6y^3$

i $-2x(5x-4) = -10x^2 + 8x$

j $(3x-5)x^2 = 3x^3 - 5x^2$

k $3(x+2) + (x-7) = 3x + 6 + x - 7$
$= 4x - 1$

l $5x - 6 - (3x-2) = 5x - 6 - 3x + 2$
$= 2x - 4$

m $4(c+3d^2) - 3(2c+d^2)$
$= 4c + 12d^2 - 6c - 3d^2$
$= -2c + 9d^2$

2 n $(r^2 + 3t^2 + 9) - (2r^2 + 3t^2 - 4)$
$= r^2 + 3t^2 + 9 - 2r^2 - 3t^2 + 4$
$= 13 - r^2$

o $x(3x^2 - 2x + 5) = 3x^3 - 2x^2 + 5x$

p $7y^2(2 - 5y + 3y^{2)} = 14y^2 - 35y^3 + 21y^4$

q $-2y^2(5 - 7y + 3y^2) = -10y^2 + 14y^3 - 6y^4$

r $7(x - 2) + 3(x + 4) - 6(x - 2)$
$= 7x - 14 + 3x + 12 - 6x + 12$
$= 4x + 10$

s $5x - 3(4 - 2x) + 6 = 5x - 12 + 6x + 6$
$= 11x - 6$

t $3x^2 - x(3 - 4x) + 7 = 3x^2 - 3x + 4x^2 + 7$
$= 7x^2 - 3x + 7$

u $4x(x + 3) - 2x(3x - 7)$
$= 4x^2 + 12x - 6x^2 + 14x$
$= 26x - 2x^2$

v $3x^2(2x + 1) - 5x^2(3x - 4)$
$= 6x^3 + 3x^2 - 15x^3 + 20x^2$
$= 23x^2 - 9x^3$

3 a $\dfrac{6x^4 + 10x^6}{2x} = \dfrac{6x^4}{2x} + \dfrac{10x^6}{2x}$
$= 3x^{4-1} + 5x^{6-1}$
$= 3x^3 + 5x^5$

b $\dfrac{3x^5 - x^7}{x} = \dfrac{3x^5}{x} - \dfrac{x^7}{x}$
$= 3x^{5-1} - x^{7-1}$
$= 3x^4 - 5x^6$

c $\dfrac{2x^4 - 4x^2}{4x} = \dfrac{2x^4}{4x} - \dfrac{4x^2}{4x}$
$= \dfrac{1}{2}x^{4-1} - x^{2-1}$
$= \dfrac{x^3}{2} - x$

3 d $\dfrac{8x^3 + 5x}{2x} = \dfrac{8x^3}{2x} + \dfrac{5x}{2x}$
$= 4x^{3-1} + \dfrac{5}{2}x^{1-1}$
$= 4x^2 + \dfrac{5}{2}$

e $\dfrac{7x^7 + 5x^2}{5x} = \dfrac{7x^7}{5x} + \dfrac{5x^2}{5x}$
$= \dfrac{7}{5}x^{7-1} + x^{2-1}$
$= \dfrac{7x^6}{5} + x$

f $\dfrac{9x^5 - 5x^3}{3x} = \dfrac{9x^5}{3x} - \dfrac{5x^3}{3x}$
$= 3x^{5-1} - \dfrac{5}{3}x^{3-1}$
$= 3x^4 - \dfrac{5x^2}{3}$

Algebraic expressions 1B

1 a $(x + 4)(x + 7)$
$= x^2 + 7x + 4x + 28$
$= x^2 + 11x + 28$

b $(x - 3)(x + 2)$
$= x^2 + 2x - 3x - 6$
$= x^2 - x - 6$

c $(x - 2)^2$
$= (x - 2)(x - 2)$
$= x^2 - 2x - 2x + 4$
$= x^2 - 4x + 4$

d $(x - y)(2x + 3)$
$= 2x^2 + 3x - 2xy - 3y$

e $(x + 3y)(4x - y)$
$= 4x^2 - xy + 12xy - 3y^2$
$= 4x^2 + 11xy - 3y^2$

f $(2x - 4y)(3x + y)$
$= 6x^2 + 2xy - 12xy - 4y^2$
$= 6x^2 - 10xy - 4y^2$

g $(2x - 3)(x - 4)$
$= 2x^2 - 8x - 3x + 12$
$= 2x^2 - 11x + 12$

h $(3x + 2y)^2$
$= (3x + 2y)(3x + 2y)$
$= 9x^2 + 6xy + 6xy + 4y^2$
$= 9x^2 + 12xy + 4y^2$

i $(2x + 8y)(2x + 3)$
$= 4x^2 + 6x + 16xy + 24y$

j $(x + 5)(2x + 3y - 5)$
$= x(2x + 3y - 5) + 5(2x + 3y - 5)$
$= 2x^2 + 3xy - 5x + 10x + 15y - 25$
$= 2x^2 + 3xy + 5x + 15y - 25$

k $(x - 1)(3x - 4y - 5)$
$= x(3x - 4y - 5) - (3x - 4y - 5)$
$= 3x^2 - 4xy - 5x - 3x + 4y + 5$
$= 3x^2 - 4xy - 8x + 4y + 5$

l $(x - 4y)(2x + y + 5)$
$= x(2x + y + 5) - 4y(2x + y + 5)$
$= 2x^2 + xy + 5x - 8xy - 4y^2 - 20y$
$= 2x^2 + 5x - 7xy - 4y^2 - 20y$

m $(x + 2y - 1)(x + 3)$
$= x(x + 3) + 2y(x + 3) - (x + 3)$
$= x^2 + 3x + 2xy + 6y - x - 3$
$= x^2 + 2x + 2xy + 6y - 3$

n $(2x + 2y + 3)(x + 6)$
$= 2x(x + 6) + 2y(x + 6) + 3(x + 6)$
$= 2x^2 + 12x + 2xy + 12y + 3x + 18$
$= 2x^2 + 15x + 2xy + 12y + 18$

o $(4 - y)(4y - x + 3)$
$= 4(4y - x + 3) - y(4y - x + 3)$
$= 16y - 4x + 12 - 4y^2 + xy - 3y$
$= -4y^2 - 4x + 12 + xy + 13y$

p $(4y + 5)(3x - y + 2)$
$= 4y(3x - y + 2) + 5(3x - y + 2)$
$= 12xy - 4y^2 + 8y + 15x - 5y + 10$
$= 12xy - 4y^2 + 3y + 15x + 10$

q $(5y - 2x + 3)(x - 4)$
$= 5y(x - 4) - 2x(x - 4) + 3(x - 4)$
$= 5xy - 20y - 2x^2 + 8x + 3x - 12$
$= 5xy - 20y - 2x^2 + 11x - 12$

r $(4y - x - 2)(5 - y)$
$= 4y(5 - y) - x(5 - y) - 2(5 - y)$
$= 20y - 4y^2 - 5x + xy - 10 + 2y$
$= 22y - 4y^2 - 5x + xy - 10$

2 a $5(x + 1)(x - 4)$
$= (5x + 5)(x - 4)$
$= 5x^2 - 20x + 5x - 20$
$= 5x^2 - 15x - 20$

b $7(x - 2)(2x + 5)$
$= (7x - 14)(2x + 5)$
$= 14x^2 + 35x - 28x - 70$
$= 14x^2 + 7x - 70$

c $3(x - 3)(x - 3)$
$= (3x - 9)(x - 3)$
$= 3x^2 - 9x - 9x + 27$
$= 3x^2 - 18x + 27$

d $x(x - y)(x + y)$
$= (x^2 - xy)(x + y)$
$= x^3 + x^2y - x^2y - xy^2$
$= x^3 - xy^2$

2 e $x(2x + y)(3x + 4)$
$= (2x^2 + xy)(3x + 4)$
$= 6x^3 + 8x^2 + 3x^2y + 4xy$

f $y(x - 5)(x + 1)$
$= (xy - 5y)(x + 1)$
$= x^2y + xy - 5xy - 5y$
$= x^2y - 4xy - 5y$

g $y(3x - 2y)(4x + 2)$
$= (3xy - 2y^2)(4x + 2)$
$= 12x^2y + 6xy - 8xy^2 - 4y^2$

h $y(7 - x)(2x - 5)$
$= (7y - xy)(2x - 5)$
$= 14xy - 35y - 2x^2y + 5xy$
$= 19xy - 35y - 2x^2y$

i $x(2x + y)(5x - 2)$
$= (2x^2 + xy)(5x - 2)$
$= 10x^3 - 4x^2 + 5x^2y - 2xy$

j $x(x + 2)(x + 3y - 4)$
$= (x^2 + 2x)(x + 3y - 4)$
$= x^2(x + 3y - 4) + 2x(x + 3y - 4)$
$= x^3 + 3x^2y - 4x^2 + 2x^2 + 6xy - 8x$
$= x^3 + 3x^2y - 2x^2 + 6xy - 8x$

k $y(2x + y - 1)(x + 5)$
$= (2xy + y^2 - y)(x + 5)$
$= 2xy(x + 5) + y^2(x + 5) - y(x + 5)$
$= 2x^2y + 10xy + xy^2 + 5y^2 - xy - 5y$
$= 2x^2y + 9xy + xy^2 + 5y^2 - 5y$

l $y(3x + 2y - 3)(2x + 1)$
$= (3xy + 2y^2 - 3y)(2x + 1)$
$= 3xy(2x + 1) + 2y^2(2x + 1) - 3y\,(2x + 1)$
$= 6x^2y + 3xy + 4xy^2 + 2y^2 - 6xy - 3y$
$= 6x^2y + 4xy^2 + 2y^2 - 3xy - 3y$

m $x\,(2x + 3)(x + y - 5)$
$= (2x^2 + 3x)(x + y - 5)$
$= 2x^2(x + y - 5) + 3x(x + y - 5)$
$= 2x^3 + 2x^2y - 10x^2 + 3x^2 + 3xy - 15x$
$= 2x^3 + 2x^2y - 7x^2 + 3xy - 15x$

n $2x\,(3x - 1)(4x - y - 3)$
$= (6x^2 - 2x)(4x - y - 3)$
$= 6x^2(4x - y - 3) - 2x(4x - y - 3)$
$= 24x^3 - 6x^2y - 18x^2 - 8x^2 + 2xy + 6x$
$= 24x^3 - 6x^2y - 26x^2 + 2xy + 6x$

o $3x\,(x - 2y)(2x + 3y + 5)$
$= (3x^2 - 6xy)(2x + 3y + 5)$
$= 3x^2(2x + 3y + 5) - 6xy(2x + 3y + 5)$
$= 6x^3 + 9x^2y + 15x^2 - 12x^2y - 18xy^2 - 30xy$
$= 6x^3 + 15x^2 - 3x^2y - 18xy^2 - 30xy$

p $(x + 3)(x + 2)(x + 1)$
$= (x^2 + 2x + 3x + 6)(x + 1)$
$= (x^2 + 5x + 6)(x + 1)$
$= x^2(x + 1) + 5x(x + 1) + 6(x + 1)$
$= x^3 + x^2 + 5x^2 + 5x + 6x + 6$
$= x^3 + 6x^2 + 11x + 6$

q $(x + 2)(x - 4)(x + 3)$
$= (x^2 - 4x + 2x - 8)(x + 3)$
$= (x^2 - 2x - 8)(x + 3)$
$= x^2(x + 3) - 2x(x + 3) - 8(x + 3)$
$= x^3 + 3x^2 - 2x^2 - 6x - 8x - 24$
$= x^3 + x^2 - 14x - 24$

r $(x + 3)(x - 1)(x - 5)$
$= (x^2 - x + 3x - 3)(x - 5)$
$= (x^2 + 2x - 3)(x - 5)$
$= x^2(x - 5) + 2x(x - 5) - 3(x - 5)$
$= x^3 - 5x^2 + 2x^2 - 10x - 3x + 15$
$= x^3 - 3x^2 - 13x + 15$

s $(x - 5)(x - 4)(x - 3)$
$= (x^2 - 4x - 5x + 20)(x - 3)$
$= (x^2 - 9x + 20)(x - 3)$
$= x^2(x - 3) - 9x(x - 3) + 20(x - 3)$
$= x^3 - 3x^2 - 9x^2 + 27x + 20x - 60$
$= x^3 - 12x^2 + 47x - 60$

t $(2x + 1)(x - 2)(x + 1)$
$= (2x^2 - 4x + x - 2)(x + 1)$
$= (2x^2 - 3x - 2)(x + 1)$
$= 2x^2(x + 1) - 3x(x + 1) - 2(x + 1)$
$= 2x^3 + 2x^2 - 3x^2 - 3x - 2x - 2$
$= 2x^3 - x^2 - 5x - 2$

u $(2x + 3)(3x - 1)(x + 2)$
$= (6x^2 - 2x + 9x - 3)(x + 2)$
$= (6x^2 + 7x - 3)(x + 2)$
$= 6x^2(x + 2) + 7x(x + 2) - 3(x + 2)$
$= 6x^3 + 12x^2 + 7x^2 + 14x - 3x - 6$
$= 6x^3 + 19x^2 + 11x - 6$

2 v $(3x-2)(2x+1)(3x-2)$
$= (6x^2 + 3x - 4x - 2)(3x - 2)$
$= (6x^2 - x - 2)(3x - 2)$
$= 6x^2(3x - 2) - x(3x - 2) - 2(3x - 2)$
$= 18x^3 - 12x^2 - 3x^2 + 2x - 6x + 4$
$= 18x^3 - 15x^2 - 4x + 4$

w $(x + y)(x - y)(x - 1)$
$= (x^2 - xy + xy - y^2)(x - 1)$
$= (x^2 - y^2)(x - 1)$
$= x^2(x - 1) - y^2(x - 1)$
$= x^3 - x^2 - xy^2 + y^2$

x $(2x - 3y)^3$
$= (2x - 3y)(2x - 3y)(2x - 3y)$
$= (4x^2 - 6xy - 6xy + 9y^2)(2x - 3y)$
$= (4x^2 - 12xy + 9y^2)(2x - 3y)$
$= 4x^2(2x - 3y) - 12xy(2x - 3y) + 9y^2(2x - 3y)$
$= 8x^3 - 12x^2y - 24x^2y + 36xy^2 + 18xy^2 - 27y^3$
$= 8x^3 - 36x^2y + 54xy^2 - 27y^3$

3 Shaded area
$= (x + 7)(3x - y + 4) - (x - 2)^2$
$= x(3x - y + 4) + 7(3x - y + 4) - (x - 2)(x - 2)$
$= 3x^2 - xy + 4x + 21x - 7y + 28 - x^2 + 2x + 2x - 4$
$= 2x^2 - xy + 29x - 7y + 24$

4 Volume $= (x + 2)(2x - 1)(2x + 3)$
$= (2x^2 - x + 4x - 2)(2x + 3)$
$= (2x^2 + 3x - 2)(2x + 3)$
$= 2x^2(2x + 3) + 3x(2x + 3) - 2(2x + 3)$
$= 4x^3 + 6x^2 + 6x^2 + 9x - 4x - 6$
$= 4x^3 + 12x^2 + 5x - 6\text{ cm}^3$

5 $(2x + 5y)(3x - y)(2x + y)$
$= (6x^2 - 2xy + 15xy - 5y^2)(2x + y)$
$= (6x^2 + 13xy - 5y^2)(2x + y)$
$= 6x^2(2x + y) + 13xy(2x + y) - 5y^2(2x + y)$
$= 12x^3 + 6x^2y + 26x^2y + 13xy^2 - 10xy^2 - 5y^3$
$= 12x^3 + 32x^2y + 3xy^2 - 5y^3$
$= ax^3 + bx^2y + cxy^2 + dy^3$
Therefore, $a = 12$, $b = 32$, $c = 3$ and $d = -5$

Challenge
$(x + y)^4$
$= (x + y)(x + y)(x + y)(x + y)$
$= (x^2 + xy + xy + y^2)(x^2 + xy + xy + y^2)$
$= (x^2 + 2xy + y^2)(x^2 + 2xy + y^2)$
$= x^2(x^2 + 2xy + y^2) + 2xy(x^2 + 2xy + y^2) + y^2(x^2 + 2xy + y^2)$
$= x^4 + 2x^3y + x^2y^2 + 2x^3y + 4x^2y^2 + 2xy^3 + x^2y^2 + 2xy^3 + y^4$
$= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

Algebraic expressions 1C

1 a $4x + 8 = 4(x + 2)$

b $6x - 24 = 6(x - 4)$

c $20x + 15 = 5(4x + 3)$

d $2x^2 + 4 = 2(x^2 + 2)$

e $4x^2 + 20 = 4(x^2 + 5)$

f $6x^2 - 18x = 6x(x - 3)$

g $x^2 - 7x = x(x - 7)$

h $2x^2 + 4x = 2x(x + 2)$

i $3x^2 - x = x(3x - 1)$

j $6x^2 - 2x = 2x(3x - 1)$

k $10y^2 - 5y = 5y(2y - 1)$

l $35x^2 - 28x = 7x(5x - 4)$

m $x^2 + 2x = x(x + 2)$

n $3y^2 + 2y = y(3y + 2)$

o $4x^2 + 12x = 4x(x + 3)$

p $5y^2 - 20y = 5y(y - 4)$

q $9xy^2 + 12x^2y = 3xy(3y + 4x)$

r $6ab - 2ab^2 = 2ab(3 - b)$

s $5x^2 - 25xy = 5x(x - 5y)$

t $12x^2y + 8xy^2 = 4xy(3x + 2y)$

u $15y - 20yz^2 = 5y(3 - 4z^2)$

v $12x^2 - 30 = 6(2x^2 - 5)$

w $xy^2 - x^2y = xy(y - x)$

x $12y^2 - 4yx = 4y(3y - x)$

2 a $x^2 + 4x = x(x + 4)$

b $2x^2 + 6x = 2x(x + 3)$

c $x^2 + 11x + 24 = x^2 + 8x + 3x + 24$
$= x(x + 8) + 3(x + 8)$
$= (x + 8)(x + 3)$

d $x^2 + 8x + 12 = x^2 + 2x + 6x + 12$
$= x(x + 2) + 6(x + 2)$
$= (x + 2)(x + 6)$

e $x^2 + 3x - 40 = x^2 + 8x - 5x - 40$
$= x(x + 8) - 5(x + 8)$
$= (x + 8)(x - 5)$

f $x^2 - 8x + 12 = x^2 - 2x - 6x + 12$
$= x(x - 2) - 6(x - 2)$
$= (x - 2)(x - 6)$

g $x^2 + 5x + 6 = x^2 + 3x + 2x + 6$
$= x(x + 3) + 2(x + 3)$
$= (x + 3)(x + 2)$

h $x^2 - 2x - 24 = x^2 - 6x + 4x - 24$
$= x(x - 6) + 4(x - 6)$
$= (x - 6)(x + 4)$

i $x^2 - 3x - 10 = x^2 - 5x + 2x - 10$
$= x(x - 5) + 2(x - 5)$
$= (x - 5)(x + 2)$

j $x^2 + x - 20 = x^2 - 4x + 5x - 20$
$= x(x - 4) + 5(x - 4)$
$= (x - 4)(x + 5)$

k $2x^2 + 5x + 2 = 2x^2 + x + 4x + 2$
$= x(2x + 1) + 2(2x + 1)$
$= (2x + 1)(x + 2)$

l $3x^2 + 10x - 8 = 3x^2 - 2x + 12x - 8$
$= x(3x - 2) + 4(3x - 2)$
$= (3x - 2)(x + 4)$

m $5x^2 - 16x + 3 = 5x^2 - 15x - x + 3$
$= 5x(x - 3) -(x - 3)$
$= (x - 3)(5x - 1)$

n $6x^2 - 8x - 8 = 6x^2 - 12x + 4x - 8$
$= 6x(x - 2) + 4(x - 2)$
$= (x - 2)(6x + 4)$
$= 2(x - 2)(3x + 2)$

2 o $2x^2 + 7x - 15 = 2x^2 + 10x - 3x - 15$
$= 2x(x + 5) - 3(x + 5)$
$= (x + 5)(2x - 3)$

p Put $y = x^2$
$2x^4 + 14x^2 + 24 = 2y^2 + 14y + 24$
$= 2y^2 + 6y + 8y + 24$
$= 2y(y + 3) + 8(y + 3)$
$= (y + 3)(2y + 8)$
$= (x^2 + 3)(2x^2 + 8)$
$= 2(x^2 + 3)(x^2 + 4)$

q $x^2 - 4 = x^2 - 2^2$
$= (x + 2)(x - 2)$

r $x^2 - 49 = x^2 - 7^2$
$= (x + 7)(x - 7)$

s $4x^2 - 25 = (2x)^2 - 5^2$
$= (2x + 5)(2x - 5)$

t $9x^2 - 25y^2 = (3x)^2 - (5y)^2$
$= (3x + 5y)(3x - 5y)$

u $36x^2 - 4 = 4(9x^2 - 1)$
$= 4(3x)^2 - 1^2$
$= 4(3x + 1)(3x - 1)$

v $2x^2 - 50 = 2(x^2 - 25)$
$= 2(x^2 - 5^2)$
$= 2(x + 5)(x - 5)$

w $6x^2 - 10x + 4 = 2(3x^2 - 5x + 2)$
$= 2(3x^2 - 3x - 2x + 2)$
$= 2\ (3x(x - 1) - 2(x - 1))$
$= 2(x - 1)(3x - 2)$

x $15x^2 + 42x - 9 = 3(5x^2 + 14x - 3)$
$= 3(5x^2 - x + 15x - 3)$
$= 3\ (x(5x - 1) + 3(5x - 1))$
$= 3(5x - 1)(x + 3)$

3 a $x^3 + 2x = x(x^2 + 2)$

b $x^3 - x^2 + x = x(x^2 - x + 1)$

c $x^3 - 5x = x(x^2 - 5)$

d $x^3 - 9x = x(x^2 - 9)$
$= x(x^2 - 3^2)$
$= x(x + 3)(x - 3)$

3 e $x^3 - x^2 - 12x = x(x^2 - x - 12)$
$= x(x^2 - 4x + 3x - 12)$
$= x(x(x - 4) + 3(x - 4))$
$= x(x - 4)(x + 3)$

f $x^3 + 11x^2 + 30x = x(x^2 + 11x + 30)$
$= x(x^2 + 5x + 6x + 30)$
$= x(x(x + 5) + 6(x + 5))$
$= x(x + 5)(x + 6)$

g $x^3 - 7x^2 + 6x = x(x^2 - 7x + 6)$
$= x(x^2 - x - 6x + 6)$
$= x(x\ (x - 1) - 6(x - 1))$
$= x(x - 1)(x - 6)$

h $x^3 - 64x = x(x^2 - 64)$
$= x(x^2 - 8^2)$
$= x(x + 8)(x - 8)$

i $2x^3 - 5x^2 - 3x = x(2x^2 - 5x - 3)$
$= x(2x^2 + x - 6x - 3)$
$= x(x\ (2x + 1) - 3(2x + 1))$
$= x(2x + 1)(x - 3)$

j $2x^3 + 13x^2 + 15x = x(2x^2 + 13x + 15)$
$= x(2x^2 + 3x + 10x + 15)$
$= x(x(2x + 3) + 5(2x + 3))$
$= x(2x + 3)(x + 5)$

k $x^3 - 4x = x(x^2 - 4)$
$= x(x^2 - 2^2)$
$= x(x + 2)(x - 2)$

l $3x^3 + 27x^2 + 60x = 3x(x^2 + 9x + 20)$
$= 3x(x^2 + 4x + 5x + 20)$
$= 3x(x(x + 4) + 5(x + 4))$
$= 3x(x + 4)(x + 5)$

4 $x^4 - y^4 = (x^2)^2 - (y^2)^2$
$= (x^2 + y^2)(x^2 - y^2)$
$= (x^2 + y^2)(x + y)(x - y)$

5 $6x^3 + 7x^2 - 5x = x(6x^2 + 7x - 5)$
$= x(6x^2 + 10x - 3x - 5)$
$= x(2x(3x + 5) - (3x + 5))$
$= x(3x + 5)(\ 2x - 1)$

Challenge

$4x^4 - 13x^2 + 9 = (4x^4 - 4x^2 - 9x^2 + 9)$
$= 4x^2(x^2 - 1) - 9(x^2 - 1)$
$= (x^2 - 1)(4x^2 - 9)$
$= (x^2 - 1^2)(\ 2x)^2 - 3^2)$
$= (x + 1)(x - 1)(2x + 3)(2x - 3)$

Algebraic expressions 1D

1 **a** $x^3 \div x^{-2} = x^{3-(-2)}$
$= x^5$

b $x^5 \div x^7 = x^{5-7}$
$= x^{-2}$

c $x^{\frac{3}{2}} \times x^{\frac{5}{2}} = x^{\frac{3}{2}+\frac{5}{2}}$
$= x^4$

d $\left(x^2\right)^{\frac{3}{2}} = x^{2\times\frac{3}{2}}$
$= x^3$

e $\left(x^3\right)^{\frac{5}{3}} = x^{3\times\frac{5}{3}}$
$= x^5$

f $3x^{0.5} \times 4x^{-0.5} = 12x^{0.5+(-0.5)}$
$= 12x^0$
$= 12$

g $9x^{\frac{2}{3}} \div 3x^{\frac{1}{6}} = 3x^{\frac{2}{3}-\frac{1}{6}}$
$= 3x^{\frac{1}{2}}$

h $5x^{\frac{7}{5}} \div x^{\frac{2}{5}} = 5x^{\frac{7}{5}-\frac{2}{5}}$
$= 5x$

i $3x^4 \times 2x^{-5} = 6x^{4+(-5)}$
$= 6x^{-1}$

j $\sqrt{x} \times \sqrt[3]{x} = x^{\frac{1}{2}} \times x^{\frac{1}{3}}$
$= x^{\frac{1}{2}+\frac{1}{3}}$
$= x^{\frac{5}{6}}$
$= (\sqrt[6]{x})^5$

k $\left(\sqrt{x}\right)^3 \times \left(\sqrt[3]{x}\right)^4 = x^{\frac{3}{2}} \times x^{\frac{4}{3}}$
$= x^{\frac{3}{2}+\frac{4}{3}}$
$= x^{\frac{17}{6}}$
$= (\sqrt[6]{x})^{17}$

l $\dfrac{\left(\sqrt[3]{x}\right)^2}{\sqrt{x}} = x^{\frac{2}{3}} \div x^{\frac{1}{2}}$
$= x^{\frac{2}{3}-\frac{1}{2}}$
$= x^{\frac{1}{6}}$
$= \sqrt[6]{x}$

2 **a** $25^{\frac{1}{2}} = \sqrt{25}$
$= 5$

b $81^{\frac{3}{2}} = \left(\sqrt{81}\right)^3$
$= 9^3$
$= 729$

c $27^{\frac{1}{3}} = \sqrt[3]{27}$
$= 3$

d $4^{-2} = \dfrac{1}{4^2}$
$= \dfrac{1}{16}$

e $9^{-\frac{1}{2}} = \dfrac{1}{9^{\frac{1}{2}}}$
$= \dfrac{1}{\sqrt{9}}$
$= \dfrac{1}{3}$

f $(-5)^{-3} = \dfrac{1}{(-5)^3}$
$= -\dfrac{1}{125}$

g $\left(\dfrac{3}{4}\right)^0 = 1$

h $1296^{\frac{3}{4}} = \left(\sqrt[4]{1296}\right)^3$
$= 6^3$
$= 216$

2 i $\left(\frac{25}{16}\right)^{\frac{3}{2}} = \frac{\left(\sqrt{25}\right)^3}{\left(\sqrt{16}\right)^3}$
$= \pm\frac{5^3}{4^3}$
$= \pm\frac{125}{64}$

j $\left(\frac{27}{8}\right)^{\frac{2}{3}} = \frac{\left(\sqrt[3]{27}\right)^2}{\left(\sqrt[3]{8}\right)^2}$
$= \frac{3^2}{2^2}$
$= \frac{9}{4}$

k $\left(\frac{6}{5}\right)^{-1} = \left(\frac{5}{6}\right)^1$
$= \frac{5}{6}$

l $\left(\frac{343}{512}\right)^{-\frac{2}{3}} = \frac{\left(\sqrt[3]{512}\right)^2}{\left(\sqrt[3]{343}\right)^2}$
$= \frac{8^2}{7^2}$
$= \frac{64}{49}$

3 a $\left(64x^{10}\right)^{\frac{1}{2}} = \sqrt{64}x^{10\times\frac{1}{2}}$
$= 8x^5$

b $\frac{5x^3-2x^2}{x^5} = \frac{5x^3}{x^5} - \frac{2x^2}{x^5}$
$= 5\times x^{3-5} - 2\times x^{2-5}$
$= 5x^{-2} - 2x^{-3}$
$= \frac{5}{x^2} - \frac{2}{x^3}$

c $\left(125x^{12}\right)^{\frac{1}{3}} = \sqrt[3]{125}x^{12\times\frac{1}{3}}$
$= 5x^4$

3 d $\frac{x+4x^3}{x^3} = \frac{x}{x^3} + \frac{4x^3}{x^3}$
$= x^{1-3} + 4\times x^{3-3}$
$= x^{-2} + 4x^0$
$= \frac{1}{x^2} + 4$

e $\frac{2x+x^2}{x^4} = \frac{2x}{x^4} + \frac{x^2}{x^4}$
$= 2\times x^{1-4} + x^{2-4}$
$= 2x^{-3} + x^{-2}$
$= \frac{2}{x^3} + \frac{1}{x^2}$

f $\left(\frac{4}{9}x^4\right)^{\frac{3}{2}} = \left(\sqrt{\frac{4}{9}}\right)^3 x^{4\times\frac{3}{2}}$
$= \frac{8}{27}x^6$

g $\frac{9x^2-15x^5}{3x^3} = \frac{9x^2}{3x^3} - \frac{15x^5}{3x^3}$
$= 3\times x^{2-3} - 5\times x^{5-3}$
$= 3x^{-1} - 5x^2$
$= \frac{3}{x} - 5x^2$

h $\frac{5x+3x^2}{15x^3} = \frac{5x}{15x^3} + \frac{3x^2}{15x^3}$
$= \frac{1}{3}\times x^{1-3} + \frac{1}{5}\times x^{2-3}$
$= \frac{1}{3}x^{-2} + \frac{1}{5}x^{-1}$
$= \frac{1}{3x^2} + \frac{1}{5x}$

4 a $81^{\frac{1}{4}} = \sqrt[4]{81}$
$= 3$

b $x(2x^{-\frac{1}{3}})^4 = x\times 2^4 x^{-\frac{1}{3}\times 4}$
$= 2^4 x^{-\frac{4}{3}+1}$
$= 16x^{-\frac{1}{3}}$

5 a $y^{\frac{1}{3}} = \left(\frac{1}{8}x^3\right)^{\frac{1}{3}}$

$= \frac{1}{\sqrt[3]{8}}x^{3\times\frac{1}{3}}$

$= \frac{x}{2}$

b $\frac{1}{2}y^{-2} = \frac{1}{2}\left(\frac{1}{8}x^3\right)^{-2}$

$= \frac{1}{2}\times\frac{1}{\left(\frac{1}{8}x^3\right)^2}$

$= \frac{1}{2}\times\frac{64}{x^{3\times 2}}$

$= \frac{32}{x^6}$

$= 32x^{-6}$

6 a $x^{\frac{1}{2}} = 7$

$\left(x^{\frac{1}{2}}\right)^2 = 7^2$

$x = 49$

b $y^{\frac{4}{3}} = 81$

$\left(y^{\frac{4}{3}}\right)^{\frac{3}{4}} = 81^{\frac{3}{4}}$

$y = 27$

c $x^{-\frac{3}{2}} = 8$

$\left(x^{-\frac{3}{2}}\right)^{-\frac{2}{3}} = 8^{\frac{-2}{3}}$

$x = \frac{1}{4}$

d $z^{-\frac{3}{4}} = 1000$

$\left(z^{-\frac{3}{4}}\right)^{-\frac{4}{3}} = 1000^{-\frac{4}{3}}$

$z = \frac{1}{10000}$

7 $27\sqrt{x} = \frac{1}{x}$

$x^{\frac{3}{2}} = \frac{1}{27}$

$\left(x^{\frac{3}{2}}\right)^{\frac{2}{3}} = \left(\frac{1}{27}\right)^{\frac{2}{3}}$

$x = \frac{1}{9}$

Algebraic expressions 1E

1 a $\sqrt{28} = \sqrt{4}\times\sqrt{7}$
$= 2\sqrt{7}$

b $\sqrt{72} = \sqrt{8}\times\sqrt{9}$
$= \sqrt{2}\times\sqrt{4}\times\sqrt{9}$
$= \sqrt{2}\times 2\times 3$
$= 6\sqrt{2}$

c $\sqrt{50} = \sqrt{25}\times\sqrt{2}$
$= 5\sqrt{2}$

d $\sqrt{32} = \sqrt{16}\times\sqrt{2}$
$= 4\sqrt{2}$

e $\sqrt{90} = \sqrt{9}\times\sqrt{10}$
$= 3\sqrt{10}$

f $\frac{\sqrt{12}}{2} = \frac{\sqrt{4}\times\sqrt{3}}{2}$
$= \frac{2\times\sqrt{3}}{2}$
$= \sqrt{3}$

g $\frac{\sqrt{27}}{3} = \frac{\sqrt{9}\times\sqrt{3}}{3}$
$= \frac{3\times\sqrt{3}}{3}$
$= \sqrt{3}$

h $\sqrt{20}+\sqrt{80} = \sqrt{4}\sqrt{5}+\sqrt{16}\sqrt{5}$
$= 2\sqrt{5}+4\sqrt{5}$
$= 6\sqrt{5}$

i $\sqrt{200}+\sqrt{18}-\sqrt{72}$
$= \sqrt{100}\sqrt{2}+\sqrt{9}\sqrt{2}-\sqrt{9}\sqrt{4}\sqrt{2}$
$= 10\sqrt{2}+3\sqrt{2}-6\sqrt{2}$
$= \sqrt{2}(10+3-6)$
$= 7\sqrt{2}$

j $\sqrt{175}+\sqrt{63}+2\sqrt{28}$
$= \sqrt{25}\times\sqrt{7}+\sqrt{9}\times\sqrt{7}+2\times\sqrt{4}\times\sqrt{7}$
$= 5\sqrt{7}+3\sqrt{7}+4\sqrt{7}$
$= \sqrt{7}(5+3+4)$
$= 12\sqrt{7}$

k $\sqrt{28}-2\sqrt{63}+\sqrt{7} = \sqrt{4}\sqrt{7}-2\sqrt{9}\sqrt{7}+\sqrt{7}$
$= 2\sqrt{7}-6\sqrt{7}+\sqrt{7}$
$= -3\sqrt{7}$

l $\sqrt{80}-2\sqrt{20}+3\sqrt{45}$
$= \sqrt{16}\sqrt{5}-2\sqrt{4}\sqrt{5}+3\sqrt{9}\sqrt{5}$
$= 4\sqrt{5}-4\sqrt{5}+9\sqrt{5}$
$= 9\sqrt{5}$

m $3\sqrt{80}-2\sqrt{20}+5\sqrt{45}$
$= 3\sqrt{16}\sqrt{5}-2\sqrt{4}\sqrt{5}+5\sqrt{9}\sqrt{5}$
$= 12\sqrt{5}-4\sqrt{5}+15\sqrt{5}$
$= 23\sqrt{5}$

n $\frac{\sqrt{44}}{\sqrt{11}} = \frac{\sqrt{4}\sqrt{11}}{\sqrt{11}}$
$= 2$

o $\sqrt{12}+3\sqrt{48}+\sqrt{75}$
$= \sqrt{4}\sqrt{3}+3\sqrt{16}\sqrt{3}+\sqrt{25}\sqrt{3}$
$= 2\sqrt{3}+12\sqrt{3}+5\sqrt{3}$
$= 19\sqrt{3}$

2 a $\sqrt{3}\left(2+\sqrt{3}\right) = 2\sqrt{3}+\sqrt{3}\sqrt{3}$
$= 2\sqrt{3}+\sqrt{9}$
$= 2\sqrt{3}+3$

b $\sqrt{5}\left(3-\sqrt{3}\right) = 3\sqrt{5}-\sqrt{3}\sqrt{5}$
$= 3\sqrt{5}-\sqrt{15}$

c $\sqrt{2}\left(4-\sqrt{5}\right) = 4\sqrt{2}-\sqrt{2}\sqrt{5}$
$= 4\sqrt{2}-\sqrt{10}$

2 d $\left(2-\sqrt{2}\right)\left(3+\sqrt{5}\right) = 6+2\sqrt{5}-3\sqrt{2}-\sqrt{2}\sqrt{5}$
$= 6+2\sqrt{5}-3\sqrt{2}-\sqrt{10}$

e $\left(2-\sqrt{3}\right)\left(3-\sqrt{7}\right) = 6-2\sqrt{7}-3\sqrt{3}+\sqrt{3}\sqrt{7}$
$= 6-2\sqrt{7}-3\sqrt{3}+\sqrt{21}$

f $\left(4+\sqrt{5}\right)\left(2+\sqrt{5}\right) = 8+4\sqrt{5}+2\sqrt{5}+\sqrt{5}\sqrt{5}$
$= 8+6\sqrt{5}+\sqrt{25}$
$= 8+6\sqrt{5}+5$
$= 13+6\sqrt{5}$

g $\left(5-\sqrt{3}\right)\left(1-\sqrt{3}\right) = 5-5\sqrt{3}-\sqrt{3}+\sqrt{3}\sqrt{3}$
$= 5-6\sqrt{3}+\sqrt{9}$
$= 5-6\sqrt{3}+3$
$= 8-6\sqrt{3}$

h $\left(4+\sqrt{3}\right)\left(2-\sqrt{3}\right) = 8-4\sqrt{3}+2\sqrt{3}-\sqrt{3}\sqrt{3}$
$= 8-2\sqrt{3}-\sqrt{9}$
$= 8-2\sqrt{3}-3$
$= 5-2\sqrt{3}$

i $\left(7-\sqrt{11}\right)\left(2+\sqrt{11}\right)$
$= 14+7\sqrt{11}-2\sqrt{11}-\sqrt{11}\sqrt{11}$
$= 14+5\sqrt{11}-\sqrt{121}$
$= 14+5\sqrt{11}-11$
$= 3+5\sqrt{11}$

3 $\sqrt{75}-\sqrt{12} = \sqrt{25}\sqrt{3}-\sqrt{4}\sqrt{3}$
$= 5\sqrt{3}-2\sqrt{3}$
$= 3\sqrt{3}$

Algebraic expressions 1F

1 a $$\frac{1}{\sqrt{5}}=\frac{1\times\sqrt{5}}{\sqrt{5}\times\sqrt{5}}=\frac{\sqrt{5}}{5}$$

b $$\frac{1}{\sqrt{11}}=\frac{1\times\sqrt{11}}{\sqrt{11}\times\sqrt{11}}=\frac{\sqrt{11}}{11}$$

c $$\frac{1}{\sqrt{2}}=\frac{1\times\sqrt{2}}{\sqrt{2}\times\sqrt{2}}=\frac{\sqrt{2}}{2}$$

d $$\frac{\sqrt{3}}{\sqrt{15}}=\frac{\sqrt{3}\times\sqrt{15}}{\sqrt{15}\times\sqrt{15}}=\frac{\sqrt{3\times15}}{15}=\frac{\sqrt{45}}{15}=\frac{\sqrt{9\times5}}{15}=\frac{\sqrt{9}\times\sqrt{5}}{15}=\frac{\sqrt{5}}{5}$$

e $$\frac{\sqrt{12}}{\sqrt{48}}=\frac{\sqrt{12}}{\sqrt{12}\times\sqrt{4}}=\frac{1}{\sqrt{4}}=\frac{1}{2}$$

f $$\frac{\sqrt{5}}{\sqrt{80}}=\frac{\sqrt{5}}{\sqrt{5}\times\sqrt{16}}=\frac{1}{\sqrt{16}}=\frac{1}{4}$$

g $$\frac{\sqrt{12}}{\sqrt{156}}=\frac{\sqrt{12}}{\sqrt{12}\times\sqrt{13}}=\frac{1}{\sqrt{13}}=\frac{1\times\sqrt{13}}{\sqrt{13}\times\sqrt{13}}=\frac{\sqrt{13}}{13}$$

h $$\frac{\sqrt{7}}{\sqrt{63}}=\frac{\sqrt{7}}{\sqrt{7}\times\sqrt{9}}=\frac{1}{\sqrt{9}}=\frac{1}{3}$$

2 a $$\frac{1}{1+\sqrt{3}}=\frac{1\times\left(1-\sqrt{3}\right)}{\left(1+\sqrt{3}\right)\left(1-\sqrt{3}\right)}=\frac{1-\sqrt{3}}{1-\sqrt{3}+\sqrt{3}-\sqrt{9}}=\frac{1-\sqrt{3}}{-2}\text{ or }\frac{-1+\sqrt{3}}{2}$$

b $$\frac{1}{2+\sqrt{5}}=\frac{1\times\left(2-\sqrt{5}\right)}{\left(2+\sqrt{5}\right)\left(2-\sqrt{5}\right)}=\frac{2-\sqrt{5}}{4-5}=\frac{2-\sqrt{5}}{-1}=-2+\sqrt{5}$$

c $$\frac{1}{3-\sqrt{7}}=\frac{3+\sqrt{7}}{\left(3-\sqrt{7}\right)\left(3+\sqrt{7}\right)}=\frac{3+\sqrt{7}}{9-7}=\frac{3+\sqrt{7}}{2}$$

2 d $\dfrac{4}{3-\sqrt{5}}=\dfrac{4\times\left(3+\sqrt{5}\right)}{\left(3-\sqrt{5}\right)\left(3+\sqrt{5}\right)}$

$=\dfrac{12+4\sqrt{5}}{9-5}$

$=\dfrac{12+4\sqrt{5}}{4}$

$=3+\sqrt{5}$

e $\dfrac{1}{\sqrt{5}-\sqrt{3}}=\dfrac{\sqrt{5}+\sqrt{3}}{\left(\sqrt{5}-\sqrt{3}\right)\left(\sqrt{5}+\sqrt{3}\right)}$

$=\dfrac{\sqrt{5}+\sqrt{3}}{5-3}$

$=\dfrac{\sqrt{5}+\sqrt{3}}{2}$

f $\dfrac{3-\sqrt{2}}{4-\sqrt{5}}=\dfrac{\left(3-\sqrt{2}\right)\left(4+\sqrt{5}\right)}{\left(4-\sqrt{5}\right)\left(4+\sqrt{5}\right)}$

$=\dfrac{\left(3-\sqrt{2}\right)\left(4+\sqrt{5}\right)}{16-5}$

$=\dfrac{\left(3-\sqrt{2}\right)\left(4+\sqrt{5}\right)}{11}$

$=\dfrac{12+3\sqrt{5}-4\sqrt{2}-\sqrt{10}}{11}$

g $\dfrac{5}{2+\sqrt{5}}=\dfrac{5\times\left(2-\sqrt{5}\right)}{\left(2+\sqrt{5}\right)\left(2-\sqrt{5}\right)}$

$=\dfrac{5\left(2-\sqrt{5}\right)}{4-5}$

$=\dfrac{5\left(2-\sqrt{5}\right)}{-1}$

$=5\left(\sqrt{5}-2\right)$

h $\dfrac{5\sqrt{2}}{\sqrt{8}-\sqrt{7}}=\dfrac{5\sqrt{2}\left(\sqrt{8}+\sqrt{7}\right)}{\left(\sqrt{8}-\sqrt{7}\right)\left(\sqrt{8}+\sqrt{7}\right)}$

$=\dfrac{5\left(\sqrt{8\times2}+\sqrt{2\times7}\right)}{8-7}$

h $\dfrac{5\sqrt{2}}{\sqrt{8}-\sqrt{7}}=\dfrac{5\left(\sqrt{16}+\sqrt{14}\right)}{1}$

$=5\left(4+\sqrt{14}\right)$

i $\dfrac{11}{3+\sqrt{11}}=\dfrac{11\left(3-\sqrt{11}\right)}{\left(3+\sqrt{11}\right)\left(3-\sqrt{11}\right)}$

$=\dfrac{11\left(3-\sqrt{11}\right)}{9-11}$

$=\dfrac{11\left(3-\sqrt{11}\right)}{-2}$

$=\dfrac{-11(3-\sqrt{11})}{2}$

j $\dfrac{\sqrt{3}-\sqrt{7}}{\sqrt{3}+\sqrt{7}}=\dfrac{\left(\sqrt{3}-\sqrt{7}\right)\left(\sqrt{3}-\sqrt{7}\right)}{\left(\sqrt{3}+\sqrt{7}\right)\left(\sqrt{3}-\sqrt{7}\right)}$

$=\dfrac{3-\sqrt{21}-\sqrt{21}+7}{3-7}$

$=\dfrac{10-2\sqrt{21}}{-4}$

$=\dfrac{5-\sqrt{21}}{-2}$ or $\dfrac{\sqrt{21}-5}{2}$

k $\dfrac{\sqrt{17}-\sqrt{11}}{\sqrt{17}+\sqrt{11}}=\dfrac{\left(\sqrt{17}-\sqrt{11}\right)\left(\sqrt{17}-\sqrt{11}\right)}{\left(\sqrt{17}+\sqrt{11}\right)\left(\sqrt{17}-\sqrt{11}\right)}$

$=\dfrac{17-\sqrt{187}-\sqrt{187}+11}{17-11}$

$=\dfrac{28-2\sqrt{187}}{6}$

$=\dfrac{14-\sqrt{187}}{3}$

l $\dfrac{\sqrt{41}+\sqrt{29}}{\sqrt{41}-\sqrt{29}}=\dfrac{\left(\sqrt{41}+\sqrt{29}\right)\left(\sqrt{41}+\sqrt{29}\right)}{\left(\sqrt{41}-\sqrt{29}\right)\left(\sqrt{41}+\sqrt{29}\right)}$

$=\dfrac{41+2\sqrt{41}\sqrt{29}+29}{41-29}$

$=\dfrac{70+2\sqrt{1189}}{12}$

$=\dfrac{35+\sqrt{1189}}{6}$

2 m $$\frac{\sqrt{2}-\sqrt{3}}{\sqrt{3}-\sqrt{2}}=\frac{\left(\sqrt{2}-\sqrt{3}\right)\left(\sqrt{3}+\sqrt{2}\right)}{\left(\sqrt{3}-\sqrt{2}\right)\left(\sqrt{3}+\sqrt{2}\right)}$$
$$=\frac{\sqrt{6}+2-3-\sqrt{6}}{3-2}$$
$$=\frac{-1}{1}$$
$$=-1$$

3 a $$\frac{1}{(3-\sqrt{2})^2}=\frac{1}{\left(3-\sqrt{2}\right)\left(3-\sqrt{2}\right)}$$
$$=\frac{1}{9-3\sqrt{2}-3\sqrt{2}+\sqrt{4}}$$
$$=\frac{1}{11-6\sqrt{2}}$$
$$=\frac{1\times(11+6\sqrt{2})}{(11-6\sqrt{2})(11+6\sqrt{2})}$$
$$=\frac{11+6\sqrt{2}}{121+66\sqrt{2}-66\sqrt{2}-72}$$
$$=\frac{11+6\sqrt{2}}{49}$$

b $$\frac{1}{\left(2+\sqrt{5}\right)^2}=\frac{1}{\left(2+\sqrt{5}\right)\left(2+\sqrt{5}\right)}$$
$$=\frac{1}{4+2\sqrt{5}+2\sqrt{5}+\sqrt{25}}$$
$$=\frac{1}{9+4\sqrt{5}}$$
$$=\frac{1\times(9-4\sqrt{5})}{(9+4\sqrt{5})(9-4\sqrt{5})}$$
$$=\frac{9-4\sqrt{5}}{81-36\sqrt{5}+36\sqrt{5}-80}$$
$$=\frac{9-4\sqrt{5}}{1}$$
$$=9-4\sqrt{5}$$

c $$\frac{4}{\left(3-\sqrt{2}\right)^2}=\frac{4}{\left(3-\sqrt{2}\right)\left(3-\sqrt{2}\right)}$$
$$=\frac{4}{9-3\sqrt{2}-3\sqrt{2}+\sqrt{4}}$$

3 c $$\frac{4}{\left(3-\sqrt{2}\right)^2}=\frac{4}{11-6\sqrt{2}}$$
$$=\frac{4\times(11+6\sqrt{2})}{(11-6\sqrt{2})(11+6\sqrt{2})}$$
$$=\frac{44+24\sqrt{2}}{121+66\sqrt{2}-66\sqrt{2}-72}$$
$$=\frac{44+24\sqrt{2}}{49}$$

d $$\frac{3}{\left(5+\sqrt{2}\right)^2}=\frac{3}{\left(5+\sqrt{2}\right)\left(5+\sqrt{2}\right)}$$
$$=\frac{3}{25+5\sqrt{2}+5\sqrt{2}+\sqrt{4}}$$
$$=\frac{3}{27+10\sqrt{2}}$$
$$=\frac{3\times\left(27-10\sqrt{2}\right)}{\left(27+10\sqrt{2}\right)\left(27-10\sqrt{2}\right)}$$
$$=\frac{3\times\left(27-10\sqrt{2}\right)}{729-270\sqrt{2}+270\sqrt{2}-200}$$
$$=\frac{81-30\sqrt{2}}{529}$$

e $$\frac{1}{\left(5+\sqrt{2}\right)\left(3-\sqrt{2}\right)}=\frac{1}{15-5\sqrt{2}+3\sqrt{2}-\sqrt{4}}$$
$$=\frac{1}{13-2\sqrt{2}}$$
$$=\frac{1\times\left(13+2\sqrt{2}\right)}{\left(13-2\sqrt{2}\right)\left(13+2\sqrt{2}\right)}$$
$$=\frac{13+2\sqrt{2}}{169+26\sqrt{2}-26\sqrt{2}-8}$$
$$=\frac{13+2\sqrt{2}}{161}$$

f $$\frac{2}{\left(5-\sqrt{3}\right)\left(2+\sqrt{3}\right)}=\frac{2}{10+5\sqrt{3}-2\sqrt{3}-\sqrt{9}}$$
$$=\frac{2}{7+3\sqrt{3}}$$
$$=\frac{2\times\left(7-3\sqrt{3}\right)}{\left(7+3\sqrt{3}\right)\left(7-3\sqrt{3}\right)}$$

3 f $$\frac{2}{\left(5-\sqrt{3}\right)\left(2+\sqrt{3}\right)}=\frac{14-6\sqrt{3}}{49-21\sqrt{3}+21\sqrt{3}-27}$$
$$=\frac{14-6\sqrt{3}}{22}$$
$$=\frac{7-3\sqrt{3}}{11}$$

4 $$\frac{3-2\sqrt{5}}{\sqrt{5}-1}=\frac{\left(3-2\sqrt{5}\right)\left(\sqrt{5}+1\right)}{\left(\sqrt{5}-1\right)\left(\sqrt{5}+1\right)}$$
$$=\frac{3\sqrt{5}+3-10-2\sqrt{5}}{5+\sqrt{5}-\sqrt{5}-1}$$
$$=\frac{\sqrt{5}-7}{4}$$
$$=\frac{-7}{4}+\frac{\sqrt{5}}{4}$$

$$p=-\frac{7}{4},q=\frac{\sqrt{5}}{4}$$

Algebraic expressions, Mixed Exercise 1

1 a $y^3 \times y^5$
$= y^{3+5}$
$= y^8$

b $3x^2 \times 2x^5$
$= 3 \times 2 \times x^{2+5}$
$= 6x^7$

c $(4x^2)^3 \div 2x^5$
$= 4^3 x^{2\times 3} \div 2x^5$
$= 64x^6 \div 2x^5$
$= 32x^{6-5}$
$= 32x$

d $4b^2 \times 3b^3 \times b^4$
$= 4 \times 3 \times b^{2+3+4}$
$= 12b^9$

2 a $(x+3)(x-5)$
$= x^2 - 5x + 3x - 15$
$= x^2 - 2x - 15$

b $(2x-7)(3x+1)$
$= 6x^2 + 2x - 21x - 7$
$= 6x^2 - 19x - 7$

c $(2x+5)(3x-y+2)$
$= 2x(3x-y+2) + 5(3x-y+2)$
$= 6x^2 - 2xy + 4x + 15x - 5y + 10$
$= 6x^2 - 2xy + 19x - 5y + 10$

3 a $x(x+4)(x-1)$
$= (x^2+4x)(x-1)$
$= x^3 - x^2 + 4x^2 - 4x$
$= x^3 + 3x^2 - 4x$

b $(x+2)(x-3)(x+7)$
$= (x^2 - 3x + 2x - 6)(x+7)$
$= (x^2 - x - 6)(x+7)$
$= x^2(x+7) - x(x+7) - 6(x+7)$
$= x^3 + 7x^2 - x^2 - 7x - 6x - 42$
$= x^3 + 6x^2 - 13x - 42$

c $(2x+3)(x-2)(3x-1)$
$= (2x^2 - 4x + 3x - 6)(3x-1)$
$= (2x^2 - x - 6)(3x-1)$
$= 2x^2(3x-1) - x(3x-1) - 6(3x-1)$
$= 6x^3 - 2x^2 - 3x^2 + x - 18x + 6$
$= 6x^3 - 5x^2 - 17x + 6$

4 a $3(5y+4)$
$= 15y + 12$

b $5x^2(3 - 5x + 2x^2)$
$= 15x^2 - 25x^3 + 10x^4$

c $5x(2x+3) - 2x(1-3x)$
$= 10x^2 + 15x - 2x + 6x^2$
$= 16x^2 + 13x$

d $3x^2(1+3x) - 2x(3x-2)$
$= 3x^2 + 9x^3 - 6x^2 + 4x$
$= 9x^3 - 3x^2 + 4x$

5 a $3x^2 + 4x$
$= x(3x+4)$

b $4y^2 + 10y$
$= 2y(2y+5)$

c $x^2 + xy + xy^2$
$= x(x + y + y^2)$

d $8xy^2 + 10x^2y$
$= 2xy(4y + 5x)$

6 a $x^2 + 3x + 2$
$= x^2 + x + 2x + 2$
$= x(x+1) + 2(x+1)$
$= (x+1)(x+2)$

b $3x^2 + 6x$
$= 3x(x+2)$

c $x^2 - 2x - 35$
$= x^2 - 7x + 5x - 35$
$= x(x-7) + 5(x-7)$
$= (x-7)(x+5)$

d $2x^2 - x - 3$
$= 2x^2 - 3x + 2x - 3$
$= x(2x-3) + (2x-3)$
$= (2x-3)(x+1)$

e $5x^2 - 13x - 6$
$= 5x^2 + 2x - 15x - 6$
$= x(5x+2) - 3(5x+2)$
$= (5x+2)(x-3)$

6 f $6 - 5x - x^2$
$= 6 + x - 6x - x^2$
$= (6 + x) - x(6 + x)$
$= (6 + x)(1 - x)$

7 a $2x^3 + 6x$
$= 2x(x^2 + 3)$

b $x^3 - 36x$
$= x(x^2 - 36)$
$= x(x^2 - 6^2)$
$= x(x + 6)(x - 6)$

c $2x^3 + 7x^2 - 15x$
$= x(2x^2 + 7x - 15)$
$= x(2x^2 - 3x + 10x - 15)$
$= x(x(2x - 3) + 5(2x - 3))$
$= x(2x - 3)(x + 5)$

8 a $9x^3 \div 3x^{-3}$
$= 3x^{3-(-3)}$
$= 3x^6$

b $\left(4^{\frac{3}{2}}\right)^{\frac{1}{3}}$
$= 4^{\frac{3}{2}\times\frac{1}{3}}$
$= 4^{\frac{1}{2}}$
$= \sqrt{4}$
$= \pm 2$

c $3x^{-2} \times 2x^4$
$= 6x^{-2+4}$
$= 6x^2$

d $3x^{\frac{1}{3}} \div 6x^{\frac{2}{3}}$
$= \frac{1}{2}x^{\frac{1}{3}-\frac{2}{3}}$
$= \frac{1}{2}x^{-\frac{1}{3}}$ or $\frac{1}{2\left(\sqrt[3]{x}\right)}$

9 a $\left(\frac{8}{27}\right)^{\frac{2}{3}}$
$= \left(\frac{\sqrt[3]{8}}{\sqrt[3]{27}}\right)^2$

9 a $\left(\frac{8}{27}\right)^{\frac{2}{3}}$
$= \left(\frac{2}{3}\right)^2$
$= \frac{4}{9}$

b $\left(\frac{225}{289}\right)^{\frac{3}{2}}$
$= \left(\frac{\sqrt{225}}{\sqrt{289}}\right)^3$
$= \pm\frac{15^3}{17^3}$
$= \pm\frac{3375}{4913}$

10 a $\frac{3}{\sqrt{63}}$
$= \frac{3}{\sqrt{9}\times\sqrt{7}}$
$= \frac{3}{3\sqrt{7}}$
$= \frac{1}{\sqrt{7}}$
$= \frac{\sqrt{7}}{7}$

b $\sqrt{20} + 2\sqrt{45} - \sqrt{80}$
$= \sqrt{4}\sqrt{5} + 2\sqrt{9}\sqrt{5} - \sqrt{16}\sqrt{5}$
$= 2\sqrt{5} + 2\times3\sqrt{5} - 4\sqrt{5}$
$= \sqrt{5}(2 + 6 - 4)$
$= 4\sqrt{5}$

11 a When $x = 25$,
$35x^2 + 2x - 48$
$= 35 \times 25^2 + 2 \times 25 - 48$
$= 21\,877$

b $35x^2 + 2x - 48$
$= 35x^2 + 42x - 40x - 48$
$= 7x(5x + 6) - 8(5x + 6)$
$= (5x + 6)(7x - 8)$
When $x = 25$, $5x + 6 = 131$
and $7x - 8 = 167$;
both 131 and 167 are prime numbers.

12 a $\sqrt{2}\left(3+\sqrt{5}\right)$

$=3\sqrt{2}+\sqrt{2}\sqrt{5}$

$=3\sqrt{2}+\sqrt{10}$

b $\left(2-\sqrt{5}\right)\left(5+\sqrt{3}\right)$

$=10+2\sqrt{3}-5\sqrt{5}-\sqrt{3}\sqrt{5}$

$=10+2\sqrt{3}-5\sqrt{5}-\sqrt{15}$

c $\left(6-\sqrt{2}\right)\left(4-\sqrt{7}\right)$

$=24-6\sqrt{7}-4\sqrt{2}+\sqrt{2}\sqrt{7}$

$=24-6\sqrt{7}-4\sqrt{2}+\sqrt{14}$

13 a $\dfrac{1}{\sqrt{3}}$

$=\dfrac{1\times\sqrt{3}}{\sqrt{3}\times\sqrt{3}}$

$=\dfrac{\sqrt{3}}{3}$

b $\dfrac{1}{\sqrt{2}-1}$

$=\dfrac{\sqrt{2}+1}{\left(\sqrt{2}-1\right)\left(\sqrt{2}+1\right)}$

$=\dfrac{\sqrt{2}+1}{2-1}$

$=\sqrt{2}+1$

c $\dfrac{3}{\sqrt{3}-2}$

$=\dfrac{3\left(\sqrt{3}+2\right)}{\left(\sqrt{3}-2\right)\left(\sqrt{3}+2\right)}$

$=\dfrac{3\sqrt{3}+6}{3-4}$

$=\dfrac{3\sqrt{3}+6}{-1}$

$=-3\sqrt{3}-6$

d $\dfrac{\sqrt{23}-\sqrt{37}}{\sqrt{23}+\sqrt{37}}$

$=\dfrac{\left(\sqrt{23}-\sqrt{37}\right)\left(\sqrt{23}-\sqrt{37}\right)}{\left(\sqrt{23}+\sqrt{37}\right)\left(\sqrt{23}-\sqrt{37}\right)}$

$=\dfrac{23-2\sqrt{23}\sqrt{37}+37}{23-37}$

$=\dfrac{60-2\sqrt{851}}{-14}$

$=\dfrac{30-\sqrt{851}}{-7}$

e $\dfrac{1}{\left(2+\sqrt{3}\right)^2}$

$=\dfrac{1}{\left(2+\sqrt{3}\right)\left(2+\sqrt{3}\right)}$

$=\dfrac{1}{4+2\sqrt{3}+2\sqrt{3}+\sqrt{9}}$

$=\dfrac{1}{4+2\sqrt{3}+2\sqrt{3}+\sqrt{9}}$

$=\dfrac{1\times\left(7-4\sqrt{3}\right)}{\left(7+4\sqrt{3}\right)\left(7-4\sqrt{3}\right)}$

$=\dfrac{7-4\sqrt{3}}{49-28\sqrt{3}+28\sqrt{3}-48}$

$=\dfrac{7-4\sqrt{3}}{1}$

$=7-4\sqrt{3}$

f $\dfrac{1}{\left(4-\sqrt{7}\right)^2}$

$=\dfrac{1}{\left(4-\sqrt{7}\right)\left(4-\sqrt{7}\right)}$

$=\dfrac{1}{16-4\sqrt{7}-4\sqrt{7}+\sqrt{49}}$

$=\dfrac{1}{23-8\sqrt{7}}$

13 f $\dfrac{1}{\left(4-\sqrt{7}\right)^2}$

$=\dfrac{1\times\left(23+8\sqrt{7}\right)}{\left(23-8\sqrt{7}\right)\left(23+8\sqrt{7}\right)}$

$=\dfrac{23+8\sqrt{7}}{529+184\sqrt{7}-184\sqrt{7}-448}$

$=\dfrac{23+8\sqrt{7}}{81}$

14 a $x^3-x^2-17x-15$
$=(x+3)(x^2+bx+c)$
$=x^3+bx^2+cx+3x^2+3bx+3c$
$=x^3+bx^2+3x^2+3bx+cx+3c$
$=x^3-x^2-17x-15$
Equating the coefficients gives $b+3=-1$,
$3b+c=-17$, $3c=-15$
Using $b+3=-1$, $b=-4$
Using $3c=-15$, $c=-5$

b $x^3-x^2-17x-15$
$=(x+3)(x^2-4x-5)$
$=(x+3)(x^2-5x+x-5)$
$=(x+3)(x(x-5)+(x-5))$
$=(x+3)(x-5)(x+1)$

15 a $y^{\frac{1}{3}}$

$=\left(\dfrac{1}{64}x^3\right)^{\frac{1}{3}}$

$=\dfrac{1}{\sqrt[3]{64}}x^{3\times\frac{1}{3}}$

$=\dfrac{x}{4}$

b $4y^{-1}$

$=4\left(\dfrac{1}{64}x^3\right)^{-1}$

$=4\times\dfrac{1}{\frac{1}{64}}x^{3\times(-1)}$

$=256x^{-3}$

16 $\dfrac{5}{\sqrt{75}-\sqrt{50}}$

$=\dfrac{5}{\sqrt{25\times3}-\sqrt{25\times2}}$

$=\dfrac{5}{5\sqrt{3}-5\sqrt{2}}$

$=\dfrac{1}{\sqrt{3}-\sqrt{2}}$

$=\dfrac{1\times(\sqrt{3}+\sqrt{2})}{(\sqrt{3}-\sqrt{2})(\sqrt{3}+\sqrt{2})}$

$=\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{9}+\sqrt{2}\sqrt{3}-\sqrt{2}\sqrt{3}-\sqrt{4}}$

$=\dfrac{\sqrt{3}+\sqrt{2}}{1}$

$=\sqrt{3}+\sqrt{2}$

17 $\left(\sqrt{11}-5\right)\left(5-\sqrt{11}\right)$
$=5\sqrt{11}-\sqrt{121}-25+5\sqrt{11}$
$=10\sqrt{11}-36$

18 $x-64x^3$
$=x(1-64x^2)$
$=x[1^2-(8x)^2]$
$=x(1+8x)(1-8x)$

19 27^{2x+1}
$=(3^3)^{2x+1}$
$=3^{3(2x+1)}$
$=3^{6x+3}$
$=3^y$
$y=6x+3$

20 $8+x\sqrt{12}=\dfrac{8x}{\sqrt{3}}$

$8\sqrt{3}+x\sqrt{12}\sqrt{3}=8x$
$8\sqrt{3}+x\sqrt{36}=8x$
$8\sqrt{3}+6x=8x$
$8\sqrt{3}=2x$
$x=4\sqrt{3}$

21 Area $= \sqrt{12}\ \text{cm}^2$, length $= \left(1+\sqrt{3}\right)$cm

$$\text{Width} = \frac{\sqrt{12}}{1+\sqrt{3}}$$
$$= \frac{\sqrt{12}\times\left(1-\sqrt{3}\right)}{\left(1+\sqrt{3}\right)\left(1-\sqrt{3}\right)}$$
$$= \frac{\sqrt{12}-\sqrt{36}}{1-\sqrt{3}+\sqrt{3}-\sqrt{9}}$$
$$= \frac{\sqrt{4\times 3}-6}{-2}$$
$$= \frac{2\sqrt{3}-6}{-2}$$
$$= \frac{-\sqrt{3}+3}{1}$$
$$= 3-\sqrt{3}\ \text{cm}$$

22
$$\frac{\left(2-\sqrt{x}\right)^2}{\sqrt{x}}$$
$$= \frac{(2-x^{\frac{1}{2}})^2}{x^{\frac{1}{2}}}$$
$$= \frac{(2-x^{\frac{1}{2}})(2-x^{\frac{1}{2}})}{x^{\frac{1}{2}}}$$
$$= \frac{4-2x^{\frac{1}{2}}-2x^{\frac{1}{2}}\quad x^{\frac{1}{2}+\frac{1}{2}}}{x^{\frac{1}{2}}}$$
$$= x^{-\frac{1}{2}}(4-4x^{\frac{1}{2}}+x)$$
$$= 4x^{-\frac{1}{2}}-4x^{-\frac{1}{2}+\frac{1}{2}}+x^{1-\frac{1}{2}}$$
$$= 4x^{-\frac{1}{2}}-4x^0+x^{\frac{1}{2}}$$
$$= 4x^{-\frac{1}{2}}-4+x^{\frac{1}{2}}$$
$$= \frac{4}{\sqrt{x}}-4+\sqrt{x}$$

23 a $243\sqrt{3}$

$$= 3^5\times 3^{\frac{1}{2}}$$
$$= 3^{5+\frac{1}{2}}$$
$$= 3^{\frac{11}{2}}$$
$$a = \frac{11}{2}$$

b From part a: $3^x\times 27^y = 243\sqrt{3} = 3^{\frac{11}{2}}$

$$3^x\times 3^{3y} = 3^{\frac{11}{2}}$$
$$3^{x+3y} = 3^{\frac{11}{2}}$$

So $x+3y = \dfrac{11}{2}$

$$3y = \frac{11-2x}{2}$$
$$y = \frac{11-2x}{6}$$

24
$$\frac{4x^3+x^{\frac{5}{2}}}{\sqrt{x}}$$
$$= \frac{4x^3+x^{\frac{5}{2}}}{x^{\frac{1}{2}}}$$
$$= x^{-\frac{1}{2}}\left(4x^3+x^{\frac{5}{2}}\right)$$
$$= 4x^{-\frac{1}{2}+3}+x^{-\frac{1}{2}+\frac{5}{2}}$$
$$= 4x^{\frac{5}{2}}+x^2$$
$$= 4x^a+x^b$$
$$a = \frac{5}{2},\ b = 2$$

Challenge

a $\left(\sqrt{a}+\sqrt{b}\right)\left(\sqrt{a}-\sqrt{b}\right)$

$= a-\sqrt{a}\sqrt{b}+\sqrt{a}\sqrt{b}-b$

$= a - b$

b Rationalising the denominators:

$$\frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{4}}+\ldots+\frac{1}{\sqrt{24}+\sqrt{25}}$$

$=$

$$\frac{\sqrt{1}-\sqrt{2}}{\left(\sqrt{1}+\sqrt{2}\right)\left(\sqrt{1}-\sqrt{2}\right)}+\frac{\sqrt{2}-\sqrt{3}}{\left(\sqrt{2}+\sqrt{3}\right)\left(\sqrt{\ }-\sqrt{3}\right)}+\frac{\sqrt{3}-\sqrt{4}}{\left(\sqrt{3}+\sqrt{4}\right)\left(\sqrt{\ }-\sqrt{4}\right)}\ \ldots+\frac{\sqrt{24}-\sqrt{25}}{\left(\sqrt{24}+\sqrt{25}\right)\left(\sqrt{24}-\sqrt{25}\right)}$$

$$=\frac{\sqrt{1}-\sqrt{2}}{1-2}+\frac{\sqrt{2}-\sqrt{3}}{2-3}+\frac{\sqrt{3}-\sqrt{4}}{3-4}+\ldots+\frac{\sqrt{24}-\sqrt{25}}{24-25}$$

$$=-\left(\sqrt{1}-\sqrt{2}\right)-\left(\sqrt{2}-\sqrt{3}\right)-\left(\sqrt{3}-\sqrt{4}\right)-\ldots-\left(\sqrt{24}-\sqrt{25}\right)$$

$$=-\sqrt{1}+\sqrt{2}-\sqrt{2}+\sqrt{3}-\sqrt{3}+\sqrt{4}-\ldots-\sqrt{24}+\sqrt{25}$$

$$=-\sqrt{1}+\sqrt{25}$$

$= -1 + 5$

$= 4$

Quadratics 2A

1 a $x^2 + 3x + 2 = 0$
$(x + 1)(x + 2) = 0$
$x + 1 = 0$ or $x + 2 = 0$
So $x = -1$ or $x = -2$

b $x^2 + 5x + 4 = 0$
$(x + 1)(x + 4) = 0$
$x + 1 = 0$ or $x + 4 = 0$
So $x = -1$ or $x = -4$

c $x^2 + 7x + 10 = 0$
$(x + 2)(x + 5) = 0$
$x + 2 = 0$ or $x + 5 = 0$
So $x = -2$ or $x = -5$

d $x^2 - x - 6 = 0$
$(x - 3)(x + 2) = 0$
$x - 3 = 0$ or $x + 2 = 0$
So $x = 3$ or $x = -2$

e $x^2 - 8x + 15 = 0$
$(x - 3)(x - 5) = 0$
$x - 3 = 0$ or $x - 5 = 0$
So $x = 3$ or $x = 5$

f $x^2 - 9x + 20 = 0$
$(x - 4)(x - 5) = 0$
$x - 4 = 0$ or $x - 5 = 0$
So $x = 4$ or $x = 5$

g $x^2 - 5x - 6 = 0$
$(x - 6)(x + 1) = 0$
$x - 6 = 0$ or $x + 1 = 0$
So $x = 6$ or $x = -1$

h $x^2 - 4x - 12 = 0$
$(x - 6)(x + 2) = 0$
$x - 6 = 0$ or $x + 2 = 0$
So $x = 6$ or $x = -2$

2 a $x^2 = 4x$
$x^2 - 4x = 0$
$x(x - 4) = 0$
$x = 0$ or $x - 4 = 0$
So $x = 0$ or $x = 4$

b $x^2 = 25x$
$x^2 - 25x = 0$
$x(x - 25) = 0$
$x = 0$ or $x - 25 = 0$
So $x = 0$ or $x = 25$

2 c $3x^2 = 6x$
$3x^2 - 6x = 0$
$3x(x - 2) = 0$
$x = 0$ or $x - 2 = 0$
So $x = 0$ or $x = 2$

d $5x^2 = 30x$
$5x^2 - 30x = 0$
$5x(x - 6) = 0$
$x = 0$ or $x - 6 = 0$
So $x = 0$ or $x = 6$

e $2x^2 + 7x + 3 = 0$
$(2x + 1)(x + 3) = 0$
$2x + 1 = 0$ or $x + 3 = 0$
$2x = -1$ or $x = -3$
So $x = -\frac{1}{2}$ or $x = -3$

f $6x^2 - 7x - 3 = 0$
$(3x + 1)(2x - 3) = 0$
$3x + 1 = 0$ or $2x - 3 = 0$
So $x = -\frac{1}{3}$ or $x = \frac{3}{2}$

g $6x^2 - 5x - 6 = 0$
$(3x + 2)(2x - 3) = 0$
$3x + 2 = 0$ or $2x - 3 = 0$
So $x = -\frac{2}{3}$ or $x = \frac{3}{2}$

h $4x^2 - 16x + 15 = 0$
$(2x - 3)(2x - 5) = 0$
$2x - 3 = 0$ or $2x - 5 = 0$
So $x = \frac{3}{2}$ or $x = \frac{5}{2}$

3 a $3x^2 + 5x = 2$
$3x^2 + 5x - 2 = 0$
$(3x - 1)(x + 2) = 0$
$3x - 1 = 0$ or $x + 2 = 0$
So $x = \frac{1}{3}$ or $x = -2$

b $(2x - 3)^2 = 9$
$2x - 3 = \pm 3$
$2x = \pm 3 + 3$
$$x = \frac{\pm 3 + 3}{2}$$
So $x = 3$ or $x = 0$

3 c $(x-7)^2 = 36$

$x - 7 = \pm 6$

$x = \pm 6 + 7$

So $x = 1$ or $x = 13$

d $2x^2 = 8$

$x^2 = 4$

$x = \pm 2$

So $x = 2$ or $x = -2$

e $3x^2 = 5$

$x = \pm\sqrt{\dfrac{5}{3}}$

So $x = \sqrt{\dfrac{5}{3}}$ or $x = -\sqrt{\dfrac{5}{3}}$

f $(x-3)^2 = 13$

$x - 3 = \pm\sqrt{13}$

$x = 3 \pm \sqrt{13}$

So $x = 3 + \sqrt{13}$ or $x = 3 - \sqrt{13}$

g $(3x-1)^2 = 11$

$3x - 1 = \pm\sqrt{11}$

$3x = 1 \pm \sqrt{11}$

$x = \dfrac{1 \pm \sqrt{11}}{3}$

h $5x^2 - 10x^2 = -7 + x + x^2$

$-6x^2 - x + 7 = 0$

$6x^2 + x - 7 = 0$

$(x-1)(6x+7) = 0$

$x - 1 = 0$ or $6x + 7 = 0$

So $x = 1$ or $x = -\frac{7}{6}$

i $6x^2 - 7 = 11x$

$6x^2 - 11x - 7 = 0$

$(3x-7)(2x+1) = 0$

$3x - 7 = 0$ or $2x + 1 = 0$

So $x = \frac{7}{3}$ or $x = -\frac{1}{2}$

j $4x^2 + 17x = 6x - 2x^2$

$6x^2 + 11x = 0$

$x(6x + 11) = 0$

$x = 0$ or $6x + 11 = 0$

So $x = 0$ or $x = -\frac{11}{6}$

4 Area of shape = 44

$x \times x + x(x+3) = 44$

$x^2 + x^2 + 3x = 44$

$2x^2 + 3x - 44 = 0$

$(2x + 11)(x - 4) = 0$

Then either $2x + 11 = 0 \Rightarrow x = -\frac{11}{2}$

or $x - 4 = 0 \Rightarrow x = 4$

x represents a length so cannot be negative, giving $x = 4$.

5 $5x + 3 = \sqrt{3x + 7}$

$(5x+3)^2 = 3x + 7$

$(5x+3)(5x+3) = 3x + 7$

$25x^2 + 15x + 15x + 9 = 3x + 7$

$25x^2 + 27x + 2 = 0$

$(25x + 2)(x + 1) = 0$

Then either $25x + 2 = 0 \Rightarrow x = -\frac{2}{25}$

or $x + 1 = 0 \Rightarrow x = -1$

Quadratics 2B

1 a $x^2 + 3x + 1 = 0$

$$x = \frac{-3 \pm \sqrt{3^2 - 4(1)(1)}}{2 \times 1}$$

$$x = \frac{-3 \pm \sqrt{9 - 4}}{2}$$

$$x = \frac{-3 \pm \sqrt{5}}{2}$$

Then, $x = \dfrac{-3 + \sqrt{5}}{2}$ or $x = \dfrac{-3 - \sqrt{5}}{2}$

b $x^2 - 3x - 2 = 0$

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(-2)}}{2 \times 1}$$

$$x = \frac{3 \pm \sqrt{9 + 8}}{2}$$

$$x = \frac{3 \pm \sqrt{17}}{2}$$

Then, $x = \dfrac{3 + \sqrt{17}}{2}$ or $x = \dfrac{3 - \sqrt{17}}{2}$

c $x^2 + 6x + 6 = 0$

$$x = \frac{-6 \pm \sqrt{6^2 - 4(1)(6)}}{2 \times 1}$$

$$x = \frac{-6 \pm \sqrt{36 - 24}}{2}$$

$$x = \frac{-6 \pm \sqrt{12}}{2}$$

$$x = \frac{-6 \pm \sqrt{4 \times 3}}{2}$$

$$x = \frac{-6 \pm 2\sqrt{3}}{2}$$

$$x = -3 \pm \sqrt{3}$$

Then, $x = -3 + \sqrt{3}$ or $x = -3 - \sqrt{3}$

d $x^2 - 5x - 2 = 0$

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(-2)}}{2 \times 1}$$

$$x = \frac{5 \pm \sqrt{25 + 8}}{2}$$

$$x = \frac{5 \pm \sqrt{33}}{2}$$

d Then, $x = \dfrac{5 + \sqrt{33}}{2}$ or $x = \dfrac{5 - \sqrt{33}}{2}$

e $3x^2 + 10x - 2 = 0$

$$x = \frac{-10 \pm \sqrt{10^2 - 4(3)(-2)}}{2 \times 3}$$

$$x = \frac{-10 \pm \sqrt{100 + 24}}{6}$$

$$x = \frac{-10 \pm \sqrt{124}}{6}$$

$$x = \frac{-10 \pm \sqrt{4 \times 31}}{6}$$

$$x = \frac{-10 \pm 2\sqrt{31}}{6}$$

$$x = \frac{-5 \pm \sqrt{31}}{3}$$

Then, $x = \dfrac{-5 + \sqrt{31}}{3}$ or $x = \dfrac{-5 - \sqrt{31}}{3}$

f $4x^2 - 4x - 1 = 0$

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(4)(-1)}}{2 \times 4}$$

$$x = \frac{4 \pm \sqrt{16 + 16}}{8}$$

$$x = \frac{4 \pm \sqrt{32}}{8}$$

$$x = \frac{4 \pm \sqrt{16 \times 2}}{8}$$

$$x = \frac{4 \pm 4\sqrt{2}}{8}$$

$$x = \frac{1 \pm \sqrt{2}}{2}$$

Then, $x = \dfrac{1 + \sqrt{2}}{2}$ or $x = \dfrac{1 - \sqrt{2}}{2}$

g $4x^2 - 7x = 2$

$4x^2 - 7x - 2 = 0$

$$x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(4)(-2)}}{2 \times 4}$$

$$x = \frac{7 \pm \sqrt{49 + 32}}{8}$$

1 g $x = \dfrac{7 \pm \sqrt{81}}{8}$

$x = \dfrac{7 \pm 9}{8}$

Then, $x = 2$ or $x = -\frac{1}{4}$

h $11x^2 + 2x - 7 = 0$

$x = \dfrac{-2 \pm \sqrt{2^2 - 4(11)(-7)}}{2 \times 11}$

$x = \dfrac{-2 \pm \sqrt{4 + 308}}{22}$

$x = \dfrac{-2 \pm \sqrt{312}}{22}$

$x = \dfrac{-2 \pm \sqrt{4 \times 78}}{22}$

$x = \dfrac{-2 \pm 2\sqrt{78}}{22}$

$x = \dfrac{-1 \pm \sqrt{78}}{11}$

Then, $x = \dfrac{-1 + \sqrt{78}}{11}$ or $x = \dfrac{-1 - \sqrt{78}}{11}$

2 a $x^2 + 4x + 2 = 0$

$x = \dfrac{-4 \pm \sqrt{4^2 - 4(1)(2)}}{2 \times 1}$

$x = \dfrac{-4 \pm \sqrt{16 - 8}}{2}$

$x = \dfrac{-4 \pm \sqrt{8}}{2}$

Then, $x = -0.586$ or $x = -3.41$

b $x^2 - 8x + 1 = 0$

$x = \dfrac{-(-8) \pm \sqrt{(-8)^2 - 4(1)(1)}}{2 \times 1}$

$x = \dfrac{8 \pm \sqrt{64 - 4}}{2}$

$x = \dfrac{8 \pm \sqrt{60}}{2}$

Then, $x = 7.87$ or $x = 0.127$

c $x^2 + 11x - 9 = 0$

$x = \dfrac{-11 \pm \sqrt{11^2 - 4(1)(-9)}}{2 \times 1}$

2 c $x = \dfrac{-11 \pm \sqrt{121 + 36}}{2}$

$x = \dfrac{-11 \pm \sqrt{157}}{2}$

Then, $x = 0.765$ or $x = -11.8$

d $x^2 - 7x - 17 = 0$

$x = \dfrac{-(-7) \pm \sqrt{(-7)^2 - 4(1)(-17)}}{2 \times 1}$

$x = \dfrac{7 \pm \sqrt{49 + 68}}{2}$

$x = \dfrac{7 \pm \sqrt{117}}{2}$

Then, $x = 8.91$ or $x = -1.91$

e $5x^2 + 9x - 1 = 0$

$x = \dfrac{-9 \pm \sqrt{9^2 - 4(5)(-1)}}{2 \times 5}$

$x = \dfrac{-9 \pm \sqrt{81 + 20}}{10}$

$x = \dfrac{-9 \pm \sqrt{101}}{10}$

Then, $x = 0.105$ or $x = -1.90$

f $2x^2 - 3x - 18 = 0$

$x = \dfrac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(-18)}}{2 \times 2}$

$x = \dfrac{3 \pm \sqrt{9 + 144}}{4}$

$x = \dfrac{3 \pm \sqrt{153}}{4}$

Then, $x = 3.84$ or $x = -2.34$

g $3x^2 + 8 = 16x$

$3x^2 - 16x + 8 = 0$

$x = \dfrac{-(-16) \pm \sqrt{(-16)^2 - 4(3)(8)}}{2 \times 3}$

$x = \dfrac{16 \pm \sqrt{256 - 96}}{6}$

$x = \dfrac{16 \pm \sqrt{160}}{6}$

Then, $x = 4.77$ or $x = 0.558$

2 h $2x^2 + 11x = 5x^2 - 18$

$3x^2 - 11x - 18 = 0$

$$x = \frac{-(-11) \pm \sqrt{(-11)^2 - 4(3)(-18)}}{2 \times 3}$$

$$x = \frac{11 \pm \sqrt{121 + 216}}{6}$$

$$x = \frac{11 \pm \sqrt{337}}{6}$$

Then, $x = 4.89$ or $x = -1.23$

3 a $x^2 + 8x + 12 = 0$

$(x + 6)(x + 2) = 0$

$x + 6 = 0$ or $x + 2 = 0$

Then $x = -6$ or $x = -2$

b $x^2 + 9x - 11 = 0$

$$x = \frac{-9 \pm \sqrt{9^2 - 4(1)(-11)}}{2 \times 1}$$

$$x = \frac{-9 \pm \sqrt{81 + 44}}{2}$$

$$x = \frac{-9 \pm \sqrt{125}}{2}$$

Then, $x = 1.09$ or $x = -10.1$

c $x^2 - 9x - 1 = 0$

$$x = \frac{-(-9) \pm \sqrt{(-9)^2 - 4(1)(-1)}}{2 \times 1}$$

$$x = \frac{9 \pm \sqrt{81 + 4}}{2}$$

$$x = \frac{9 \pm \sqrt{85}}{2}$$

Then, $x = 9.11$ or $x = -0.110$

d $2x^2 + 5x + 2 = 0$

$(2x + 1)(x + 2) = 0$

$2x + 1 = 0$ or $x + 2 = 0$

Then $x = -\frac{1}{2}$ or $x = -2$

e $(2x + 8)^2 = 100$

$2x + 8 = \pm 10$

$x + 4 = \pm 5$

$x = -4 \pm 5$

Then, $x = 1$ or $x = -9$

3 f $6x^2 + 6 = 12x$

$6x^2 - 12x + 6 = 0$

$6(x^2 - 2x + 1) = 0$

$6(x - 1)(x - 1) = 0$

$x - 1 = 0$

Then, $x = 1$

g $2x^2 - 11 = 7x$

$2x^2 - 7x - 11 = 0$

$$x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(2)(-11)}}{2 \times 2}$$

$$x = \frac{7 \pm \sqrt{49 + 88}}{4}$$

$$x = \frac{7 \pm \sqrt{137}}{4}$$

Then $x = 4.68$ or $x = -1.18$

h $x = \sqrt{8x - 15}$

$x^2 = 8x - 15$

$x^2 - 8x + 15 = 0$

$(x - 3)(x - 5) = 0$

$x - 3 = 0$ or $x - 5 = 0$

Then, $x = 3$ or $x = 5$

4 Area of trapezium = 50

$\frac{1}{2}(2x)(x + (x + 10)) = 50$

$x(2x + 10) = 50$

$x^2 + 5x - 25 = 0$

$$x = \frac{-5 \pm \sqrt{5^2 - 4(1)(-25)}}{2 \times 1}$$

$$x = \frac{-5 \pm \sqrt{25 + 100}}{2}$$

$$x = \frac{-5 \pm \sqrt{125}}{2}$$

$$x = \frac{-5 \pm \sqrt{25 \times 5}}{2}$$

$$x = \frac{-5 \pm 5\sqrt{5}}{2}$$

Height = $2x = -5 \pm 5\sqrt{5} = 5\left(\pm\sqrt{5} - 1\right)$

Height cannot be negative, so height is $5\left(\sqrt{5} - 1\right)$ m.

Challenge

$$\frac{1}{x}+\frac{1}{x+2}=\frac{28}{195}$$

$$\frac{195}{x}+\frac{195}{x+2}=28$$

$$195+\frac{195}{x+2}x=28x$$

$$195(x+2)+195x=28x(x+2)$$

$$28x^2-334x-390=0$$

$$x=\frac{-(-334)\pm\sqrt{(-334)^2-4(28)(-390)}}{2\times 28}$$

$$x=\frac{334\pm\sqrt{111\ 556+43\ 680}}{56}$$

$$x=\frac{334\pm\sqrt{155\ 236}}{56}$$

x is positive, so $x = 13$

Quadratics 2C

1 **a** $x^2 + 4x = (x + 2)^2 - 2^2$
$= (x + 2)^2 - 4$

b $x^2 - 6x = (x - 3)^2 - 3^2$
$= (x - 3)^2 - 9$

c $x^2 - 16x = (x - 8)^2 - 8^2$
$= (x - 8)^2 - 64$

d $x^2 + x = \left(x+\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2$
$= \left(x+\frac{1}{2}\right)^2 - \frac{1}{4}$

e $x^2 - 14 = (x - 7)^2 - 7^2$
$= (x - 7)^2 - 49$

2 **a** $2x^2 + 16x = 2(x^2 + 8x)$
$= 2((x + 4)^2 - 4^2)$
$= 2((x + 4)^2 - 16)$
$= 2(x + 4)^2 - 32$

b $3x^2 - 24x = 3(x^2 - 8x)$
$= 3((x - 4)^2 - 4^2)$
$= 3((x - 4)^2 - 16)$
$= 3(x - 4)^2 - 48$

c $5x^2 + 20x = 5(x^2 + 4x)$
$= 5((x + 2)^2 - 2^2)$
$= 5((x + 2)^2 - 4)$
$= 5(x + 2)^2 - 20$

d $2x^2 - 5x = 2\left(x^2 - \frac{5}{2}x\right)$
$= 2\left(\left(x-\frac{5}{4}\right)^2 - \left(\frac{5}{4}\right)^2\right)$
$= 2\left(\left(x-\frac{5}{4}\right)^2 - \frac{25}{16}\right)$
$= 2\left(x-\frac{5}{4}\right)^2 - \frac{25}{8}$

e $8x - 2x^2 = -2x^2 + 8x$
$= -2(x^2 - 4x)$
$= -2((x - 2)^2 - 2^2)$
$= -2((x - 2)^2 - 4)$
$= -2(x - 2)^2 + 8$

3 **a** $2x^2 + 8x + 1 = 2(x^2 + 4x) + 1$
$= 2((x + 2)^2 - 2^2) + 1$
$= 2(x + 2)^2 - 8 + 1$
$= 2(x + 2)^2 - 7$
So $p = 2$, $q = 2$ and $r = -7$

b $5x^2 - 15x + 3 = 5(x^2 - 3x) + 3$
$= 5\left(\left(x-\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2\right) + 3$
$= 5\left(x-\frac{3}{2}\right)^2 - \frac{45}{4} + 3$
$= 5\left(x-\frac{3}{2}\right)^2 - \frac{33}{4}$
So $p = 5$, $q = \frac{3}{2}$ and $r = -\frac{33}{4}$

c $3x^2 + 2x - 1 = 3\left(x^2 + \frac{2}{3}x\right) - 1$
$= 3\left(\left(x+\frac{1}{3}\right)^2 - \left(\frac{1}{3}\right)^2\right) - 1$
$= 3\left(x+\frac{1}{3}\right)^2 - \frac{1}{3} - 1$
$= 3\left(x+\frac{1}{3}\right)^2 - \frac{4}{3}$
So $p = 3$, $q = \frac{1}{3}$ and $r = -\frac{4}{3}$

d $10 - 16x - 4x^2 = -4x^2 - 16x + 10$
$= -4(x^2 + 4x) + 10$
$= -4((x + 2)^2 - 2^2) + 10$
$= -4(x + 2)^2 + 16 + 10$
$= -4(x + 2)^2 + 26$
So $p = -4$, $q = 2$ and $r = 26$

e $2x - 8x^2 + 10 = -8x^2 + 2x + 10$
$= -8\left(x^2 - \frac{1}{4}x\right) + 10$
$= -8\left(\left(x-\frac{1}{8}\right)^2 - \left(\frac{1}{8}\right)^2\right) + 10$
$= -8\left(x-\frac{1}{8}\right)^2 + \frac{1}{8} + 10$
$= -8\left(x-\frac{1}{8}\right)^2 + \frac{81}{8}$
So $p = -8$, $q = -\frac{1}{8}$ and $r = \frac{81}{8}$

4 $x^2 + 3x + 6 = \left(x-\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + 6$
$= \left(x-\frac{3}{2}\right)^2 + \frac{15}{4}$
$a = \frac{3}{2}$ and $b = \frac{15}{4}$

5 $2 + 0.8x - 0.04x^2 = -0.04x^2 + 0.8x + 2$
$= -0.04(x^2 - 20x) + 2$
$= -0.04((x - 10)^2 - 10^2) + 2$
$= -0.04(x - 10)^2 + 4 + 2$
$= 6 - 0.04(x - 10)^2$
$A = 6$, $B = 0.04$ and $C = -10$

Quadratics 2D

1 a
$$x^2 + 6x + 1 = 0$$
$$x^2 + 6x = -1$$
$$(x + 3)^2 - 3^2 = -1$$
$$(x + 3)^2 = -1 + 9$$
$$(x + 3)^2 = 8$$
$$x + 3 = \pm\sqrt{8}$$
$$x = -3 \pm \sqrt{8}$$
$$x = -3 \pm \sqrt{4 \times 2}$$
$$x = -3 \pm 2\sqrt{2}$$
$$x = -3 + 2\sqrt{2} \text{ or } x = -3 - 2\sqrt{2}$$

b
$$x^2 + 12x + 3 = 0$$
$$x^2 + 12x = -3$$
$$(x + 6)^2 - 6^2 = -3$$
$$(x + 6)^2 = -3 + 36$$
$$(x + 6)^2 = 33$$
$$x + 6 = \pm\sqrt{33}$$
$$x = -6 \pm \sqrt{33}$$
$$x = -6 + \sqrt{33} \text{ or } x = -6 - \sqrt{33}$$

c
$$x^2 + 4x - 2 = 0$$
$$x^2 + 4x = 2$$
$$(x + 2)^2 - 2^2 = 2$$
$$(x + 2)^2 = 2 + 4$$
$$(x + 2)^2 = 6$$
$$x + 2 = \pm\sqrt{6}$$
$$x = -2 \pm \sqrt{6}$$
$$x = -2 + \sqrt{6} \text{ or } x = -2 - \sqrt{6}$$

d
$$x^2 - 10x = 5$$
$$(x - 5)^2 - 5^2 = 5$$
$$(x - 5)^2 = 5 + 25$$
$$(x - 5)^2 = 30$$
$$x - 5 = \pm\sqrt{30}$$
$$x = 5 \pm \sqrt{30}$$
$$x = 5 + \sqrt{30} \text{ or } x = 5 - \sqrt{30}$$

2 a
$$2x^2 + 6x - 3 = 0$$
$$x^2 + 3x - \frac{3}{2} = 0$$
$$x^2 + 3x = \frac{3}{2}$$
$$\left(x + \frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 = \frac{3}{2}$$

2 a
$$\left(x + \frac{3}{2}\right)^2 = \frac{3}{2} + \frac{9}{4}$$
$$\left(x + \frac{3}{2}\right)^2 = \frac{15}{4}$$
$$x + \frac{3}{2} = \pm\sqrt{\frac{15}{4}}$$
$$x = -\frac{3}{2} \pm \sqrt{\frac{15}{2}}$$
$$x = -\frac{3}{2} + \frac{\sqrt{15}}{2} \text{ or } x = -\frac{3}{2} - \frac{\sqrt{15}}{2}$$

b
$$5x^2 + 8x - 2 = 0$$
$$x^2 + \frac{8}{5}x - \frac{2}{5} = 0$$
$$x^2 + \frac{8}{5}x = \frac{2}{5}$$
$$\left(x + \frac{4}{5}\right)^2 - \left(\frac{4}{5}\right)^2 = \frac{2}{5}$$
$$\left(x + \frac{4}{5}\right)^2 = \frac{2}{5} + \frac{16}{25}$$
$$\left(x + \frac{4}{5}\right)^2 = \frac{26}{25}$$
$$x + \frac{4}{5} = \pm\sqrt{\frac{26}{25}}$$
$$x = -\frac{4}{5} \pm \frac{\sqrt{26}}{5}$$
$$x = -\frac{4}{5} + \frac{\sqrt{26}}{5} \text{ or } x = -\frac{4}{5} - \frac{\sqrt{26}}{5}$$

c
$$4x^2 - x - 8 = 0$$
$$x^2 - \frac{1}{4}x - 2 = 0$$
$$x^2 - \frac{1}{4}x = 2$$
$$\left(x - \frac{1}{8}\right)^2 - \left(\frac{1}{8}\right)^2 = 2$$
$$\left(x - \frac{1}{8}\right)^2 = 2 + \frac{1}{64}$$
$$\left(x - \frac{1}{8}\right)^2 = \frac{129}{64}$$

2 c $x-\frac{1}{8}=\pm\sqrt{\frac{129}{64}}$

$x=\frac{1}{8}\pm\frac{\sqrt{129}}{8}$

$x=\frac{1}{8}+\frac{\sqrt{129}}{8}$ or $x=\frac{1}{8}-\frac{\sqrt{129}}{8}$

d $15-6x-2x^2=0$

$-2x^2-6x+15=0$

$x^2+3x-\frac{15}{2}=0$

$x^2+3x=\frac{15}{2}$

$\left(x+\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2=\frac{15}{2}$

$\left(x+\frac{3}{2}\right)^2=\frac{15}{2}+\frac{9}{4}$

$\left(x+\frac{3}{2}\right)^2=\frac{39}{4}$

$x+\frac{3}{2}=\pm\sqrt{\frac{39}{4}}$

$x=-\frac{3}{2}\pm\frac{\sqrt{39}}{2}$

$x=-\frac{3}{2}+\frac{\sqrt{39}}{2}$ or $x=-\frac{3}{2}-\frac{\sqrt{39}}{2}$

3 a $x^2-14x+1=(x-7)^2-7^2+1$

$=(x-7)^2-49+1$

$=(x-7)^2-48$

$p=-7$ and $q=-48$

b $x^2-14x+1=0$

$(x-7)^2-48=0$

$(x-7)^2=48$

$x-7=\pm\sqrt{48}$

$x=7\pm\sqrt{16\times 3}$

$x=7\pm 4\sqrt{3}$

$r=7$ and $s=4$

4 $x^2+2bx+c=0$

$(x+b)^2-b^2+c=0$

$(x+b)^2=b^2-c$

$x+b=\pm\sqrt{b^2-c}$

$x=-b\pm\sqrt{b^2-c}$

Challenge

a $ax^2+2bx+c=0$

$x^2+\frac{2b}{a}x+\frac{c}{a}=0$

$\left(x+\frac{b}{a}\right)^2-\left(\frac{b}{a}\right)^2+\frac{c}{a}=0$

$\left(x+\frac{b}{a}\right)^2-\frac{b^2}{a^2}+\frac{c}{a}=0$

$\left(x+\frac{b}{a}\right)^2=\frac{b^2}{a^2}-\frac{c}{a}$

$\left(x+\frac{b}{a}\right)^2=\frac{b^2-ac}{a^2}$

$x+\frac{b}{a}=\pm\sqrt{\frac{b^2-ac}{a^2}}$

$x=-\frac{b}{a}\pm\sqrt{\frac{b^2-ac}{a^2}}$

b $ax^2+bx+c=0$

$x^2+\frac{b}{a}x+\frac{c}{a}=0$

$\left(x+\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2+\frac{c}{a}=0$

$\left(x+\frac{b}{2a}\right)^2-\frac{b^2}{4a^2}+\frac{c}{a}=0$

$\left(x+\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{c}{a}$

$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$

$x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}$

$x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$

$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

Quadratics 2E

1 a $f(1) = 5(1) + 3$
$= 5 + 3$
$= 8$

b $g(3) = 3^2 - 2$
$= 9 - 2$
$= 7$

c $h(8) = \sqrt{8+1}$
$= \sqrt{9}$
$= 3$

d $f(1.5) = 5(1.5) + 3$
$= 7.5 + 3$
$= 10.5$

e $g\left(\sqrt{2}\right) = \left(\sqrt{2}\right)^2 - 2$
$= 2 - 2$
$= 0$

f $h(-1) = \sqrt{-1+1}$
$=$
$= 0$

g $f(4) + g(2) = 5(4) + 3 + 2^2 - 2$
$= 20 + 3 + 4 - 2$
$= 25$

h $f(0) + g(0) + h(0) = 5(0) + 3 + 0^2 - 2 + \sqrt{0+1}$
$= 0 + 3 + 0 - 2 + 1$
$= 2$

i $\dfrac{g(4)}{h(3)} = \dfrac{4^2 - 2}{\sqrt{3+1}}$
$= \dfrac{16-2}{\sqrt{4}}$
$= \dfrac{14}{2}$
$= 7$

2 $f(a) = a^2 - 2a = 8$
$a^2 - 2a - 8 = 0$
$(a\ - 4a + 2) = 0$
$a = 4$ and $a = -2$

3 a $f(x) = 0$
$10 - 15x = 0$
$5(2 - 3x) = 0$
$x = \frac{2}{3}$
The root of $f(x)$ is $\frac{2}{3}$.

b $g(x) = 0$
$(x + 9)(x - 2) = 0$
$(x + 9x - 2) = 0$
$x = -9$ or $x = 2$
The roots of $g(x)$ are -9 and 2.

c $h(x) = 0$
$x^2 + 6x - 40 = 0$
$(x + 10x - 4) = 0$
$x = -10$ or $x = 4$
The roots of $h(x)$ are -10 and 4.

d $j(x) = 0$
$144 - x^2 = 0$
$(12 + 12x - x) = 0$
$x = -12$ or 12
The roots of $j(x)$ are 12 and -12.

e $k(x) = 0$
$x(x + 5)(x + 7) = 0$
$x(x + 5x + 7) = 0$
$x = 0$, $x = -5$ or $x = -7$
The roots of $k(x)$ are 0, -5 and -7.

f $m(x) = 0$
$x^3 + 5x^2 - 24x = 0$
$x(x^2 + 5x - 24) = 0$
$x(x + 8x - 3) = 0$
$x = 0$, $x = -8$ or $x = 3$
The roots of $m(x)$ are 0, -8 and 3.

4 $p(x) = q(x)$
$x^2 - 3x = 2x - 6$
$x^2 - 5x + 6 = 0$
$(x - 3x - 2) = 0$
$x = 3$ and $x = 2$

5 $f(x) = g(x)$
$2x^3 + 30x = 17x^2$
$2x^3 - 17x^2 + 30x = 0$
$x(2x^2 - 17x + 30) = 0$
$x(2x - 5x - 6) = 0$
$x = 0$, $x = \frac{5}{2}$ and $x = 6$

6 a $f(x) = x^2 - 2x + 2$
$= (x - 1)^2 - 1^2 + 2$
$= (x - 1)^2 + 1$
$p = -1$ and $q = 1$

b $(x - 1)^2$ is a squared term so is always ≥ 0.
Therefore, the minimum value of
$f(x) = 0 + 1 = 1$, so $f(x) > 0$.

7 a $f(x) = 0$
$x^6 + 9x^3 + 8 = 0$
$(x^{3)2} + 9(x^{3)} + 8 = 0$
$(x^3 + 1x^3 + 8) = 0$
So $x^3 = -1$ or $x^3 = -8$
$x^3 = -1 \Rightarrow x = -1$
$x^3 = -8 \Rightarrow x = -2$
The roots of $f(x)$ are -1 and -2.

b $g(x) = 0$
$x^4 - 12x^2 + 32 = 0$
$(x^{2)2} - 12(x^{2)} + 32 = 0$
$(x^2 - 4x^2 - 8) = 0$
So $x^2 = 4$ or $x^2 = 8$
$x^2 = 4 \Rightarrow x = \pm 2$
$x^2 = 8 \Rightarrow x = \pm\sqrt{8} = \pm\sqrt{4 \times 2} = \pm 2\sqrt{2}$
The roots of $g(x)$ are -2, 2, $-2\sqrt{2}$ and $2\sqrt{2}$

c $h(x) = 0$
$27x^6 + 26x^3 - 1 = 0$
$27(x^3)^2 + 26(x^{3)} - 1 = 0$
$(27x^3 - 1x^3 + 1) = 0$
$x^3 = \frac{1}{27} \Rightarrow x = \frac{1}{3}$
$x^3 = -1 \Rightarrow x = -1$
The roots of $h(x)$ are -1 and $\frac{1}{3}$.

d $j(x) = 0$
$32x^{10} - 33x^5 + 1 = 0$
$32(x^{5)2} - 33(x^5) + 1 = 0$
$(32x^5 - 1x^5 - 1) = 0$
So $x^5 = \frac{1}{32}$ or $x^5 = 1$
$x^5 = \frac{1}{32} \Rightarrow x = \frac{1}{2}$
$x^5 = 1 \Rightarrow x = 1$
The roots of $j(x)$ are $\frac{1}{2}$ and 1.

e $k(x) = 0$
$x - 7\sqrt{x} + 10 = 0$

$$\left(x^{\frac{1}{2}}\right)^2 - 7\left(x^{\frac{1}{2}}\right) + 10 = 0$$

7 e $$\left(x^{\frac{1}{2}} - 2\right)\left(x^{\frac{1}{2}} - 5\right) = 0$$

So $x^{\frac{1}{2}} = 2$ or $x^{\frac{1}{2}} = 5$
$x^{\frac{1}{2}} = 2 \Rightarrow x = 4$
$x^{\frac{1}{2}} = 5 \Rightarrow x = 25$
The roots of $k(x)$ are 4 and 25.

f $m(x) = 0$
$2x^{\frac{2}{3}} + 2x^{\frac{1}{3}} - 12 = 0$

$$\left(x^{\frac{1}{3}}\right)^2 + \left(x^{\frac{1}{3}}\right) - 6 = 0$$
$$\left(x^{\frac{1}{3}} - 2\right)\left(x^{\frac{1}{3}} + 3\right) = 0$$

So $x^{\frac{1}{3}} = 2$ or $x^{\frac{1}{3}} = -3$
$x^{\frac{1}{3}} = 2 \Rightarrow x = 8$
$x^{\frac{1}{3}} = -3 \Rightarrow x = -27$
The roots of $m(x)$ are 8 and -27.

8 a $3^{2x} - 28(3^x) + 27 = (3^x)^2 - 28(3^x) + 27$
$= (3^x - 27)(3^x - 1)$

b $f(x) = 0$
$(3^x - 27)(3^x - 1) = 0$
$3^x = 27 \Rightarrow x = 3$
$3^x = 1 \Rightarrow x = 0$
The roots of $f(x)$ are 0 and 3.

Quadratics 2F

1 a $y = x^2 - 6x + 8$

As $a = 1$ is positive, the graph has a $\cup$ shape and a minimum point.

When $x = 0$, $y = 8$, so the graph crosses the y-axis at $(0, 8)$.

When $y = 0$,

$x^2 - 6x + 8 = 0$

$(x - 2)(x - 4) = 0$

$x = 2$ or $x = 4$, so the graph crosses the x-axis at $(2, 0)$ and $(4, 0)$.

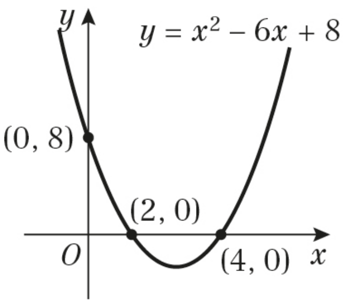
Completing the square:

$x^2 - 6x + 8 = (x - 3)^2 - 9 + 8$

$= (x - 3)^2 - 1$

So the minimum point has coordinate $(3, -1)$.

The sketch of the graph is:



b $y = x^2 + 2x - 15$

As $a = 1$ is positive, the graph has a $\cup$ shape and a minimum point.

When $x = 0$, $y = -15$, so the graph crosses the y-axis at $(0, -15)$.

When $y = 0$,

$x^2 + 2x - 15 = 0$

$(x - 3)(x + 5) = 0$

$x = 3$ or $x = -5$, so the graph crosses the x-axis at $(3, 0)$ and $(-5, 0)$.

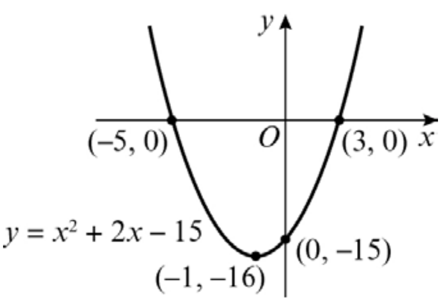
Completing the square:

$x^2 + 2x - 15 = (x + 1)^2 - 1 - 15$

$= (x + 1)^2 - 16$

So the minimum point has coordinate $(-1, -16)$.

The sketch of the graph is:



c $y = 25 - x^2$

As $a = -1$ is negative, the graph has a $\cap$ shape and a maximum point.

When $x = 0$, $y = 25$, so the graph crosses the y-axis at $(0, 25)$.

When $y = 0$,

$25 - x^2 = 0$

$(5 + x)(5 - x) = 0$

$x = -5$ or $x = 5$, so the graph crosses the x-axis at $(-5, 0)$ and $(5, 0)$.

Completing the square:

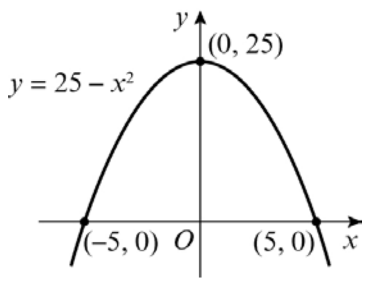
$25 - x^2 = -x^2 + 0x + 25$

$= -(x^2 - 0x - 25)$

$= -(x - 0)^2 + 25$

So the maximum point has coordinate $(0, 25)$.

The sketch of the graph is:



d $y = x^2 + 3x + 2$

As $a = 1$ is positive, the graph has a $\cup$ shape and a minimum point.

When $x = 0$, $y = 2$, so the graph crosses the y-axis at $(0, 2)$.

When $y = 0$,

$x^2 + 3x + 2 = 0$

$(x + 2)(x + 1) = 0$

$x = -2$ or $x = -1$, so the graph crosses the x-axis at $(-2, 0)$ and $(-1, 0)$.

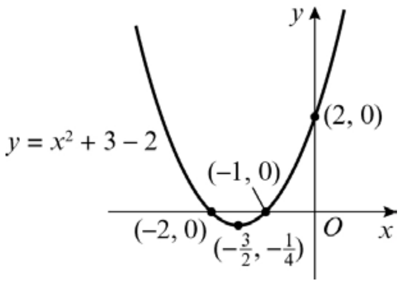
Completing the square:

$$x^2 + 3x + 2 = \left(x + \frac{3}{2}\right)^2 - \frac{9}{4} + 2$$

$$= \left(x + \frac{3}{2}\right)^2 - \frac{1}{4}$$

So the minimum point has coordinate $\left(-\frac{3}{2}, -\frac{1}{4}\right)$.

1 d The sketch of the graph is:



e $y = -x^2 + 6x + 7$

As $a = -1$ is negative, the graph has a $\cap$ shape and a maximum point.

When $x = 0$, $y = 7$, so the graph crosses the y-axis at (0, 7).

When $y = 0$,

$-x^2 + 6x + 7 = 0$

$(-x - 1)(x - 7) = 0$

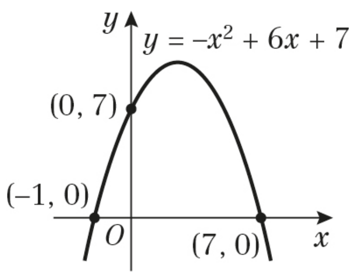
$x = -1$ or $x = 7$, so the graph crosses the x-axis at (−1, 0) and (7, 0).

Completing the square:

$$\begin{aligned}-x^2 + 6x + 7 &= -(x^2 - 6x) + 7\\ &= -((x-3)^2 - 9) + 7\\ &= -(x-3)^2 + 16\end{aligned}$$

So the maximum point has coordinate (3, 16).

The sketch of the graph is:



f $y = 2x^2 + 4x + 10$

As $a = 2$ is positive, the graph has a $\cup$ shape and a minimum point.

When $x = 0$, $y = 10$, so the graph crosses the y-axis at (0, 10).

When $y = 0$,

$2x^2 + 4x + 10 = 0$

Using the quadratic formula,

$$x = \frac{-4 \pm \sqrt{4^2 - 4(2)(10)}}{2 \times 2}$$

$$x = \frac{-4 \pm \sqrt{-64}}{4}$$

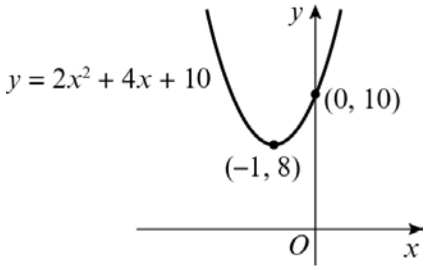
There are no real solutions, so the graph does not cross the x-axis.

f Completing the square:

$$\begin{aligned}2x^2 + 4x + 10 &= 2(x^2 + 2x) + 10\\ &= 2((x + 1)^2 - 1) + 10\\ &= 2(x + 1)^2 + 8\end{aligned}$$

So the minimum point has coordinate (−1, 8).

The sketch of the graph is:



g $y = 2x^2 + 7x - 15$

As $a = 2$ is positive, the graph has a $\cup$ shape and a minimum point.

When $x = 0$, $y = -15$, so the graph crosses the y-axis at (0, −15).

When $y = 0$,

$2x^2 + 7x - 15 = 0$

$(2x - 3)(x + 5) = 0$

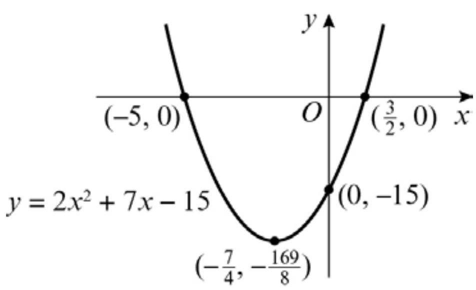
$x = \frac{3}{2}$ or $x = -5$, so the graph crosses the x-axis at $\left(\frac{3}{2}, 0\right)$ and (−5, 0).

Completing the square:

$$\begin{aligned}2x^2 + 7x - 15 &= 2\left(x^2 + \frac{7}{2}x\right) - 15\\ &= 2\left(\left(x + \frac{7}{4}\right)^2 - \frac{49}{16}\right) - 15\\ &= 2\left(x + \frac{7}{4}\right)^2 - \frac{169}{8}\end{aligned}$$

So the minimum point has coordinate $\left(-\frac{7}{4}, -\frac{169}{8}\right)$.

The sketch of the graph is:



1 h $y = 6x^2 - 19x + 10$

As $a = 6$ is positive, the graph has a $\cup$ shape and a minimum point.

When $x = 0$, $y = 10$, so the graph crosses the y-axis at (0, 10).

When $y = 0$,

$6x^2 - 19x + 10 = 0$

$(3x - 2)(2x - 5) = 0$

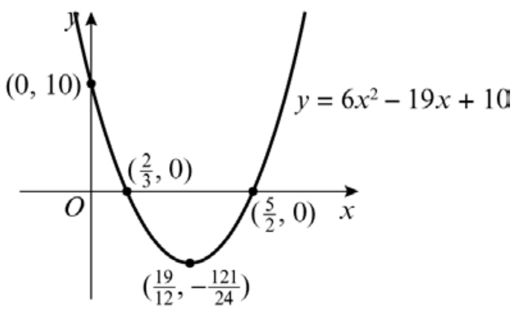
$x = \frac{2}{3}$ or $x = \frac{5}{2}$, so the graph crosses the x-axis at $\left(\frac{2}{3}, 0\right)$ and $\left(\frac{5}{2}, 0\right)$.

Completing the square:

$$6x^2 - 19x + 10 = 6\left(x^2 - \frac{19}{6}x\right) + 10$$
$$= 6\left(\left(x - \frac{19}{12}\right)^2 - \frac{361}{144}\right) + 10$$
$$= 6\left(x - \frac{19}{12}\right)^2 - \frac{121}{24}$$

So the minimum point has coordinate $\left(\frac{19}{12}, -\frac{121}{24}\right)$.

The sketch of the graph is:



i $y = 4 - 7x - 2x^2$

As $a = -2$ is negative, the graph has a $\cap$ shape and a maximum point.

When $x = 0$, $y = 4$, so the graph crosses the y-axis at (0, 4).

When $y = 0$,

$-2x^2 - 7x + 4 = 0$

$(-2x + 1)(x + 4) = 0$

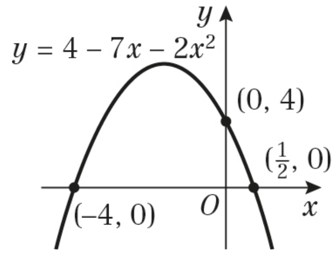
$x = \frac{1}{2}$ or $x = -4$, so the graph crosses the x-axis at $\left(\frac{1}{2}, 0\right)$ and (−4, 0).

Completing the square:

$$-2x^2 - 7x + 4 = -2\left(x^2 + \frac{7}{2}x\right) + 4$$
$$= -2\left(\left(x + \frac{7}{4}x\right)^2 - \frac{49}{16}\right) + 4$$
$$= -2\left(x + \frac{7}{4}x\right)^2 + \frac{81}{8}$$

i So the maximum point has coordinate $\left(-\frac{7}{4}, \frac{81}{8}\right)$.

The sketch of the graph is:



j $y = 0.5x^2 + 0.2x + 0.02$

As a = 0.5 is positive, the graph has a $\cup$ shape and a minimum point.

When $x = 0$, $y = 0.02$, so the graph crosses the y-axis at (0, 0.02).

When $y = 0$,

$0.5x^2 + 0.2x + 0.02 = 0$

Using the quadratic formula,

$$x = \frac{-0.2 \pm \sqrt{0.2^2 - 4(0.5)(0.02)}}{2 \times 0.5}$$
$$x = -0.2 \pm \sqrt{0}$$
$$= -0.2$$

There is only one solution, so the graph touches the x-axis.

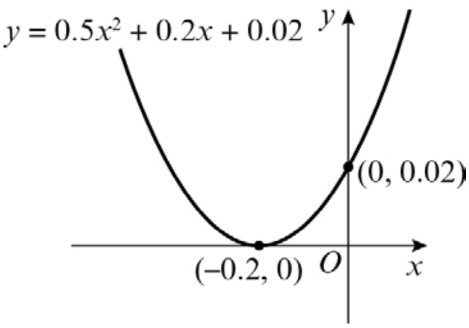
Completing the square:

$0.5x^2 + 0.2x + 0.02$

$$= 0.5(x^2 + 0.4x) + 0.02$$
$$= 0.5((x + 0.2)^2 - 0.04) + 0.02$$
$$= 0.5(x + 0.2)^2$$

So the minimum point has coordinate (−0.2, 0).

The sketch of the graph is:



2 a The graph crosses the y-axis at (0, 15), so $c = 15$.
The graph crosses the x-axis at (3, 0) and (5, 0) and has a minimum value.
$(x - 3)(x - 5) = 0$
$x^2 - 8x + 15 = 0$
$a = 1$, $b = -8$ and $c = 15$

b The graph crosses the y-axis at (0, 10), so $c = 10$.
The graph crosses the x-axis at (−2, 0) and (5, 0) and has a maximum value.
$-(x + 2)(x - 5) = 0$
$-x^2 + 3x + 10 = 0$
$a = -1$, $b = 3$ and $c = 10$

c The graph crosses the y-axis at (0, −18), so $c = -18$.
The graph crosses the x-axis at (−3, 0) and (3, 0) and has a minimum value.
$(x + 3)(x - 3) = 0$
$x^2 + 0x - 9 = 0$
But $c = -18$, not -9, so $2(x^2 + 0x - 9) = 0$
$a = 2$, $b = 0$ and $c = -18$

d The graph crosses the y-axis at (0, −1), so $c = -1$.
The graph crosses the x-axis at (−1, 0) and (4, 0) and has a minimum value.
$(x + 1)(x - 4) = 0$
$x^2 - 3x - 4 = 0$
But $c = -1$, not -4, so $\frac{1}{4}(x^2 - 3x - 4) = 0$
$a = \frac{1}{4}$, $b = -\frac{3}{4}$ and $c = -1$

3 Minimum value = (5, −3), so the line of symmetry is at $x = 5$.
The reflection of (4, 0) in the line $y = 5$ is (6, 0).
$(x - 6)(x - 4) = 0$
$x^2 - 10x + 24 = 0$
Completing the square:
$x^2 - 10x + 24 = (x - 5)^2 - 25 + 24$
$= (x - 5)^2 - 1$
But the minimum value is (5, − 3), therefore:
$y = 3(x - 5)^2 - 3$
$= 3x^2 - 30x + 72$
$a = 3$, $b = -30$ and $c = 72$

Quadratics 2G

1 a i $f(x) = x^2 + 8x + 3$
$b^2 - 4ac$
$= 8^2 - 4(1)(3)$
$= 64 - 12$
$= 52$

ii $g(x) = 2x^2 - 3x + 4$
$b^2 - 4ac$
$= (-3)^2 - 4(2)(4)$
$= 9 - 32$
$= -23$

iii $h(x) = -x^2 + 7x - 3$
$b^2 - 4ac$
$= 7^2 - 4(-1)(-3)$
$= 49 - 12$
$= 37$

iv $j(x) = x^2 - 8x + 16$
$b^2 - 4ac$
$= (-8)^2 - 4(1)(16)$
$= 64 - 64$
$= 0$

v $k(x) = 2x - 3x^2 - 4$
$= - 3x^2 + 2x - 4$
$b^2 - 4ac$
$= (2)^2 - 4(-3)(-4)$
$= 4 - 48$
$= -44$

b i This graph has two distinct real roots and has a maximum, so $a < 0$: $h(x)$.

ii This graph has two distinct real roots and has a minimum, so $a > 0$: $f(x)$.

iii This graph has no real roots and has a maximum, so $a < 0$: $k(x)$.

iv This graph has one repeated root and has a minimum, so $a > 0$: $j(x)$.

v This graph has no real roots and has a minimum, so $a > 0$: $g(x)$.

2 $x^2 + 6x + k = 0$
$a = 1$, $b = 6$ and $c = k$
For two real solutions, $b^2 - 4ac > 0$
$6^2 - 4 \times 1 \times k > 0$
$36 - 4k > 0$
$36 > 4k$
$9 > k$
So $k < 9$

3 $2x^2 - 3x + t = 0$
$a = 2$, $b = -3$ and $c = t$
For exactly one solution, $b^2 - 4ac = 0$
$(-3)^2 - 4 \times 2 \times t = 0$
$9 - 8t = 0$
So $t = \frac{9}{8}$

4 $f(x) = sx^2 + 8x + s$
$a = s$, $b = 8$ and $c = s$
For equal solutions, $b^2 - 4ac = 0$
$8^2 - 4 \times s \times s = 0$
$64 - 4s^2 = 0$
$64 = 4s^2$
$16 = s^2$
So $s = \pm 4$
The positive solution is $s = 4$.

5 $3x^2 - 4x + k = 0$
$a = 3$, $b = -4$ and $c = k$
For no real solutions, $b^2 - 4ac < 0$
$(-4)^2 - 4 \times 3 \times k < 0$
$16 - 12k < 0$
$16 < 12k$
$4 < 3k$
So $k > \frac{4}{3}$

6 a $g(x) = x^2 + 3px + (14p - 3) = 0$
$a = 1$, $b = 3p$ and $c = 14p - 3$
For two equal roots, $b^2 - 4ac = 0$
$(3p)^2 - 4 \times 1 \times (14p - 3) = 0$
$9p^2 - 56p + 12 = 0$
$(p - 6)(9p - 2) = 0$
$p = 6$ or $p = \frac{2}{9}$
p is an integer, so $p = 6$

6 b When $p = 6$,
$x^2 + 3px + (14p - 3)$
$= x^2 + 3(6)x + (14(6) - 3)$
$= x^2 + 18x + 81$
$x^2 + 18x + 81 = 0$
$(x + 9)(x + 9) = 0$
So $x = -9$

7 a $\mathrm{h}(x) = 2x^2 + (k + 4)x + k$
$a = 2$, $b = k + 4$ and $c = k$
$b^2 - 4ac = (k + 4)^2 - 4 \times 2 \times k$
$= k^2 + 8k + 16 - 8k = k^2 + 16$

b $k^2 \geqslant 0$, therefore $k^2 + 16$ is also > 0. If $b^2 - 4ac > 0$, then $\mathrm{h}(x)$ has two distinct real roots.

Challenge

a For distinct real roots, $b^2 - 4ac > 0$
$b^2 > 4ac$
If $a > 0$ and $c > 0$, or $a < 0$ and $c < 0$, choose b such that $b > \sqrt{4ac}$
If $a > 0$ and $c < 0$, or $a < 0$ and $c > 0$, $4ac < 0$, therefore $4ac < b^2$ for all b

b For equal roots, $b^2 - 4ac = 0$
$b^2 = 4ac$
If $4ac < 0$, then there is no value for b to satisfy $b^2 = 4ac$ as b^2 is always positive.

Quadratics 2H

1 a The bridge is 200 m above ground level, since this is the height at the centre of the bridge.

b $0.000\,12x^2 + 200 = 346$

$0.000\,12x^2 = 146$

$x^2 = \dfrac{146}{0.000\,12}$

$x = \pm\sqrt{\dfrac{146}{0.000\,12}}$

So $x = 1103$ and $x = -1103$

c length $= 1103 \times 2 = 2206$ m

2 a $-0.01x^2 + 0.975x + 16 = 32.5$

$-0.01x^2 + 0.975x - 16.5 = 0$

Using the formula, where $a = -0.01$, $b = 0.975$ and $c = -16.5$,

$$x = \frac{-0.975 \pm \sqrt{0.975^2 - 4(-0.01)(-16.5)}}{2(-0.01)}$$

$$x = \frac{0.975 \pm \sqrt{0.290\,625}}{0.02}$$

$x = 75.7$ and $x = 21.8$ (to 3 s.f.)

21.8 mph and 75.7 mph

b $y = -0.01x^2 + 0.975x + 16$

$= -0.01(x^2 - 97.5x) + 16$

$= -0.01((x - 48.75)^2 - 2376.5625) + 16$

$= -0.01(x - 48.75)^2 + 39.765\,625$

$A = 39.77$ (to 4 s.f.), $B = 0.01$ and $C = 48.75$

c The greatest fuel efficiency is the maximum, when $x = 48.75$

48.75 mph

d When $x = 120$,

$y = -0.01(120)^2 + 0.975(120) + 16$

$= -11$

A negative fuel consumption is impossible, so this model is not valid for very high speeds.

3 a Without any fertliser, $f = 0$, so each hectare would yield 6 tonnes of grain.

b When $f = 20$,

$g = 6 + 0.03(20) - 0.00006(20)^2$

$= 6.576$

For an extra tonne yield, $g = 6.576 + 1$

$= 7.576$

$6 + 0.03f - 0.000\,06f^2 = 7.576$

$1.576 - 0.03f + 0.000\,06f^2 = 0$

Using the formula, where $a = 0.000\,06$, $b = -0.03$ and $c = 1.576$,

$$x = \frac{-(-0.03) \pm \sqrt{(-0.03)^2 - 4(0.000\,06)(1.576)}}{2(0.000\,06)}$$

$$x = \frac{0.03 \pm \sqrt{0.000\,521\,76}}{0.000\,12}$$

$x = 440.4$ and $x = 59.6$ (to 1 d.p.)

$59.6 - 20 = 39.6$

39.6 kilograms per hectare

4 a $t = M - 1000p$, $t = 10\,000$ when $p = £30$

$10\,000 = M - 1000 \times 30$

$M = 40\,000$

b $r = p(40\,000 - 1000p)$

$= -1000p^2 + 40\,000p$

$= -1000(p^2 - 40p)$

$= -1000((p - 20)^2 - 400)$

$= -1000(p - 20)^2 + 400\,000$

$A = 400\,000$, $B = 1000$ and $C = 20$

c $r = = -1000(p - 20)^2 + 400\,000$

maximum = £400 000 when $p = 20$

They should charge £20 per ticket.

Challenge

a $d(s) = as^2 + bs + c$
When $s = 20$, $d = 6$:
$6 = a(20)^2 + b(20) + c$
$6 = 400a + 20b + c$ (**1**)

When $s = 30$, d = 14:
$14 = a(30)^2 + b(30) + c$
$14 = 900a + 30b + c$ (**2**)

When $s = 20$, $d = 24$:
$24 = a(40)^2 + b(40) + c$
$24 = 1600a + 40b + c$ (**3**)

(**2**) − (**1**):
$(14 = 900a + 30b + c) - (6 = 400a + 20b + c)$
$\Rightarrow 8 = 500a + 10b$ (**4**)

(**3**) − (**1**):
$(24 = 1600a + 40b + c) - (6 = 400a + 20b + c)$
$\Rightarrow 18 = 1200a + 20b$ (**5**)

(**5**) − 2 × (**4**):
$(18 = 1200a + 20b) - 2(8 = 500a + 10b)$
$\Rightarrow 2 = 200a$, so $a = 0.01$

$8 = 500(0.01) + 10b$
$8 = 5 + 10b \Rightarrow b = 0.3$
$6 = 400(0.01) + 20(0.3) + c$
$6 = 4 + 6 + c \Rightarrow c = -4$
$a = 0.01$, $b = 0.3$ and $c = -4$

b $0.01s^2 + 0.3s - 4 = 20$
$0.01s^2 + 0.3s - 24 = 0$
Using the formula, where $a = 0.01$, $b = 0.3$ and $c = -24$,

$$x = \frac{-0.3 \pm \sqrt{0.3^2 - 4(0.01)(-24)}}{2(0.01)}$$

$$x = \frac{-0.3 \pm \sqrt{1.05}}{0.02}$$

$x = 36.2$ or -66.2 (to 3 s.f.)
The speed of the car must be positive, so is 36.2 mph.

Quadratics 2 Mixed Exercise

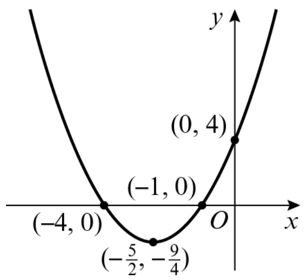
1 a $y^2 + 3y + 2 = 0$
$(y + 1)(y + 2) = 0$
$y = -1$ or $y = -2$

b $3x^2 + 13x - 10 = 0$
$(3x - 2)(x + 5) = 0$
$x = \frac{2}{3}$ or $x = -5$

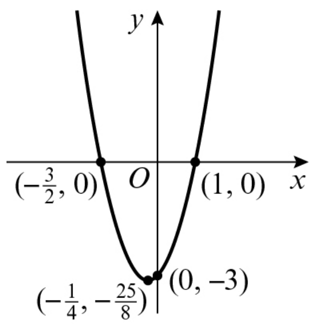
c $5x^2 - 10x = 4x + 3\left(-\frac{1}{4}, -\frac{25}{8}\right)$
$5x^2 - 14x - 3 = 0$
$(5x + 1)(x - 3) = 0$
$x = -\frac{1}{5}$ or $x = 3$

d $(2x - 5)^2 = 7$
$2x - 5 = \pm\sqrt{7}$
$2x = \pm\sqrt{7} + 5$
$x = \dfrac{5 \pm \sqrt{7}}{2}$

2 a $y = x^2 + 5x + 4$
As $a = 1$ is positive, the graph has a $\cup$ shape and a minimum point.
When $x = 0$, $y = 4$, so the graph crosses the y-axis at (0, 4).
When $y = 0$,
$x^2 + 5x + 4 = 0$
$(x + 1)(x + 4) = 0$
$x = -1$ or $x = -4$, so the graph crosses the x-axis at $(-1, 0)$ and $(-4, 0)$.
Completing the square:
$x^2 + 5x + 4 = \left(x + \frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^2 + 4$
$= \left(x + \frac{5}{2}\right)^2 - \frac{9}{4}$
So the minimum point has coordinates $\left(-\frac{5}{2}, -\frac{9}{4}\right)$.
The sketch of the graph is:



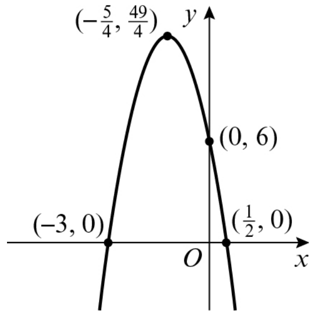
2 b $y = 2x^2 + x - 3$
As $a = 2$ is positive, the graph has a $\cup$ shape and a minimum point.
When $x = 0$, $y = -3$, so the graph crosses the y-axis at $(0, -3)$.
When $y = 0$,
$2x^2 + x - 3 = 0$
$(2x + 3)(x - 1) = 0$
$x = -\frac{3}{2}$ $2x = \pm\sqrt{7} + 5$ or $x = 1$, so the graph crosses the
x-axis at $\left(-\frac{3}{2}, 0\right)$ and (1, 0).
Completing the square:
$2x^2 + x - 3 = 2\left(x^2 + \frac{1}{2}x\right) - 3$
$= 2\left(\left(x + \frac{1}{4}\right)^2 - \left(\frac{1}{4}\right)^2\right) - 3$
$= 2\left(x + \frac{1}{4}\right)^2 - \frac{25}{8}$
So the minimum point has coordinates $\left(-\frac{1}{4}, \frac{25}{8}\right)$.
The sketch of the graph is:



c $y = 6 - 10x - 4x^2$
As $a = -4$ is negative, the graph has a $\cap$ shape and a maximum point.
When $x = 0$, $y = 6$, so the graph crosses the y-axis at (0, 6).
When $y = 0$,
$6 - 10x - 4x^2 = 0$
$(1 - 2x)(6 + 2x) = 0$
$x = \frac{1}{2}$ or $x = -3$, so the graph crosses the x-axis at $\left(\frac{1}{2}, 0\right)$ and $(-3, 0)$.
Completing the square:
$6 - 10x - 4x^2 = -4x^2 - 10x + 6$
$= -4\left(x^2 + \frac{5}{2}x\right) + 6$
$= -4\left(\left(x + \frac{5}{4}\right)^2 - \left(\frac{5}{4}\right)^2\right) + 6$
$= -4\left(x + \frac{5}{4}\right)^2 + \frac{49}{4}$

2 c So the minimum point has coordinates $\left(-\frac{5}{4}, \frac{49}{4}\right)$.

The sketch of the graph is:



d $y = 15x - 2x^2$

As $a = -2$ is negative, the graph has a $\cap$ shape and a maximum point.

When $x = 0$, $y = 0$, so the graph crosses the y-axis at (0, 0).

When $y = 0$,

$15x - 2x^2 = 0$

$x(15 - 2x) = 0$

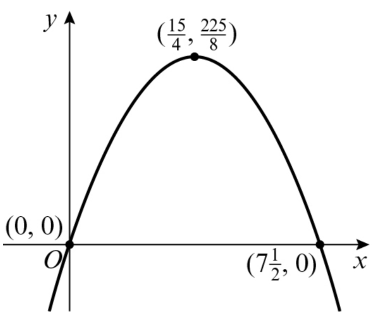
$x = 0$ or $x = 7\frac{1}{2}$, so the graph crosses the x-axis at (0, 0) and .$\left(7\frac{1}{2}, 0\right)$..

Completing the square:

$$\begin{aligned}15x - 2x^2 &= -2x^2 + 15x \\ &= -2\left(x^2 - \tfrac{15}{2}x\right) \\ &= -2\left(\left(x - \tfrac{15}{4}\right)^2 - \left(\tfrac{15}{4}\right)^2\right) \\ &= -2\left(x - \tfrac{15}{4}\right)^2 + \tfrac{225}{8}\end{aligned}$$

So the minimum point has coordinates $\left(\frac{15}{4}, \frac{225}{8}\right)$.

The sketch of the graph is:



3 a $f(3) = 3^2 + 3(3) - 5 = 13$

$g(3) = 4(3) + k = 12 + k$

$f(3) = g(3)$

$13 = 12 + k$

$k = 1$

3 b $x^2 + 3x - 5 = 4x + 1$

$x^2 - x - 6 = 0$

$(x - 3)(x + 2) = 0$

$x = 3$ and $x = -2$

4 a $k^2 + 11k - 1 = 0$

$a = 1$, $b = 11$ and $c = -1$

Using the quadratic formula:

$$k = \frac{-11 \pm \sqrt{11^2 - 4(1)(-1)}}{2(1)}$$

$$k = \frac{-11 \pm \sqrt{125}}{2}$$

$k = 0.0902$ or $k = -11.1$

b $2t^2 - 5t + 1 = 0$

$a = 2$, $b = -5$ and $c = 1$

Using the quadratic formula:

$$t = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(2)(1)}}{2(2)}$$

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(-1)(3)}}{2(-1)}$$

$$t = \frac{5 \pm \sqrt{17}}{4}$$

$t = 2.28$ or $t = 0.219$

c $10 - x - x^2 = 7$

$3 - x - x^2 = 0$

$a = -1$, $b = -1$ and $c = 3$

Using the quadratic formula:

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(-1)(3)}}{2(-1)}$$

$$x = \frac{1 \pm \sqrt{13}}{-2}$$

$x = -2.30$ or $x = 1.30$

d $(3x - 1)^2 = 3 - x^2$

$9x^2 - 3x - 3x + 1 = 3 - x^2$

$10x^2 - 6x - 2 = 0$

$a = 10$, $b = -6$ and $c = -2$

Using the quadratic formula:

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(10)(-2)}}{2(10)}$$

$$x = \frac{6 \pm \sqrt{116}}{20}$$

$x = 0.839$ or $x = -0.239$

5 a $x^2 + 12x - 9 = (x + 6)^2 - 36 - 9$
$= (x + 6)^2 - 45$
$p = 1$, $q = 6$ and $r = -45$

b $5x^2 - 40x + 13 = 5(x^2 - 8x) + 13$
$= 5((x - 4)^2 - 16) + 13$
$= 5(x - 4)^2 - 67$
$p = 5$, $q = -4$ and $r = -67$

c $8x - 2x^2 = -2x^2 + 8x$
$= -2(x^2 - 4x)$
$= -2((x - 2)^2 - 4)$
$= -2(x - 2)^2 + 8$
$p = -2$, $q = -2$ and $r = 8$

d $3x^2 - (x + 1)^2 = 3x^2 - (x^2 + x + x + 1)$
$= 2x^2 - 2x - 1$
$= 2(x^2 - x) - 1$
$= 2\left(\left(x - \frac{1}{2}\right)^2 - \frac{1}{4}\right) - 1$
$= 2\left(x - \frac{1}{2}\right) - \frac{3}{2}$
$p = 2$, $q = -\frac{1}{2}$ and $r = -\frac{3}{2}$

6 $5x^2 - 2x + k = 0$
$a = 5$, $b = -2$ and $c = k$
For exactly one solution, $b^2 - 4ac = 0$
$(-2)^2 - 4 \times 5 \times k = 0$
$4 - 20k = 0$
$4 = 20k$
$k = \frac{1}{5}$

7 a $3x^2 + 12x + 5 = p(x + q)^2 + r$
$3x^2 + 12x + 5 = p(x^2 + 2qx + q^2) + r$
$3x^2 + 12x + 5 = px^2 + 2pqx + pq^2 + r$
Comparing x^2: $p = 3$ **(1)**
Comparing x: $2pq = 12$ **(2)**
Comparing constants: $pq^2 + r = 5$ **(3)**
Substitute **(1)** into **(2)**:
$2 \times 3 \times q = 12$
$q = 2$
Substitute $p = 3$ and $q = 2$ into **(3)**
$3 \times 2^2 + r = 5$
$12 + r = 5$
$r = -7$
So $p = 3$, $q = 2$ and $r = -7$

b $3x^2 + 12x + 5 = 0$
$3(x+2)^2 - 7 = 0$
$3(x+2)^2 = 7$
$(x+2)^2 = \frac{7}{3}$

7 b $x + 2 = \pm\sqrt{\frac{7}{3}}$

So $x = -2 \pm \sqrt{\frac{7}{3}}$

8 a $2^{2x} - 20(2^x) + 64 = (2^x)^2 - 20(2^x) + 64$
$= (2^x - 16)(2^x - 4)$

b $f(x) = (2^x - 16)(2^x - 4)$
Then either $2^x = 16 \Rightarrow x = 4$
or $2^x = 4 \Rightarrow x = 2$
$x = 2$ or $x = 4$

9 $2(x + 1)(x - 4) - (x - 2)^2 = 0$
$2(x^2 - 3x - 4) - (x^2 - 4x + 4) = 0$
$2x^2 - 6x - 8 - x^2 + 4x - 4 = 0$
$x^2 - 2x - 12 = 0$
$a = 1$, $b = -2$, $c = -12$
Using the quadratic formula:
$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-12)}}{2}$$
$$x = \frac{2 \pm \sqrt{52}}{2}$$
$$x = \frac{2 \pm \sqrt{4 \times 13}}{2}$$
$$x = \frac{2 \pm 2\sqrt{13}}{2}$$
$$x = 1 \pm \sqrt{13}$$

10 $(x - 1)(x + 2) = 18$
$x^2 + x - 2 = 18$
$x^2 + x - 20 = 0$
$(x + 5)(x - 4) = 0$
$x = -5$ or $x = 4$

11 a The springboard is 10 m above the water, since this is the height at time 0.

b When $h = 0$, $5t - 10t^2 + 10 = 0$
$-10t^2 + 5t + 10 = 0$
$a = -10$, $b = 5$ and $c = 10$
Using the quadratic formula:
$$t = \frac{-5 \pm \sqrt{5^2 - 4(-10)(10)}}{2(-10)}$$
$$= \frac{-5 \pm \sqrt{425}}{-20}$$

11 b $t = -0.78$ or $t = 1.28$ (to 3 s.f.)
t cannot be negative, so the time is 1.28 seconds.

c $-10t^2 + 5t + 10$
$= -10(t^2 - 0.5t) + 10$
$= -10((t - 0.25)^2 - 0.0625) + 10$
$= -10(t - 0.25)^2 + 10.625$
$A = 10.625$, $B = 10$ and $C = 0.25$

d The maximum height is when $t - 0.25 = 0$, therefore when $t = 0.25$ s, $h = 10.625$ m.

12 a $f(x) = 4kx^2 + (4k + 2)x + 1$
$a = 4k$, $b = 4k + 2$ and $c = 1$
$b^2 - 4ac = (4k + 2)^2 - 4 \times 4k \times 1$
$= 16k^2 + 8k + 8k + 4 - 16k$
$= 16k^2 + 4$

b $16k^2 + 4$
$k^2 \geqslant 0$ for all values of k, therefore
$16k^2 + 4 > 0$
As $b^2 - 4ac = 16k^2 + 4 > 0$, $f(x)$ has two distinct real roots.

c When $k = 0$,
$f(x) = 4(0)x^2 + (4(0) + 2)x + 1 = 2x + 1$
$2x + 1$ is a linear function with only one root, so $f(x)$ cannot have two distinct real roots when $k = 0$.

13 i Let $\sqrt{x} = u$
$2u^2 + u - 6 = 0$
$(2u - 3)(u + 2) = 0$
u is positive so $u = \frac{3}{2}$
$x = u^2 = \frac{9}{4}$

ii $x^8 - 17x^4 + 16 = 0$
$(x^4)^2 - 17(x^4) + 16 = 0$
$(x^4 - 1)(x^4 - 16) = 0$
Then either $x^4 = 1 \Rightarrow x = \pm 1$
or $x^4 = 16 \Rightarrow x = \pm 2$
$x = -2$, $x = -1$, $x = 1$ and $x = 2$

14 a $c = 230 - Hp$, $c = 80$ and $p = 15$
$80 = 230 - 15H$
$H = 10$

14 b $r = p(230 - 10p)$
$= -10p^2 + 230p$
$= -10(p^2 - 23p)$
$= -10((p - 11.5)^2 - 132.25)$
$= -10(p - 11.5)^2 + 1322.5$
$A = 1322.5$, $B = 10$ and $C = 11.5$

c The maximum value is when $(p - 11.5) = 0$.
This is a maximum revenue of £1322.50 at £11.50 per cushion.
Original revenue $= 80 \times 15 =$ £1200
Increase in revenue = £1322.50 − £1200
= £122.50

Challenge

a $\frac{a}{b} = \frac{b}{c}$
$\frac{b + c}{b} = \frac{b}{c}$
$b^2 - bc - c^2 = 0$
Using the quadratic formula:
$$b = \frac{c \pm \sqrt{c^2 - 4(1)(-c^2)}}{2(1)}$$
$$= \frac{c \pm \sqrt{5c^2}}{2}$$
$$= \frac{c \pm c\sqrt{5}}{2}$$
So $b : c = \frac{c \pm c\sqrt{5}}{2} : c$
Dividing by c:
$\frac{1 \pm \sqrt{5}}{2} : 1$
The length cannot be negative so
$b : c = \frac{1 + \sqrt{5}}{2} : 1$

Challenge

b Let $x = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\ldots}}}}$

So $x = \sqrt{1+x}$

Squaring both sides:

$x^2 = 1 + x$

$x^2 - x - 1 = 0$

Using the quadratic formula:

$$x = \frac{1 \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)}$$

$$= \frac{1 \pm \sqrt{5}}{2}$$

The square root cannot be negative so

$$\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\ldots}}}} = \frac{1 \pm \sqrt{5}}{2}$$

Equations and inequalities 3A

1 a Multiply $2x - y = 6$ by 3:
$6x - 3y = 18$
$4x + 3y = 22$
Add:
$10x = 40$
$x = 4$
Substitute into $2x - y = 6$:
$8 - y = 6$
$y = 2$
So solution is $x = 4,\ y = 2$.

b Multiply $7x + 3y = 16$ by 3:
$21x + 9y = 48$
$2x + 9y = 29$
Subtract:
$19x = 19$
$x = 1$
Substitute into $7x + 3y = 16$:
$7 + 3y = 16$
$3y = 9$
$y = 3$
So solution is $x = 1,\ y = 3$.

c Multiply $5x + 2y = 6$ by 5:
$25x + 10y = 30$
$3x - 10y = 26$
Add:
$28x = 56$
$x = 2$
Substitute into $5x + 2y = 6$:
$10 + 2y = 6$
$2y = -4$
$y = -2$
So solution is $x = 2,\ y = -2$.

d Multiply $2x - y = 12$ by 2:
$4x - 2y = 24$
$6x + 2y = 21$
Add:
$10x = 45$
$x = 4\tfrac{1}{2}$
Substitute into $2x - y = 12$:
$9 - y = 12$
$y = -3$
So solution is $x = 4\tfrac{1}{2},\ y = -3$.

e Multiply $3x - 2y = -6$ by 2:
$6x - 4y = -12$
$6x + 3y = 2$
Subtract:
$-7y = -14$
$y = 2$
Substitute into $3x - 2y = -6$:
$3x - 4 = -6$
$3x = -2$
$x = -\tfrac{2}{3}$
So solution is $x = -\tfrac{2}{3},\ y = 2$.

f Multiply $3x + 8y = 33$ by 2:
$6x + 16y = 66$
$6x = 3 + 5y$
$6x + 16y = 66$
$6x - 5y = 3$
Subtract:
$21y = 63$
$y = 3$
Substitute into $3x + 8y = 33$:
$3x + 24 = 33$
$3x = 9$
$x = 3$
So solution is $x = 3,\ y = 3$.

2 a Rearrange $x + 3y = 11$ to give:
$x = 11 - 3y$
Substitute into $4x - 7y = 6$:
$4(11 - 3y) - 7y = 6$
$44 - 12y - 7y = 6$
$-19y = -38$
$y = 2$
Substitute into $x = 11 - 3y$:
$x = 11 - 6$
$x = 5$
So solution is $x = 5,\ y = 2$.

b Rearrange $2x + y = 5$ to give:
$y = 5 - 2x$

2 b Substitute into $4x-3y=40$:

$4x-3(5-2x)=40$

$4x-15+6x=40$

$10x=55$

$x=5\frac{1}{2}$

Substitute into $y=5-2x$:

$y=5-11$

$y=-6$

So solution is $x=5\frac{1}{2},\ y=-6$.

c Rearrange $3x-y=7$ to give:

$-y=7-3x$

$y=3x-7$

Substitute into $10x+3y=-2$:

$10x+3(3x-7)=-2$

$10x+9x-21=-2$

$19x=19$

$y=1$

Substitute into $y=3x-7$:

$y=3-7$

$y=-4$

So solution is $x=1,\ y=-4$.

d Rearrange $3y=x-1$ to give:

$x=3y+1$

Substitute into $2y=2x-3$:

$2y=2(3y+1)-3$

$2y=6y+2-3$

$-4y=-1$

$y=\frac{1}{4}$

Substitute into $x=3y+1$:

$x=\frac{3}{4}+1$

$x=1\frac{3}{4}$

So solution is $x=1\frac{3}{4},\ y=\frac{1}{4}$.

3 a Rearrange $3x-2y+5=0$ to give:

$3x-2y=-5$ **(1)**

Expand and rearrange $5(x+y)=6(x+1)$ to give

$5x+5y=6x+6$

$x-5y=-6$ **(2)**

Multiply (**2**) by 3 to give:

$3x-15y=-18$ **(3)**

Subtract (**3**) from (**1**) to give:

$13y=13$

$y=1,\ x=5(1)-6=-1$

$x=-1$ and $y=1$

b Rearrange $\dfrac{x-2y}{3}=4$ to give:

$x-2y=12$ **(1)**

Rearrange $2x+3y+4=0$ to give:

$2x+3y=-4$ **(2)**

Multiply **(1)** by 2 to give:

$2x-4y=24$ **(3)**

Subtract **(2)** from **(3)** to give:

$-7y=28$

$y=-4,\ x=2(-4)+12=4$

So solution is $x=4$ and $y=-4$

c Expand and rearrange $3y=5(x-2)$ to give:

$5x-3y=10$ **(1)**

Expand and rearrange $3(x-1)+y+4=0$ to give:

$3x+y=-1$ **(2)**

Multiply (**2**) by 3 to give:

$9x+3y=-3$ **(3)**

Add (**1**) and (**3**) to give:

$14x=7$

$x=\frac{1}{2},\ y=-3(\frac{1}{2})-1=-\frac{5}{2}$

So solution is $x=\frac{1}{2}$ and $y=-2\frac{1}{2}$

4 a $3x+ky=8$ **(1)**

$x-2ky=5$ **(2)**

Multiply **(1)** by 2 to give:

$6x+2ky=16$ **(3)**

Add (**2**) and (**3**) to give:

$7x=21$

$x=3$

b Using (**1**), $3(3)+k(\frac{1}{2})=8$

$\frac{1}{2}k=-1$

$k=-2$

5 Substitute $x=q$ and $y=-1$ into both equations to give:

$2q+p=5$ **(1)**

$4q-5+q=0$ **(2)**

From (**2**), $5q=5,\ q=1$

Substituting $q=1$ into (**1**) gives:

$2(1)+p=5$

$p=3$

So $p=3$ and $q=1$

Equations and inequalities 3B

1 a Rearrange $x + y = 11$ to give:
$y = 11 - x$
Substitute into $xy = 30$:
$$x(11 - x) = 30$$
$$11x - x^2 = 30$$
$$0 = x^2 - 11x + 30$$
$$0 = (x-5)(x-6)$$
$x = 5$ or $x = 6$
Substitute into $y = 11 - x$:
When $x = 5$, $y = 11 - 5 = 6$
When $x = 6$, $y = 11 - 6 = 5$
Solutions are $x = 5$, $y = 6$ or $x = 6$, $y = 5$

b Rearrange $2x + y = 1$ to give:
$y = 1 - 2x$
Substitute into $x^2 + y^2 = 1$:
$$x^2 + (1 - 2x)^2 = 1$$
$$x^2 + 1 - 4x + 4x^2 = 1$$
$$5x^2 - 4x = 0$$
$$x(5x - 4) = 0$$
$x = 0$ or $x = \frac{4}{5}$
Substitute into $y = 1 - 2x$:
When $x = 0$, $y = 1$
When $x = \frac{4}{5}$, $y = 1 - \frac{8}{5} = -\frac{3}{5}$
Solutions are
$x = 0$, $y = 1$ or $x = \frac{4}{5}$, $y = -\frac{3}{5}$

c $y = 3x$
Substitute into $2y^2 - xy = 15$:
$$2(3x)^2 - x(3x) = 15$$
$$18x^2 - 3x^2 = 15$$
$$15x^2 = 15$$
$$x^2 = 1$$
$x = -1$ or $x = 1$
Substitute into $y = 3x$:
When $x = -1$, $y = -3$
When $x = 1$, $y = 3$
Solutions are
$x = -1$, $y = -3$ or $x = 1$, $y = 3$

d Rearrange $3a + b = 8$ to give:
$b = 8 - 3a$
Substitute into $3a^2 + b^2 = 28$:
$$3a^2 + (8 - 3a)^2 = 28$$
$$3a^2 + 64 - 48a + 9a^2 = 28$$
$$12a^2 - 48a + 36 = 0$$
Divide by 12:
$$a^2 - 4a + 3 = 0$$
$$(a-1)(a-3) = 0$$
$a = 1$ or $a = 3$
Substitute into $b = 8 - 3a$:
When $a = 1$, $b = 8 - 3 = 5$
When $a = 3$, $b = 8 - 9 = -1$
Solutions are
$a = 1$, $b = 5$ or $a = 3$, $b = -1$.

e Rearrange $2u + v = 7$ to give:
$v = 7 - 2u$
Substitute into $uv = 6$:
$$u(7 - 2u) = 6$$
$$7u - 2u^2 = 6$$
$$0 = 2u^2 - 7u + 6$$
$$0 = (2u - 3)(u - 2)$$
$u = \frac{3}{2}$ or $u = 2$
Substitute into $v = 7 - 2u$:
When $u = \frac{3}{2}$, $v = 7 - 3 = 4$
When $u = 2$, $v = 7 - 4 = 3$
Solutions are
$u = \frac{3}{2}$, $v = 4$ or $u = 2$, $v = 3$

f Rearrange $3x + 2y = 7$ to give:
$$2y = 7 - 3x$$
$$y = \tfrac{1}{2}(7 - 3x)$$
Substitute into $x^2 + y = 8$:
$$x^2 + \tfrac{1}{2}(7 - 3x) = 8$$
Multiply by 2:
$$2x^2 + (7 - 3x) = 16$$
$$2x^2 - 3x - 9 = 0$$
$$(2x + 3)(x - 3) = 0$$
$x = -\frac{3}{2}$ or $x = 3$

1 f Substitute into $y = \frac{1}{2}(7 - 3x)$:

When $x = -\frac{3}{2}$, $y = \frac{1}{2}\left(7 + \frac{9}{2}\right) = \frac{23}{4}$

When $x = 3$, $y = \frac{1}{2}(7 - 9) = -1$

Solutions are

$x = -1\frac{1}{2}$, $y = 5\frac{3}{4}$ or $x = 3$, $y = -1$

2 a Rearrange $2x + 2y = 7$ to give:

$2x = 7 - 2y$

$x = \frac{1}{2}(7 - 2y)$

Substitute into $x^2 - 4y^2 = 8$:

$\left(\frac{1}{2}(7 - 2y)\right)^2 - 4y^2 = 8$

$\frac{1}{4}(7 - 2y)^2 - 4y^2 = 8$

Multiply by 4:

$$(7 - 2y)^2 - 16y^2 = 32$$

$$49 - 28y + 4y^2 - 16y^2 = 32$$

$$0 = 12y^2 + 28y - 17$$

$$0 = (6y + 17)(2y - 1)$$

$y = -\frac{17}{6}$ or $y = \frac{1}{2}$

Substitute into $x = \frac{1}{2}(7 - 2y)$:

When $y = -\frac{17}{6}$, $x = \frac{1}{2}\left(7 + \frac{17}{3}\right) = \frac{19}{3}$

When $y = \frac{1}{2}$, $x = \frac{1}{2}(7 - 1) = 3$

Solutions are

$x = 6\frac{1}{3}$, $y = -2\frac{5}{6}$ or $x = 3$, $y = \frac{1}{2}$

b Rearrange $x + y = 9$ to give:

$x = 9 - y$

Substitute into $x^2 - 3xy + 2y^2 = 0$:

$$(9 - y)^2 - 3y(9 - y) + 2y^2 = 0$$

$$81 - 18y + y^2 - 27y + 3y^2 + 2y^2 = 0$$

$$6y^2 - 45y + 81 = 0$$

Divide by 3:

$2y^2 - 15y + 27 = 0$

$(2y - 9)(y - 3) = 0$

$y = \frac{9}{2}$ or $y = 3$

Substitute into $x = 9 - y$:

When $y = \frac{9}{2}$, $x = 9 - \frac{9}{2} = \frac{9}{2}$

When $y = 3$, $x = 9 - 3 = 6$

Solutions are

$x = 4\frac{1}{2}$, $y = 4\frac{1}{2}$ or $x = 6$, $y = 3$

2 c Rearrange $5y - 4x = 1$ to give:

$5y = 4x + 1$

Substitute $y = \frac{4}{5}x + \frac{1}{5}$ into

$x^2 - y^2 + 5x = 41$:

$$x^2 - \left(\tfrac{4}{5}x + \tfrac{1}{5}\right)^2 + 5x = 41$$

$$x^2 - \tfrac{16}{25}x^2 - \tfrac{8}{25}x - \tfrac{1}{25} + 5x = 41$$

$$25x^2 - 16x^2 - 8x - 1 + 125x = 1025$$

$$9x^2 + 117x - 1026 = 0$$

$$x^2 + 13x - 114 = 0$$

$$(x + 19)(x - 6) = 0$$

So $x = -19$ or $x = 6$

Substitute into $y = \frac{4}{5}x + \frac{1}{5}$

Solutions are $x = -19$, $y = -15$

or $x = 6$, $y = 5$

3 a Rearrange $x - y = 6$ to give:

$x = 6 + y$

Substitute into $xy = 4$:

$y(6 + y) = 4$

$6y + y^2 = 4$

$y^2 + 6y - 4 = 0$

Use the quadratic formula.

$a = 1, b = 6, c = -4$

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-6 \pm \sqrt{36 + 16}}{2}$$

$$= \frac{-6 \pm \sqrt{52}}{2}$$

$$= \frac{-6 \pm \sqrt{4 \times 13}}{2}$$

$$= -3 \pm \sqrt{13}$$

Substitute into $x = 6 + y$:

When $y = -3 - \sqrt{13}$,

$x = 6 - 3 - \sqrt{13} = 3 - \sqrt{13}$

When $y = -3 + \sqrt{13}$,

$x = 6 - 3 + \sqrt{13} = 3 + \sqrt{13}$

Solutions are

$x = 3 - \sqrt{13}$, $y = -3 - \sqrt{13}$

or $x = 3 + \sqrt{13}$, $y = -3 + \sqrt{13}$

3 b Rearrange $2x + 3y = 13$ to give:

$2x = 13 - 3y$

$x = \frac{1}{2}(13 - 3y)$

Substitute into $x^2 + y^2 = 78$:

$\left(\frac{1}{2}(13-3y)\right)^2 + y^2 = 78$

$\frac{1}{4}(13-3y)^2 + y^2 = 78$

Multiply by 4:

$(13-3y)^2 + 4y^2 = 312$

$169 - 78y + 9y^2 + 4y^2 = 312$

$13y^2 - 78y - 143 = 0$

Divide by 13:

$y^2 - 6y - 11 = 0$

Use the quadratic formula.

$a = 1,\ b = -6,\ c = -11$

$$y = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$$

$$= \frac{6 \pm \sqrt{(36+44)}}{2}$$

$$= \frac{6 \pm \sqrt{80}}{2}$$

$$= \frac{6 \pm \sqrt{16 \times 5}}{2}$$

$$= \frac{6 \pm 4\sqrt{5}}{2}$$

$$= 3 \pm 2\sqrt{5}$$

Substitute into $x = \frac{1}{2}(13 - 3y)$:

When $y = 3 - 2\sqrt{5}$,

$x = \frac{1}{2}\left(13 - 3\left(3 - 2\sqrt{5}\right)\right)$

$= \frac{1}{2}\left(13 - 9 + 6\sqrt{5}\right)$

$= 2 + 3\sqrt{5}$

When $y = 3 + 2\sqrt{5}$,

$x = \frac{1}{2}\left(13 - 3\left(3 + 2\sqrt{5}\right)\right)$

$= \frac{1}{2}\left(13 - 9 - 6\sqrt{5}\right)$

$= 2 - 3\sqrt{5}$

Solutions are

$x = 2 - 3\sqrt{5},\ y = 3 + 2\sqrt{5}$

or $x = 2 + 3\sqrt{5},\ y = 3 - 2\sqrt{5}$

4 Rearrange $x + y = 3$ to give:

$y = 3 - x$

Substitute into $x^2 - 3y = 1$.

$x^2 - 3(3 - x) = 1$

$x^2 + 3x - 10 = 0$

$(x + 5)(x - 2) = 0$

So $x = -5$ or $x = 2$

Solutions are $x = -5,\ y = 8$ or $x = 2$ and $y = 1$

5 a $3x^2 + x(2 - 4x) + 11 = 0$

$3x^2 + 2x - 4x^2 + 11 = 0$

$-x^2 + 2x + 11 = 0$

$x^2 - 2x - 11 = 0$

b Using the quadratic formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{2 \pm \sqrt{4+44}}{2}$$

$$x = \frac{2 \pm \sqrt{48}}{2}$$

$$x = \frac{2 \pm \sqrt{16 \times 3}}{2}$$

$$x = \frac{2 \pm 4\sqrt{3}}{2}$$

$x = 1 \pm 2\sqrt{3}$

Substitute into $y = 2 - 4x$:

$x = 1 + 2\sqrt{3},\ y = -2 - 8\sqrt{3}$

or $x = 1 - 2\sqrt{3},\ y = -2 + 8\sqrt{3}$

6 a At the point $(1, p)$, $x = 1$ and $y = p$.

Substituting these values into the first equation gives:

$p = k - 5$ **(1)**

Substituting these values into the second equation gives:

$4 - p = 6$ **(2)**

$p = -2$

When $p = -2$, $k = 3$

$k = 3,\ p = -2$

b Substitute for k into $y = kx - 5$:

$y = 3x - 5$

Substitute into $4x^2 - xy = 6$:

$4x^2 - x(3x - 5) = 6$

$4x^2 - 3x^2 + 5x - 6 = 0$

$x^2 + 5x - 6 = 0$

$(x - 1)(x + 6) = 0$

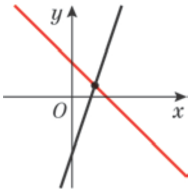
6 b $x = 1$ or $x = -6$
When $x = -6$, $y = -23$ and $x = 1$, $y = -2$
The solutions are $x = -6$, $y = -23$
or $x = 1$, $y = -2$

Challenge

Rearrange $y - x = k$ into $x^2 + y^2 = 4$:
$y = x + k$
$x^2 + (x + k)^2 = 4$
$x^2 + x^2 + 2kx + k^2 - 4 = 0$
$2x^2 + 2kx + k^2 - 4 = 0$
Using the discriminant for one pair of solutions,
$b^2 - 4ac = 0$
$(2k)^2 - 4(2)(k^2 - 4) = 0$
$4k^2 - 8k^2 + 32 = 0$
$-4k^2 = -32$
$k^2 = 8$
$k = \pm\sqrt{8}$
$= \pm\sqrt{4\times 2}$
$= \pm 2\sqrt{2}$

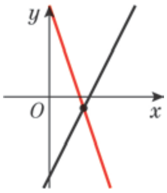
Equations and inequalities 3C

1 a i



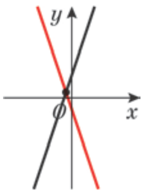
ii (2, 1)

b i



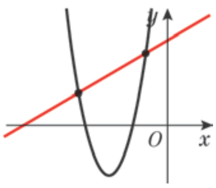
ii (3, −1)

c i Rearrange $3x + y + 1 = 0$ to give
$y = -3x - 1$



ii (−0.5, 0.5)

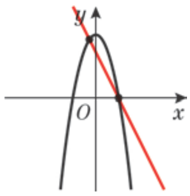
2 a Rearrange $2y = 2x + 11$ to give $y = x + \frac{11}{2}$



b (−1.5, 4) and (3.5, 9)

c Substitute values for x into each equation.
When $x = -1.5$:
$2y = 2(-1\frac{1}{2}) + 11 = 8, y = 4$
When $x = 3.5$:
$2y = 2(3\frac{1}{2}) + 11 = 18, y = 9$
When $x = -1.5$:
$y = 2(-1\frac{1}{2})^2 - 3(-1\frac{1}{2}) - 5 = \frac{9}{2} + \frac{9}{2} - 5 = 4$
When $x = 3.5$:
$y = 2(3\frac{1}{2})^2 - 3(3\frac{1}{2}) - 5 = \frac{49}{2} - \frac{21}{2} - 5 = 9$

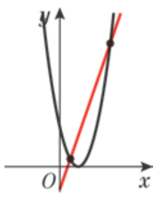
3 a $y = 9 - x^2$
$y = -2x + 6$



b (−1, 8) and (3, 0)

c Substitute each value of x and y into each equation:
$(-1)^2 + 8 = 1 + 8 = 9$
$2(-1) + 8 = -2 + 8 = 6$
$(3)^2 + 0 = 9$
$2(3) + 0 = 6$

4 a $y = (x - 2)^2$
$0 = (x - 2)^2$
$x = 2$
When $x = 0, y = 4$



b $(x - 2)^2 = 3x - 2$
$x^2 - 4x + 4 - 3x + 2 = 0$
$x^2 - 7x + 6 = 0$
$(x - 6)(x - 1) = 0$
$x = 6$ or $x = 1$
When $x = 1, y = 1$
When $x = 6, y = 16$
(1, 1) and (6, 16) are the points of intersection.

5 $y = x - 4$
Substitute into $y^2 = 2x^2 - 17$:
$$(x-4)^2 = 2x^2 - 17$$
$$x^2 - 8x + 16 = 2x^2 - 17$$
$$0 = x^2 + 8x - 33$$
$$0 = (x+11)(x-3)$$
$x = -11$ or $x = 3$

5 Substitute into $y = x - 4$:
When $x = -11$, $y = -11 - 4 = -15$
When $x = 3$, $y = 3 - 4 = -1$
Intersection points:
$(-11, -15)$ and $(3, -1)$

6 $y = 3x - 1$
Substitute into $y^2 - xy = 15$:
$$(3x-1)^2 - x(3x-1) = 15$$
$$9x^2 - 6x + 1 - 3x^2 + x = 15$$
$$6x^2 - 5x - 14 = 0$$
$$(6x+7)(x-2) = 0$$
$x = -\frac{7}{6}$ or $x = 2$
Substitute into $y = 3x - 1$:
When $x = -\frac{7}{6}$, $y = -\frac{21}{6} - 1 = -\frac{9}{2}$
When $x = 2$, $y = 6 - 1 = 5$
Intersection points:
$\left(-1\frac{1}{6}, -4\frac{1}{2}\right)$ and $(2, 5)$

7 a $6x^2 + 3x - 7 = 2x + 8$
$6x^2 + x - 15 = 0$
Using the discriminant:
$b^2 - 4ac = 1 + 360 = 361$
$361 > 0$
Therefore, there are 2 points of intersection.

b $4x^2 - 18x + 40 = 10x - 9$
$4x^2 - 28x + 49 = 0$
Using the discriminant:
$b^2 - 4ac = 784 - 784 = 0$
Therefore, there is 1 point of intersection.

c Rearrange $7x + y + 3 = 0$ to give:
$y = -7x - 3$
$3x^2 - 2x + 4 = -7x - 3$
$3x^2 + 5x + 7 = 0$
Using the discriminant:
$b^2 - 4ac = 25 - 84 = -59$
$-59 < 0$
Therefore, there are 0 points of intersection.

8 a Rearrange $2x - y = 1$ and then substitute into $x^2 + 4ky + 5k = 0$:
$y = 2x - 1$
$x^2 + 4k(2x - 1) + 5k = 0$
$x^2 + 8kx - 4k + 5k = 0$
$x^2 + 8kx + k = 0$

b Using the discriminant,
$b^2 - 4ac = 0$
$(8k)^2 - 4(1)(k) = 0$
$64k^2 - 4k = 0$
$4k(16k - 1) = 0$
$k = 0$ or $k = \frac{1}{16}$
As k is a non-zero constant, $k = \frac{1}{16}$

c $x^2 + 8(\frac{1}{16})x + \frac{1}{16} = 0$
$16x^2 + 8x + 1 = 0$
$(4x + 1)^2 = 0$
$x = -\frac{1}{4}$, $y = -\frac{3}{2}$

9 $p = 0.3x - 6$
If the swimmer touches the bottom of the pool, then
$0.5x^2 - 3x = 0.3x - 6$
$0.5x^2 - 3.3x + 6 = 0$
Using the discriminant:
$b^2 - 4ac = (-3.3)^2 - 4 \times 0.5 \times 6$
$= -1.11$
As -1.11 is negative, there are no solutions, so the swimmer does not reach the bottom of the pool.

Equations and inequalities 3D

1 a $2x < 5+3$
$2x < 8$
$x < 4$

b $5x \geq 39-4$
$5x \geq 35$
$x \geq 7$

c $6x-2x > 7+3$
$4x > 10$
$x > 2\frac{1}{2}$

d $5x+x \leq -12-6$
$6x \leq -18$
$x \leq -3$

e $-x > 4-15$
$-x > -11$
$x < 11$

f $21-8 > 3x+2x$
$13 > 5x$
$5x < 13$
$x < 2\frac{3}{5}$

g $x-3x < 25-1$
$-2x < 24$
$x > -12$

h $7x+7x < 7+7$
$14x < 14$
$x < 1$

i $-0.5x \geq 1-5$
$-0.5x \geq -4$
$x \leq 8$

j $5x+2x > 12-4$
$7x > 8$
$x > 1\frac{1}{7}$

2 a $2x-6 \geq 0$
$2x \geq 6$
$x \geq 3$

2 b $8-8x > x-1$
$8+1 > x+8x$
$9 > 9x$
$1 > x$
$x < 1$

c $3x+21 \leq 8-x$
$3x+x \leq 8-21$
$4x \leq -13$
$x \leq -3\frac{1}{4}$

d $2x-6-x-12 < 0$
$2x-x < 6+12$
$x < 18$

e $1+22-11x < 10x-40$
$1+22+40 < 10x+11x$
$63 < 21x$
$3 < x$
$x > 3$

f $2x-10 \geq 12-3x$
$2x+3x \geq 12+10$
$5x \geq 22$
$x \geq 4\frac{2}{5}$

g $12x-3x+9 < 45$
$12x-3x < 45-9$
$9x < 36$
$x < 4$

h $x-10-4x < 11$
$x-4x < 11+10$
$-3x < 21$
$x > -7$

i $x^2-4x \geq x^2+2$
$x^2-x^2-4x \geq 2$
$-4x \geq 2$
$x \leq -\frac{1}{2}$

2 j $5x - x^2 \geq 3 + x - x^2$

$5x - x - x^2 + x^2 \geq 3$

$4x \geq 3$

$x \geq \frac{3}{4}$

k $3x + 2x^2 - 6x \leq 10 + 2x^2$

$-3x \leq 10$

$x \geq -\frac{10}{3}$

l $2x^2 - 5x \leq \dfrac{4x^2 + 12x}{2} - 9$

$4x^2 - 10x \leq 4x^2 + 12x - 18$

$18 \leq 22x$

$x \geq \frac{9}{11}$

3 a $3x - 6 > x - 4$

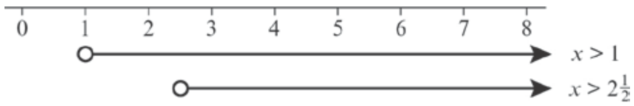
$2x > 2$

$x > 1$

$4x + 12 > 2x + 17$

$2x > 5$

$x > 2\frac{1}{2}$



So the required set of values is $x > 2\frac{1}{2}$

$\{x: x > 2\frac{1}{2}\}$

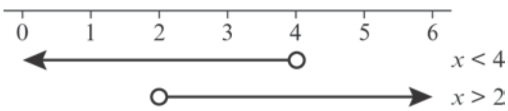
b $2x - 5 < x - 1$

$x < 4$

$7x + 7 > 23 - x$

$8x > 16$

$x > 2$



So the required set of values is $2 < x < 4$

$\{x: 2 < x < 4\}$

c $2x - 3 > 2$

$2x > 5$

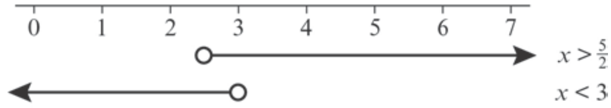
$x > \dfrac{5}{2}$

$3x + 6 < 12 + x$

$2x < 6$

$x < 3$

3 c



So the required set of values is $\frac{5}{2} < x < 3$

$\{x: \frac{5}{2} < x < 3\}$

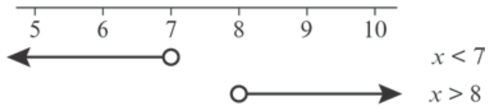
d $15 - x < 22 - 2x$

$x < 7$

$15x - 5 > 12x + 19$

$3x > 24$

$x > 8$



So there are no values that satisfy the inequality.

e $3x + 8 \leq 20$

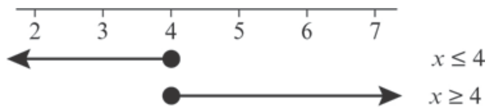
$3x \leq 12$

$x \leq 4$

$6x - 14 \geq x + 6$

$5x \geq 20$

$x \geq 4$



So the required set of values is $x = 4$

$\{x: x = 4\}$

f $5x + 3 < 9$

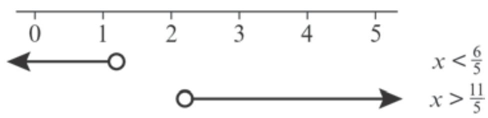
$5x < 6$

$x < \frac{6}{5}$

$10x + 5 > 27$

$10x > 22$

$x > \frac{11}{5}$



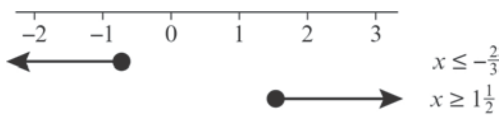
So the required set of values is

$x < \frac{6}{5}$ or $x > \frac{11}{5}$

$\{x: x < \frac{6}{5}\} \cup \{x: x > \frac{11}{5}\}$

3 g $12x + 28 \leq 20$
$12x \leq -8$
$x \leq -\frac{2}{3}$

$6x - 10 \geq \dfrac{7-6x}{2}$
$12x - 20 \geq 7 - 6x$
$18x \geq 27$
$x \geq 1\frac{1}{2}$



So the required set of values is
$x \leq -\frac{2}{3}$ or $x \geq 1\frac{1}{2}$
$\{x: x \leq -\frac{2}{3}\} \cup \{x: x \geq 1\frac{1}{2}\}$

Challenge

A: $3x + 5 > 2$
$3x > -3$
$x > -1$

B: $\dfrac{x}{2} + 1 \leq 3$
$\dfrac{x}{2} \leq 2$
$x \leq 4$

C: $11 < 2x - 1$
$12 < 2x$
$x > 6$

$p = -1$, $q = 4$, $r = 6$

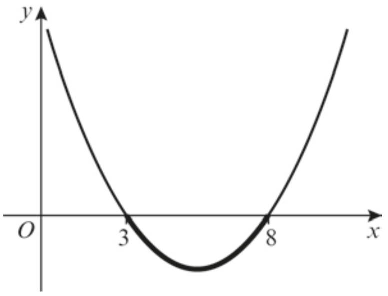
Equations and inequalities 3E

1 a $x^2 - 11x + 24 = 0$

$(x-3)(x-8) = 0$

$x = 3,\ x = 8$

Sketch of $y = x^2 - 11x + 24$:



$x^2 - 11 = 24 < 0$ when $3 < x < 8$

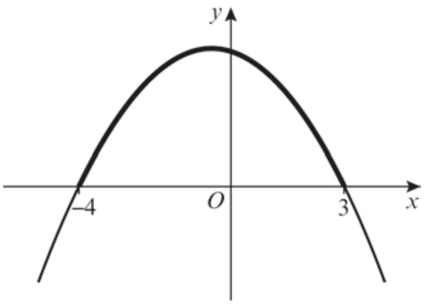
b $12 - x - x^2 = 0$

$0 = x^2 + x - 12$

$0 = (x+4)(x-3)$

$x = -4,\ x = 3$

Sketch of $y = 12 - x - x^2$:



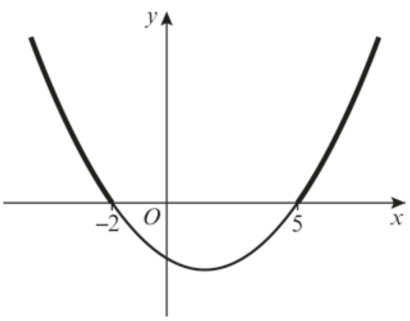
$12 - x - x^2 > 0$ when $-4 < x < 3$

c $x^2 - 3x - 10 = 0$

$(x+2)(x-5) = 0$

$x = -2,\ x = 5$

Sketch of $y = x^2 - 3x - 10$:



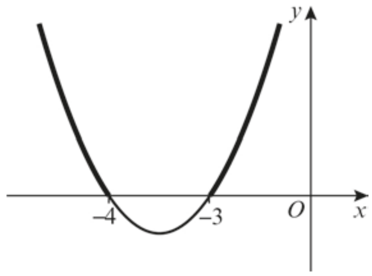
$x^2 - 3x - 10 > 0$ when $x < -2$ or $x > 5$

d $x^2 + 7x + 12 = 0$

$(x+4)(x+3) = 0$

$x = -4,\ x = -3$

Sketch of $y = x^2 + 7x + 12$:



$x^2 + 7x + 12 \geq 0$ when $x \leq -4$ or $x \geq -3$

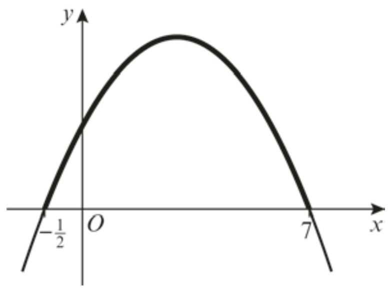
e $7 + 13x - 2x^2 = 0$

$2x^2 - 13x - 7 = 0$

$(2x+1)(x-7) = 0$

$x = -\frac{1}{2},\ x = 7$

Sketch of $y = 7 + 13x - 2x^2$:



$7 + 13x - 2x^2 > 0$ when $-\frac{1}{2} < x < 7$

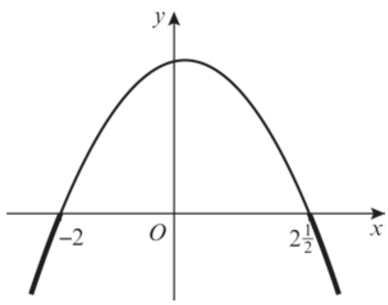
f $10 + x - 2x^2 = 0$

$2x^2 - x - 10 = 0$

$(2x-5)(x+2) = 0$

$x = \frac{5}{2},\ x = -2$

Sketch of $y = 10 + x - 2x^2$:



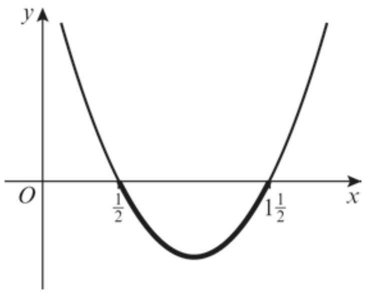
$10 + x - 2x^2 < 0$ when $x < -2$ or $x > \frac{5}{2}$

1 g $4x^2-8x+3=0$

$(2x-1)(2x-3)=0$

$x=\frac{1}{2},\ x=\frac{3}{2}$

Sketch of $y=4x^2-8x=3$:



$4x^2-8x+3\le 0$ when $\frac{1}{2}\le\ x\le\frac{3}{2}$

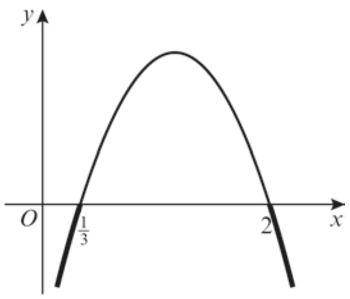
h $-2+7x-3x^2=0$

$3x^2-7x+2=0$

$(3x-1)(x-2)=0$

$x=\frac{1}{3},\ x=2$

Sketch of $y=-2+7x-3x^2$:



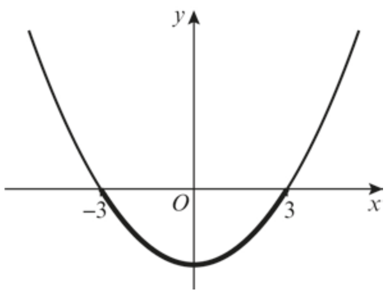
$-2+7x-3x^2<0$ when $x<\frac{1}{3}$ or $x>2$

i $x^2-9=0$

$(x+3)(x-3)=0$

$x=-3,\ x=3$

Sketch of $y=x^2-9$:



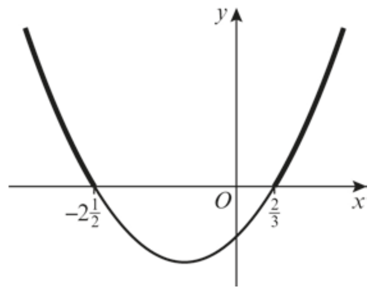
$x^2-9<0$ when $-3<x<3$

j $6x^2+11x-10=0$

$(3x-2)(2x+5)=0$

$x=\frac{2}{3},\ x=-\frac{5}{2}$

Sketch of $y=6x^2+11x-10$:



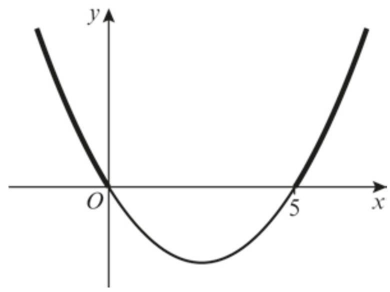
$6x^2+11x-10>0$ when $x<-\frac{5}{2}$ or $x>\frac{2}{3}$

k $x^2-5x=0$

$x(x-5)=0$

$x=0,\ x=5$

Sketch of $y=x^2-5x$:



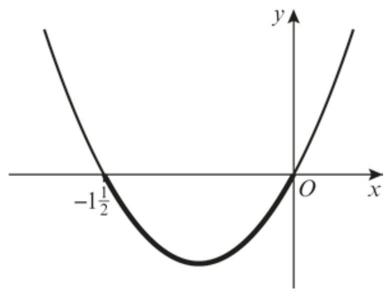
$x^2-5x>0$ when $x<0$ or $x>5$

l $2x^2+3x=0$

$x(2x+3)=0$

$x=0,\ x=-\frac{3}{2}$

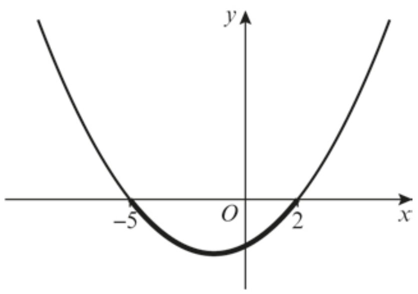
Sketch of $y=2x^2+3x$:



$2x^2+3x\le 0$ when $-\frac{3}{2}\le x\le 0$

2 a
$$x^2 = 10 - 3x$$
$$x^2 + 3x - 10 = 0$$
$$(x+5)(x-2) = 0$$
$$x = -5,\ \ x = 2$$
$$x^2 < 10 - 3x \Rightarrow x^2 + 3x - 10 < 0$$

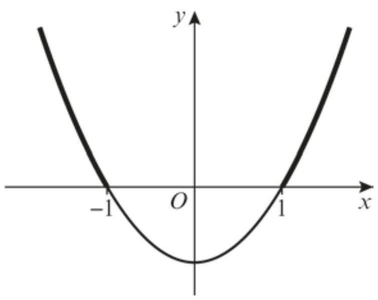
Sketch of $y = x^2 + 3x - 10$:



$x^2 + 3x - 10 < 0$ when $-5 < x < 2$

b
$$11 < x^2 + 10$$
$$0 < x^2 + 10 - 11$$
$$x^2 - 1 > 0$$

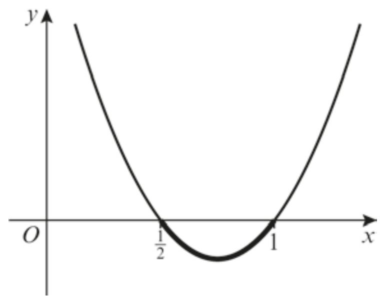
Sketch of $y = x^2 - 1$:



$x^2 - 1 > 0$ when $x < -1$ or $x > 1$

c
$$x(3-2x) = 1$$
$$3x - 2x^2 = 1$$
$$0 = 2x^2 - 3x + 1$$
$$0 = (2x-1)(x-1)$$
$$x = \tfrac{1}{2},\ \ x = 1$$
$$x(3-2x) > 1$$
$$\Rightarrow -2x^2 + 3x - 1 > 0$$
$$\Rightarrow 2x^2 - 3x + 1 < 0$$

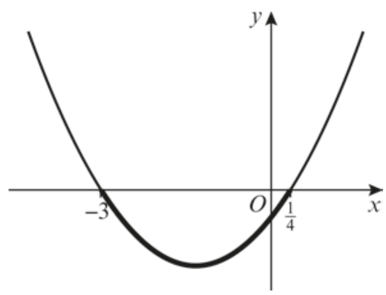
c Sketch of $y = 2x^2 - 3x + 1$:



$2x^2 - 3x + 1 < 0$ when $\tfrac{1}{2} < x < 1$

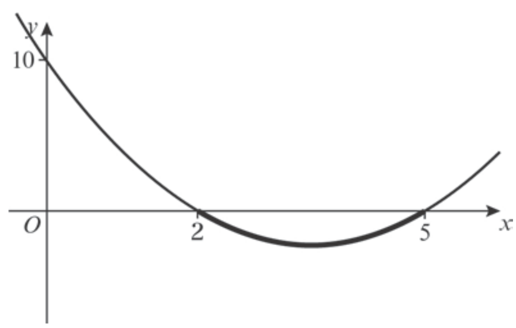
d
$$x(x+11) = 3(1-x^2)$$
$$x^2 + 11x = 3 - 3x^2$$
$$x^2 + 3x^2 + 11x - 3 = 0$$
$$4x^2 + 11x - 3 = 0$$
$$(4x-1)(x+3) = 0$$
$$x = \tfrac{1}{4},\ \ x = -3$$
$$x(x+11) < 3(1-x^2)$$
$$\Rightarrow 4x^2 + 11x - 3 < 0$$

Sketch of $y = 4x^2 + 11x - 3$:



$4x^2 + 11x - 3 < 0$ when $-3 < x < \tfrac{1}{4}$

3 a $x^2 - 7x + 10 < 0$
$$x^2 - 7x + 10 = 0$$
$$(x-2)(x-5) = 0$$
$$x = 2 \text{ or } x = 5$$

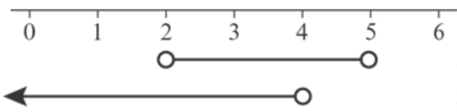


3 **a** So $2 < x < 5$

$3x + 5 < 17$

$3x < 12$

$x < 4$



So the required values are $2 < x < 4$

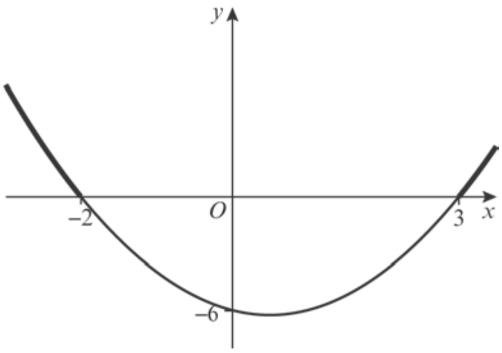
$\{x: 2 < x < 4\}$

b $x^2 - x - 6 > 0$

$x^2 - x - 6 = 0$

$(x - 3)(x + 2) = 0$

$x = 3$ or $x = -2$

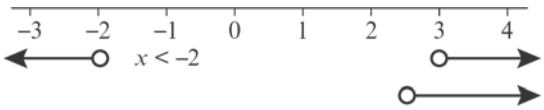


So $x < -2$ or $x > 3$

$10 - 2x < 5$

$-2x < -5$

$x > 2\frac{1}{2}$



So the required values are $x > 3$

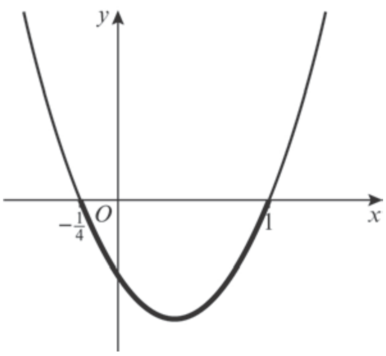
$\{x: x > 3\}$

c $4x^2 - 3x - 1 < 0$

$4x^2 - 3x - 1 = 0$

$(x - 1)(4x + 1) = 0$

$x = 1$ or $x = -\frac{1}{4}$

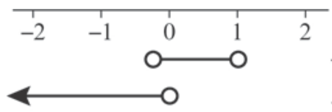


c So $-\frac{1}{4} < x < 1$

$4x + 8 < 15 - x - 7$

$5x < 0$

$x < 0$



So the required values are $-\frac{1}{4} < x < 0$

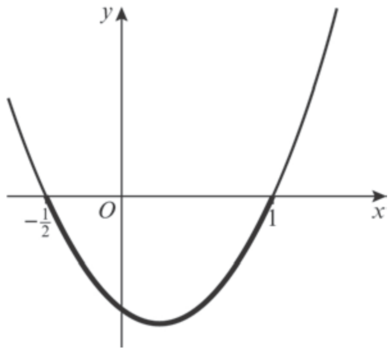
$\{x: -\frac{1}{4} < x < 0\}$

d $2x^2 - x - 1 < 0$

$2x^2 - x - 1 = 0$

$(2x + 1)(x - 1) = 0$

$x = -\frac{1}{2}$ or $x = 1$

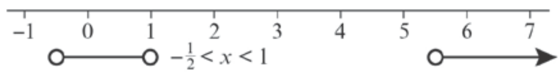


So $-\frac{1}{2} < x < 1$

$14 < 3x - 2$

$3x > 16$

$x > 5\frac{1}{3}$



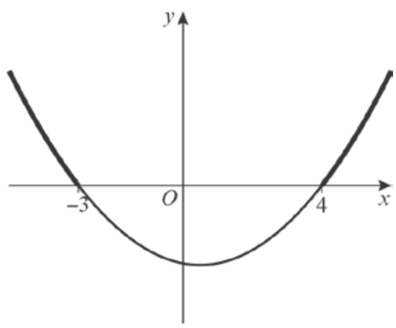
So there are no values.

e $x^2 - x - 12 > 0$

$x^2 - x - 12 = 0$

$(x - 4)(x + 3) = 0$

$x = 4$ or $x = -3$

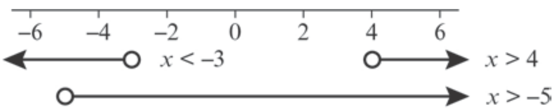


3 e So $x < -3$ or $x > 4$

$3x + 17 > 2$

$3x > -15$

$x > -5$



So the required values are $-5 < x < -3$ and $x > 4$

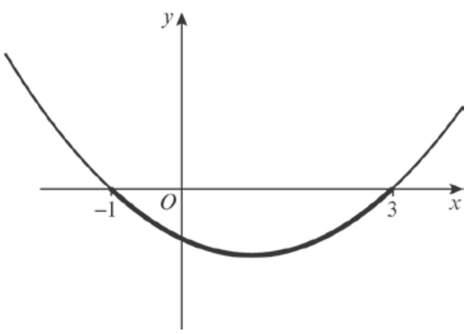
$\{x: -5 < x < -3\} \cup \{x: x > 4\}$

f $x^2 - 2x - 3 < 0$

$x^2 - 2x - 3 = 0$

$(x - 3)(x + 1) = 0$

$x = 3$ or $x = -1$



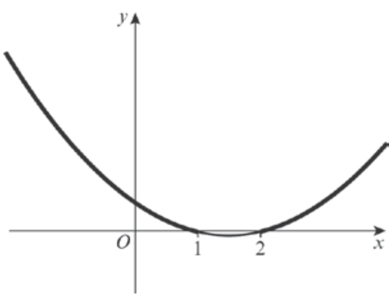
So $-1 < x < 3$

$x^2 - 3x + 2 > 0$

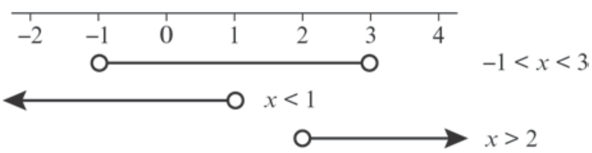
$x^2 - 3x + 2 = 0$

$(x - 2)(x - 1) = 0$

$x = 2$ or $x = 1$



f So $x < 1$ or $x > 2$



So the required values are $-1 < x < 1$ and $2 < x < 3$

$\{x: -1 < x < 1\} \cup \{x: 2 < x < 3\}$

4 a $\dfrac{2}{x} < 1$

Multiply both sides by x^2:

$2x < x^2$

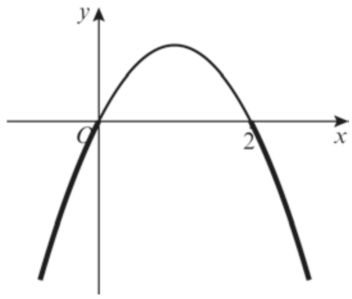
$2x - x^2 < 0$

Solve the quadratic to find the critical values:

$2x - x^2 = 0$

$x(2 - x) = 0$

$x = 0$ or $x = 2$



The solution is $x < 0$ or $x > 2$

b $5 > \dfrac{4}{x}$

Multiply both sides by x^2:

$5x^2 > 4x$

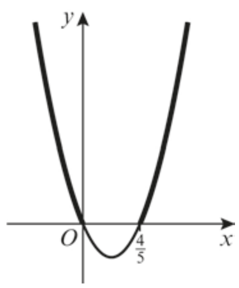
$5x^2 - 4x > 0$

Solve the quadratic to find the critical values:

$5x^2 - 4x = 0$

$x(5x - 4) = 0$

$x = 0$ or $x = \frac{4}{5}$



The solution is $x < 0$ or $x > \frac{4}{5}$.

4 c $\dfrac{1}{x}+3>2$

$\dfrac{1}{x}>-1$

Multiply both sides by x^2:

$x>-x^2$

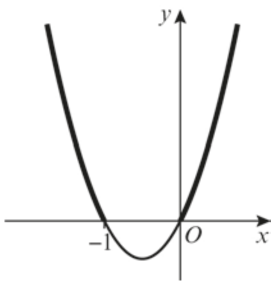
$x^2+x>0$

Solve the quadratic to find the critical values:

$x^2+x=0$

$x(x+1)=0$

$x=0$ or $x=-1$



The solution is $x<-1$ or $x>0$.

d $6+\dfrac{5}{x}>\dfrac{8}{x}$

$6>\dfrac{3}{x}$

Multiply both sides by x^2:

$6x^2>3x$

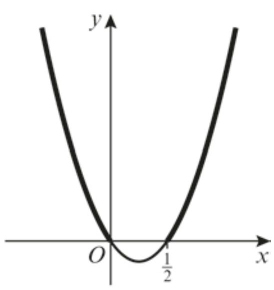
$6x^2-3x>0$

Solve the quadratic to find the critical values:

$6x^2-3x=0$

$3x(2x-1)=0$

$x=0$ or $x=\frac{1}{2}$



The solution is $x<0$ or $x>\frac{1}{2}$.

e $25>\dfrac{1}{x^2}$

$25x^2>1$

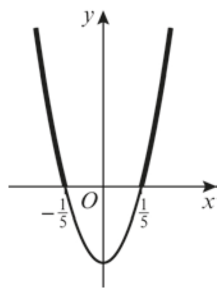
$25x^2-1>0$

Solve the quadratic to find the critical values:

$25x^2-1=0$

$(5x-1)(5x+1)=0$

$x=\frac{1}{5}$ or $x=-\frac{1}{5}$



The solution is $x<-\frac{1}{5}$ or $x>\frac{1}{5}$.

f $\dfrac{6}{x^2}+\dfrac{7}{x}\leq 3$

Multiply both sides by x^2:

$6+7x\leq 3x^2$

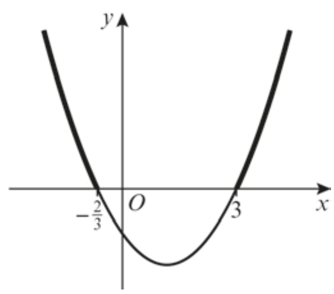
$3x^2-7x-6\geq 0$

Solve the quadratic to find the critical values:

$3x^2-7x-6=0$

$(3x+2)(x-3)=0$

$x=-\frac{2}{3}$ or $x=3$



The solution is $x\leq-\frac{2}{3}$ or $x\geq 3$.

5 a Using the quadratic formula:

$a=1,\ b=-k,\ c=k+3$

$b^2-4ac<0$ for no real roots, so

$k^2-4(k+3)<0$

$k^2-4k-12<0$

$(k-6)(k+2)<0$

$-2<k<6$

5 b Using the quadratic formula:

$a = p,\ b = p,\ c = -2$

$b^2 - 4ac > 0$ for real roots, so

$p^2 + 8p > 0$

$p(p+8) > 0$

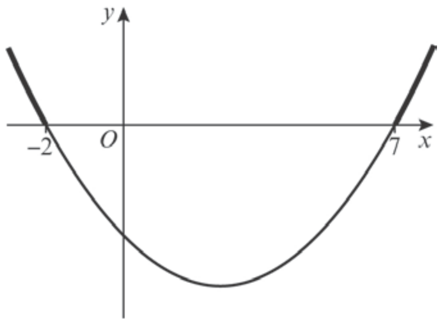
$p > 0$ or $p < -8$

6 $x^2 - 5x - 14 > 0$

$x^2 - 5x - 14 = 0$

$(x - 7)(x + 2) = 0$

$x = 7$ or $x = -2$



So the required values are $x < -2$ or $x > 7$

$\{x: x < -2\} \cup \{x: x > 7\}$

7 a $2(3x - 1) < 4 - 3x$

$6x - 2 < 4 - 3x$

$9x < 6$

$x < \frac{2}{3}$

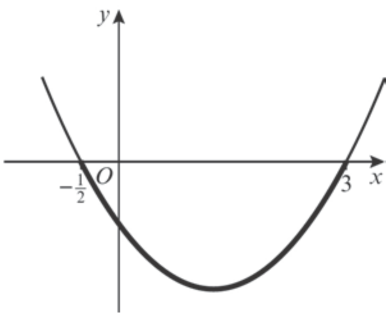
$\{x: x < \frac{2}{3}\}$

b $2x^2 - 5x - 3 < 0$

$2x^2 - 5x - 3 = 0$

$(2x + 1)(x - 3) = 0$

$x = -\frac{1}{2}$ or $x = 3$



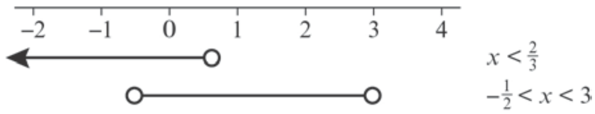
So $-\frac{1}{2} < x < 3$

$\{x: -\frac{1}{2} < x < 3\}$

c $6x - 2 < 4 - 3x \Rightarrow x < \frac{2}{3}$

$2x^2 - 5x - 3 < 0 \Rightarrow -\frac{1}{2} < x < 3$

7 c



So the required values are $-\frac{1}{2} < x < \frac{2}{3}$

$\{x: -\frac{1}{2} < x < \frac{2}{3}\}$

8 $\dfrac{5}{x-3} < 2$

Multiply both sides by $(x - 3)^2$:

$5(x - 3) < 2(x - 3)^2$

$5x - 15 < 2x^2 - 12x + 18$

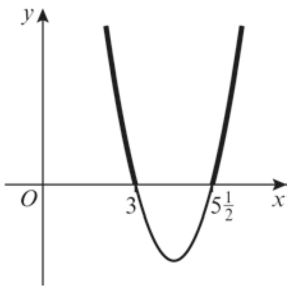
$2x^2 - 17x + 33 > 0$

Solve the quadratic to find the critical values:

$2x^2 - 17x + 33 = 0$

$(2x - 11)(x - 3) = 0$

$x = \frac{11}{2}$ or $x = 3$



The solution is $x < 3$ or $x > 5\frac{1}{2}$

9 $kx^2 - 2kx + 3 = 0$

For no real roots, using the discriminant:

$b^2 - 4ac < 0$

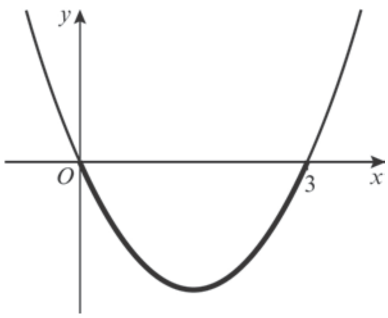
$(-2k)^2 - 4(k)(3) < 0$

$4k^2 - 12k < 0$

$4k^2 - 12k = 0$

$4k(k - 3) = 0$

$k = 0$ or $k = 3$



So $0 < k < 3$

When $k = 0$, the equation gives $3 = 0$

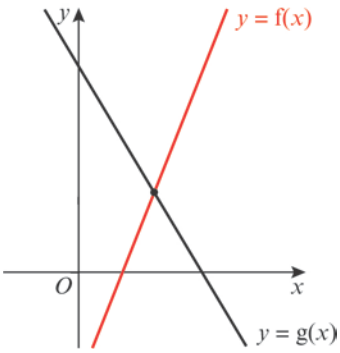
Therefore, $0 \le k < 3$.

Equations and inequalities 3F

1 a $3x + 2y = 6$ **(1)**
$x - y = 5$ **(2)**
Multiply equation (**2**) by 2:
$2x - 2y = 10$ **(3)**
Add equations (**1**) and (**3**):
$5x = 16$
$x = \frac{16}{5}, y = -\frac{9}{5}$
The solution is $P\left(\frac{16}{5}, -\frac{9}{5}\right)$.

b $2y + 3x > x - y$ when the line L_1 is above the line L_2:
$x < \frac{16}{5}$

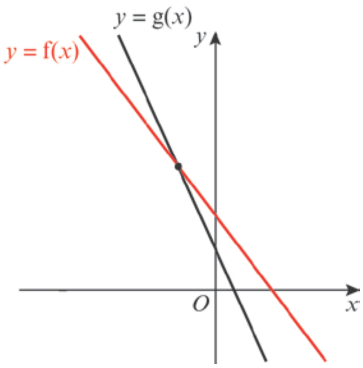
2 a i



ii $3x - 7 = 13 - 2x$
$5x = 20$
$x = 4, y = 5$
The lines intersect at (4, 5).

iii $f(x) \leq g(x)$ when the f(x) is below g(x), so $x \leq 4$

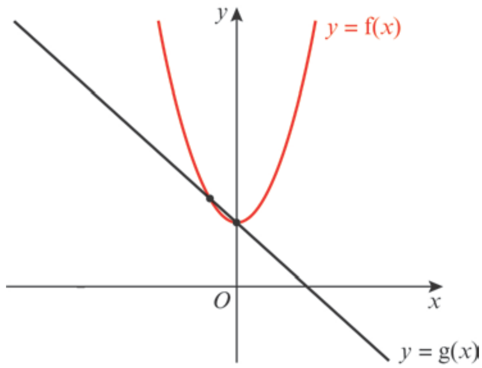
b i



ii $8 - 5x = 14 - 3x$
$-2x = 6$
$x = -3, y = 23$
The lines intersect at (−3, 23).

iii $f(x) \leq g(x)$ when f(x) is below g(x), so $x \geq -3$

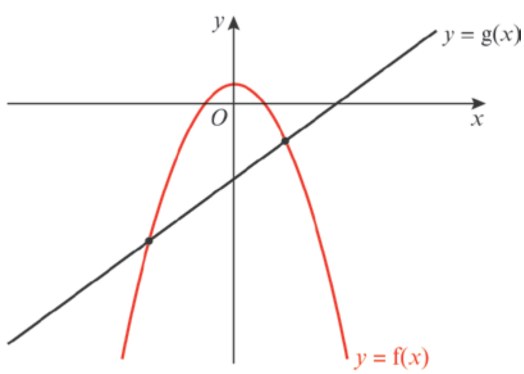
c i



ii $x^2 + 5 = 5 - 2x$
$x^2 + 2x = 0$
$x(x + 2) = 0$
$x = 0$ or $x = -2$
When $x = 0, y = 5$
When $x = -2, y = 9$
The lines intersect at (0, 5) and (−2, 9).

iii $f(x) \leq g(x)$ when f(x) is below g(x), so $-2 \leq x \leq 0$

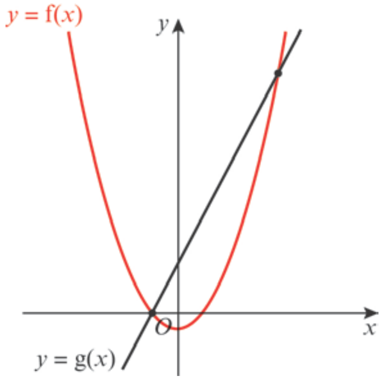
d i



2 d ii $3 - x^2 = 2x - 12$
$x^2 + 2x - 15 = 0$
$(x + 5)(x - 3) = 0$
$x = -5$ or $x = 3$
When $x = -5$, $y = -22$
When $x = 3$, $y = -6$
The lines intersect at $(-5, -22)$ and $(3, -6)$.

iii $f(x) \le g(x)$ when $f(x)$ is below $g(x)$, so $x \le -5$ or $x \ge 3$

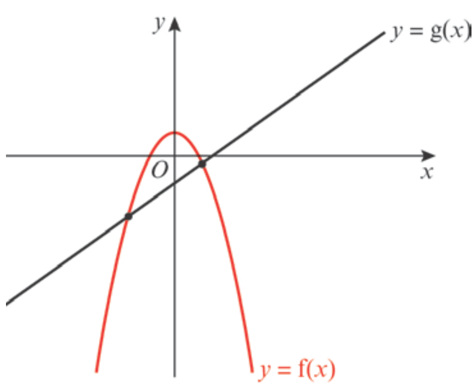
e i



ii $x^2 - 5 = 7x + 13$
$x^2 - 7x - 18 = 0$
$(x - 9)(x + 2) = 0$
$x = 9$ or $x = -2$
When $x = 9$, $y = 76$
When $x = -2$, $y = -1$
The lines intersect at $(-2, -1)$ and $(9, 76)$.

iii $f(x) \le g(x)$ when $f(x)$ is below $g(x)$, so $-2 \le x \le 9$

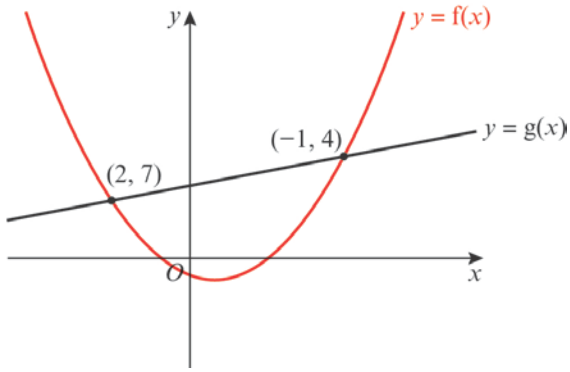
f i



f ii $7 - x^2 = 2x - 8$
$x^2 + 2x - 15 = 0$
$(x + 5)(x - 3) = 0$
$x = -5$ or $x = 3$
When $x = -5$, $y = -18$
When $x = 3$, $y = -2$
The lines intersect at $(-5, -18)$ and $(3, -2)$

iii $f(x) \le g(x)$ when $f(x)$ is below $g(x)$, so $x \le -5$ or $x \ge 3$

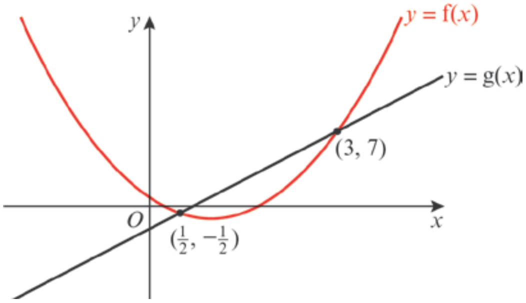
3 a $3x^2 - 2x - 1 = x + 5$
$3x^2 - 3x - 6 = 0$
$x^2 - x - 2 = 0$
$(x - 2)(x + 1) = 0$
$x = 2, x = -1$
The points of intersection are $(2, 7)$ and $(-1, 4)$.



So the required values are $-1 < x < 2$

b $2x^2 - 4x + 1 = 3x - 2$
$2x^2 - 7x + 3 = 0$
$(2x - 1)(x - 3) = 0$
$x = \frac{1}{2}$ or $x = 3$
The points of intersection are $(\frac{1}{2}, -\frac{1}{2})$ and $(3, 7)$.

3 b



So the required values are $\frac{1}{2} < x < 3$

c $5x - 2x^2 - 4 = -2x - 1$

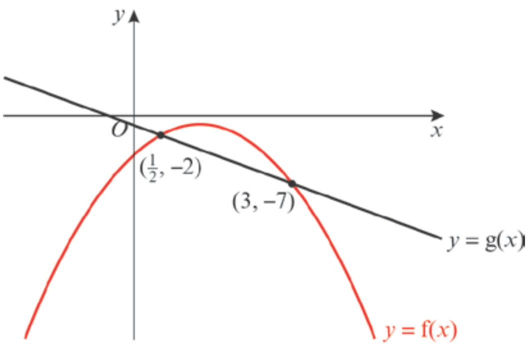
$2x^2 - 7x + 3 = 0$

$(2x - 1)(x - 3) = 0$

$x = \frac{1}{2}$ or $x = 3$

The points of intersection are

$(\frac{1}{2}, -2)$ and $(3, -7)$.



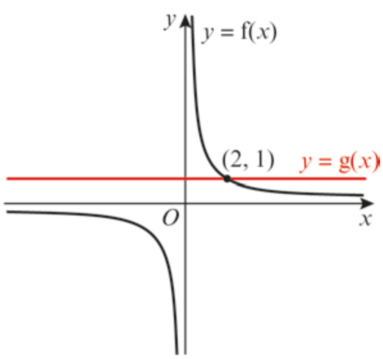
So the required values are

$x < \frac{1}{2}$ or $x > 3$

d $\frac{2}{x} = 1$

$x = 2$

Point of intersection is (2, 1)



d So the required values are

$x < 0$ or $x > 2$

e $\frac{3}{x^2} - \frac{4}{x} = -1$

Multiply both sides by x^2

$3 - 4x = -x^2$

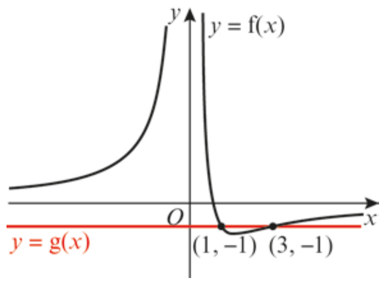
$x^2 - 4x + 3 = 0$

$(x - 1)(x - 3) = 0$

$x = 1$ or $x = 3$

Points of intersection are

$(1, -1)$ and $(3, -1)$



So the required values are $1 < x < 3$

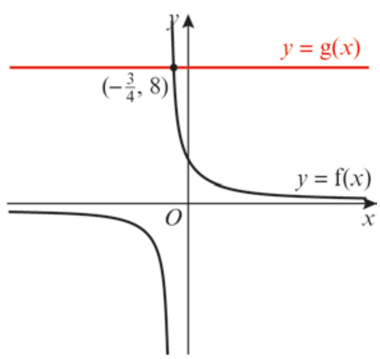
f $\frac{2}{x+1} = 8$

$2 = 8(x + 1)$

$8x + 6 = 0$

$x = -\frac{3}{4}$

Point of intersection is $(-\frac{3}{4}, 8)$



So the required values are

$x < -1$ or $x > -\frac{3}{4}$

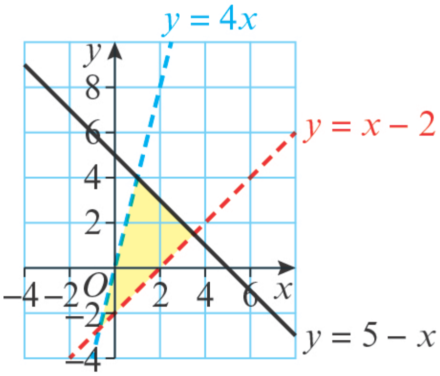
Challenge

a $x^2 - 4x - 12 = 6 + 5x - x^2$
$2x^2 - 9x - 18 = 0$
$(2x + 3)(x - 6) = 0$
$x = -\frac{3}{2}$ or $x = 6$
The points of intersection are
$(-\frac{3}{2}, -\frac{15}{4})$ and (6, 0).

b So the required values are $-\frac{3}{2} < x < 6$
$\{x: -\frac{3}{2} < x < 6\}$

Equations and inequalities 3G

1



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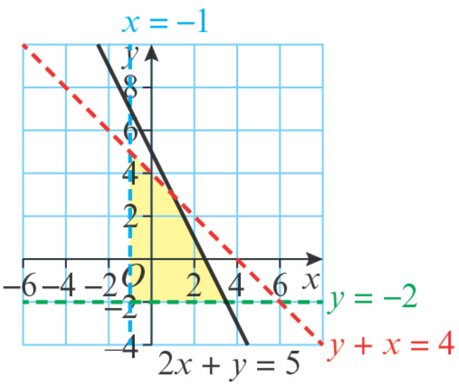

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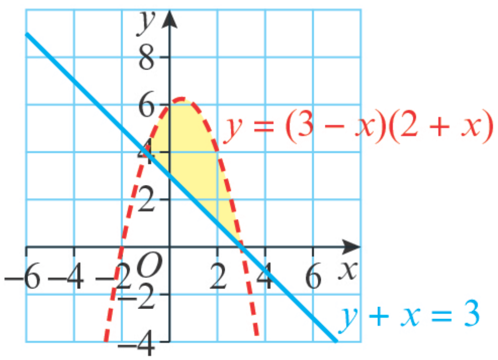
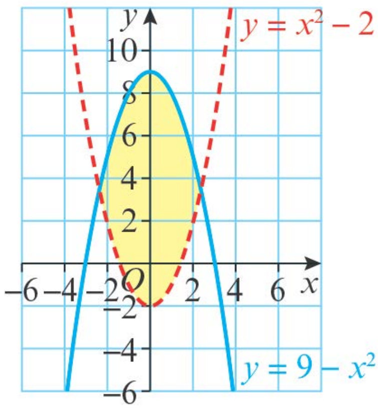
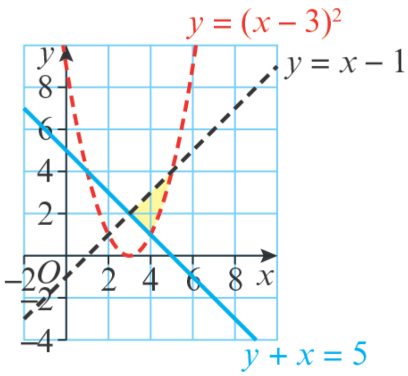


4



5



6 **a** For $y = x + 1$ and $y = 7 - x$:

$x + 1 = 7 - x$

$2x = 6$

$x = 3, y = 4$

For $y = 7 - x$ and $x = 1$:

$x = 1, y = 6$

For $x = 1$ and $y = x + 1$

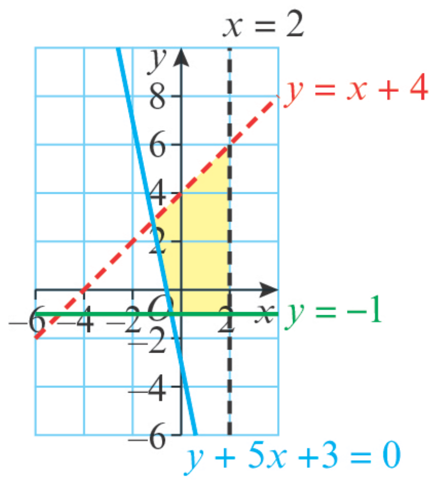
$x = 1, y = 2$

The points of intersection are $(3, 4)$, $(1, 6)$ and $(1, 2)$.

b $y \geq x + 1$, $y \leq 7 - x$ and $x \geq 1$

7 $y < 2 - 5x - x^2$, $2x + y \geq 0$ and $x + y \leq 4$

8 a



b For $y = x + 4$ and $y = -5x - 3$:
$x + 4 = -5x - 3$
$6x = -7$
$x = -\frac{7}{6}$, $y = \frac{17}{6}$
For $y = -5x - 3$ and $y = -1$:
$y = -1$, $x = -\frac{2}{5}$
For $y = -1$ and $x = 2$:
$x = 2$, $y = -1$
For $x = 2$ and $y = x + 4$:
$x = 2$, $y = 6$
The vertices are at the points
$(-\frac{7}{6}, \frac{17}{6})$, $(-\frac{2}{5}, -1)$, $(2, -1)$ and $(2, 6)$.

c $(-\frac{2}{5}, -1)$ is the only vertex formed by two solid lines.

d Area of shaded region = area of right-angled triangle − area of unshaded triangle
Area of right-angled triangle
$= \frac{1}{2} \times 7 \times 7$
$= \frac{49}{2}$
Area of unshaded triangle
$= \frac{1}{2} \times (-\frac{2}{5} - -5) \times (\frac{17}{6} - (-1))$
$= \frac{529}{60}$
Area of shaded region
$= \frac{49}{2} - \frac{529}{60}$
$= \frac{941}{60}$

Equations and inequalities, Mixed Exercise 3

1 a $2kx - y = 4$ (**1**)
$4kx + 3y = -2$ (**2**)
Multiply equation (**1**) by 2 to give
$4kx - 2y = 8$ (**3**)
Subtract equation (**2**) from equation (**3**)
$-5y = 10$
$y = -2$

b Using (**1**), $2kx + 2 = 4$:
$2kx = 2$
$x = \dfrac{1}{k}$

2 Rearrange $x + 2y = 3$ to give:
$x = 3 - 2y$
Substitute into $x^2 - 4y^2 = -33$:
$$(3-2y)^2 - 4y^2 = -33$$
$$9 - 12y + 4y^2 - 4y^2 = -33$$
$$-12y = -33 - 9$$
$$-12y = -42$$
$$y = \tfrac{7}{2}$$
Substitute into $x = 3 - 2y$:
$x = 3 - 7 = -4$
So the solution is $x = -4, y = \frac{7}{2}$

3 a Rearrange $x - 2y = 1$ to give:
$x = 1 + 2y$
Substitute into $3xy - y^2 = 8$:
$$3y(1+2y) - y^2 = 8$$
$$3y + 6y^2 - y^2 = 8$$
$$5y^2 + 3y - 8 = 0$$

b $(5y+8)(y-1) = 0$
$y = -\frac{8}{5}$ or $y = 1$
Substitute into $x = 1 + 2y$.
When $y = -\frac{8}{5}$, $x = 1 - \frac{16}{5} = -\frac{11}{5}$
When $y = 1$, $x = 1 + 2 = 3$

3 b So the solutions are
$\left(-\frac{11}{5}, -\frac{8}{5}\right)$ and $(3, 1)$

4 a Rearrange $x + y = 2$ to give:
$y = 2 - x$
$$x^2 + x(2-x) - (2-x)^2 = -1$$
$$x^2 + 2x - x^2 - 4 + 4x - x^2 + 1 = 0$$
$$-x^2 + 6x - 3 = 0$$
$$x^2 - 6x + 3 = 0$$

b Using the quadratic formula:
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{6 \pm \sqrt{36-12}}{2}$$
$$x = \frac{6 \pm \sqrt{24}}{2}$$
$$x = \frac{6 \pm \sqrt{4\times 6}}{2}$$
$$x = \frac{6 \pm 2\sqrt{6}}{2}$$
$$x = 3 \pm \sqrt{6}$$

Substitute into $y = 2 - x$.
$$y = 2 - \left(3 \pm \sqrt{6}\right)$$
$$y = -1 \pm \sqrt{6}$$

5 a $9 = 3^2$, so $3^x = \left(3^2\right)^{y-1}$
$\Rightarrow 3^x = 3^{2(y-1)}$

Equate powers:
$x = 2(y-1)$
$\Rightarrow x = 2y - 2$

5 b $x = 2y - 2$

Substitute into $x^2 = y^2 + 7$:

$$(2y-2)^2 = y^2 + 7$$
$$4y^2 - 8y + 4 = y^2 + 7$$
$$4y^2 - y^2 - 8y + 4 - 7 = 0$$
$$3y^2 - 8y - 3 = 0$$
$$(3y+1)(y-3) = 0$$
$$y = -\tfrac{1}{3} \text{ or } y = 3$$

Substitute into $x = 2y - 2$.

When $y = -\frac{1}{3}$, $x = -\frac{2}{3} - 2 = -2\frac{2}{3}$

When $y = 3$, $x = 6 - 2 = 4$

The solutions are:

$x = -\frac{8}{3}, y = -\frac{1}{3}$ and $x = 4, y = 3$

6 Rearrange $x + 2y = 3$ to give:

$x = 3 - 2y$

Substitute into $x^2 - 2y + 4y^2 = 18$:

$$(3-2y)^2 - 2y + 4y^2 = 18$$
$$9 - 12y + 4y^2 - 2y + 4y^2 = 18$$
$$8y^2 - 14y + 9 - 18 = 0$$
$$8y^2 - 14y - 9 = 0$$
$$(4y-9)(2y+1) = 0$$
$$y = \tfrac{9}{4} \text{ or } y = -\tfrac{1}{2}$$

Substitute into $x = 3 - 2y$.

When $y = \frac{9}{4}$, $x = 3 - \frac{9}{2} = -\frac{3}{2}$

When $y = -\frac{1}{2}$, $x = 3 + 1 = 4$

The solutions are: $x = -\frac{3}{2}, y = \frac{9}{4}$

and $x = 4$, $y = -\frac{1}{2}$

7 a Rearrange $-\dfrac{k}{2}x + y = 1$ and substitute into the quadratic equation:

$$y = 1 + \frac{k}{2}x$$

7 a $kx^2 - x\left(1 + \dfrac{k}{2}x\right) + (k+1)x = 1$

$$kx^2 - x - \frac{k}{2}x^2 + kx + x - 1 = 0$$
$$kx^2 + 2kx - 2 = 0$$

Using the discriminant for one solution:

$$b^2 - 4ac = 0$$
$$(2k)^2 - 4(k)(-2) = 0$$
$$4k^2 + 8k = 0$$
$$4k(k+2) = 0$$

k is non-zero, so $k = -2$

b Substituting into $kx^2 + 2kx - 2 = 0$ gives:

$$-2x^2 + 4x - 2 = 0$$
$$-x^2 - 2x - 1 = 0$$
$$x^2 + 2x + 1 = 0$$
$$(x+1)^2 = 0$$
$$x = -1$$

Substitute for x and k:

$$y = 1 + \frac{k}{2}x = 1 - (-1) = 2$$

Therefore, the coordinates are $(-1, 2)$.

8 The sloping ceiling can be modelled by $h = \frac{15}{2} - \frac{1}{5}x$

If the ball hits the ceiling, then

$$\tfrac{15}{2} - \tfrac{1}{5}x = -\tfrac{3}{10}x^2 + \tfrac{5}{2}x + \tfrac{3}{2}$$
$$-\tfrac{3}{10}x^2 + \tfrac{27}{10}x - 6 = 0$$
$$-3x^2 + 27x - 60 = 0$$
$$x^2 - 9x + 20 = 0$$

Using the discriminant:

$b^2 - 4ac = 81 - 80 = 1$

As $1 > 0$, there are two solutions.

Therefore, the model does predict that the ball hits the ceiling.

9 a $3x - x > 13 + 8$

$2x > 21$

$x > 10\frac{1}{2}$

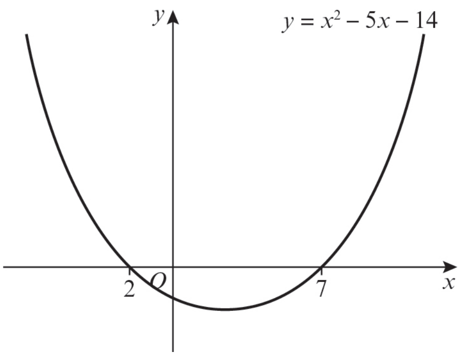
In set notation, the solution is

$\{x: x > \frac{21}{2}\}$

b $x^2 - 5x - 14 = 0$

$(x+2)(x-7) = 0$

$x = -2$ or $x = 7$



$x^2 - 5x - 14 > 0$ when $x < -2$

or $x > 7$

In set notation, the solution is

$\{x : x < -2\} \cup \{x : x > 7\}$

10 Multiplying out the brackets:

$x^2 - 5x + 4 < 2x - 8$

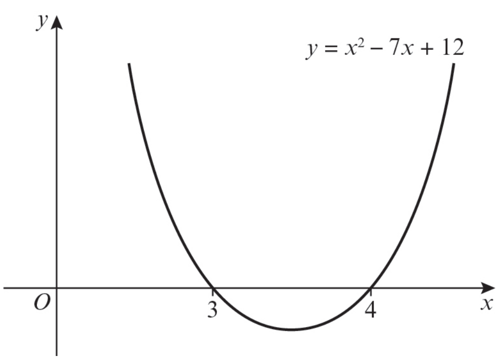
$x^2 - 5x - 2x + 4 + 8 < 0$

$x^2 - 7x + 12 < 0$

$x^2 - 7x + 12 = 0$

$(x-3)(x-4) = 0$

$x = 3$ or $x = 4$



$x^2 - 7x + 12 < 0$ when $3 < x < 4$

11 a $x^2 + x - 2 = 18$

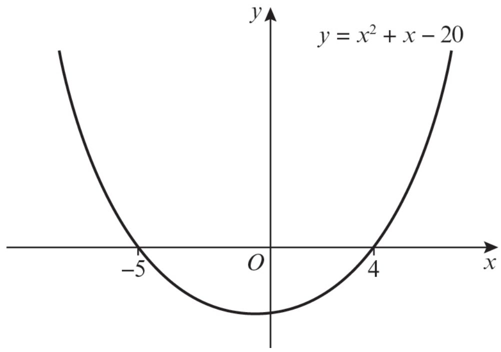
$x^2 + x - 20 = 0$

$(x+5)(x-4) = 0$

$x = -5$ or $x = 4$

b $(x-1)(x+2) > 18$

$\Rightarrow x^2 + x - 20 > 0$



$x^2 + x - 20 > 0$ when $x < -5$ or $x > 4$

In set notation, the solution is

$\{x : x < -5\} \cup \{x : x > 4\}$

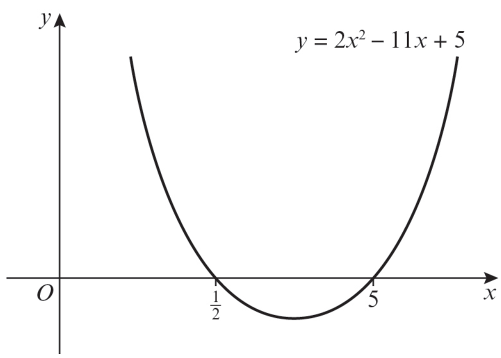
12 a $6x - 2x < 3 + 7$

$4x < 10$

$x < \frac{5}{2}$

b $(2x-1)(x-5) = 0$

$x = \frac{1}{2}$ or $x = 5$



$2x^2 - 11x + 5 < 0$ when $\frac{1}{2} < x < 5$

12 c $5 < \dfrac{20}{x}$

Multiply both sides by x^2

$5x^2 < 20x$

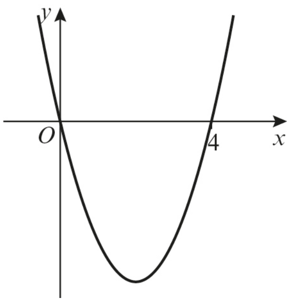
$5x^2 - 20x < 0$

Solve the quadratic to find the critical values:

$5x^2 - 20x = 0$

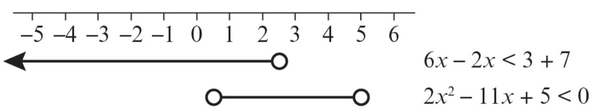
$5x(x - 4) = 0$

$x = 0$ or $x = 4$



The solution is $0 < x < 4$

d



Intersection is $\frac{1}{2} < x < \frac{5}{2}$

13 $\dfrac{8}{x^2} + 1 \leq \dfrac{9}{x}$

Multiply both sides by x^2:

$8 + x^2 \leq 9x$

$x^2 - 9x + 8 \leq 0$

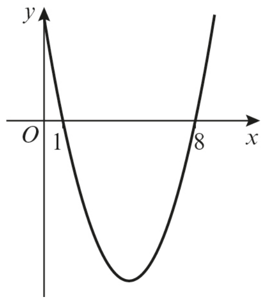
Solve the quadratic to find the critical values:

$x^2 - 9x + 8 = 0$

$(x - 1)(x - 8) = 0$

$x = 1$ or $x = 8$

13



The solution is $1 \leq x \leq 8$

14 $a = k, b = 8, c = 5$

Using the discriminant $b^2 - 4ac \geq 0$:

$8^2 - 4k \times 5 \geq 0$

$64 - 20k \geq 0$

$64 \geq 20k$

$\frac{64}{20} \geq k$

$k \leq \frac{16}{5}$

15 $a = 2,\ b = 4k,\ c = -5k$

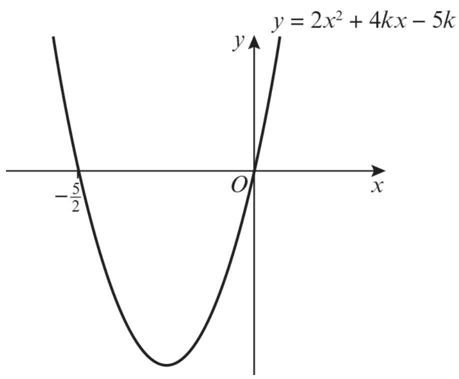
Using the discriminant $b^2 - 4ac < 0$:

$(4k)^2 - 4(2)(-5k) < 0$

$16k^2 + 40k < 0$

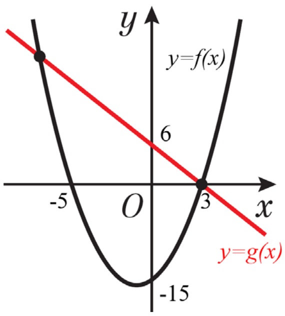
$8k(2k + 5) < 0$

$k = 0$ or $k = -\frac{5}{2}$



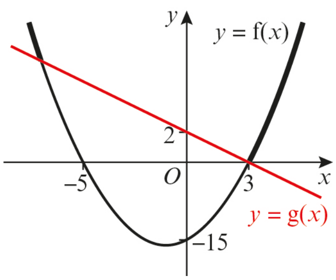
$-\frac{5}{2} < k < 0$

16 a $y = x^2 + 2x - 15$
$y = (x + 5)(x - 3)$
$0 = (x + 5)(x - 3)$
$x = -5$ or $x = 3$
When $x = 0$, $y = -15$



b $x^2 + 2x - 15 = 6 - 2x$
$x^2 + 4x - 21 = 0$
$(x + 7)(x - 3) = 0$
$x = -7$ or $x = 3$
When $x = -7$, $y = 20$
When $x = 3$, $y = 0$
The points of intersection are $(-7, 20)$ and $(3, 0)$.

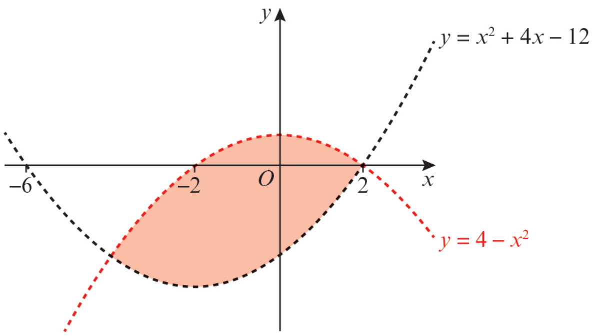
c



From the graph and the calculated points of intersection, the required values are
$x < -7$ or $x > 3$.

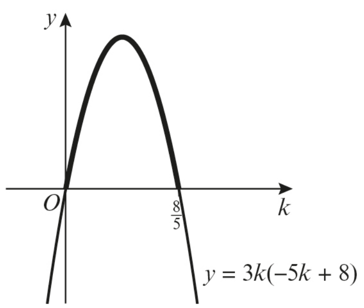
17 $2x^2 + 3x - 15 = 8 + 2x$
$2x^2 + x - 23 = 0$
$$x = \frac{-1 \pm \sqrt{185}}{4} = \tfrac{1}{4}\left(-1 \pm \sqrt{185}\right)$$
$$\tfrac{1}{4}\left(-1 - \sqrt{185}\right) < x < \tfrac{1}{4}\left(-1 + \sqrt{185}\right)$$

18 $y = x^2 + 4x - 12$
$x^2 + 4x - 12 = 0$
$(x + 6)(x - 2) = 0$
$x = -6$ or $x = 2$
$y = 4 - x^2$
$4 - x^2 = 0$
$(2 + x)(2 - x) = 0$
$x = -2$ or $x = 2$



Challenge

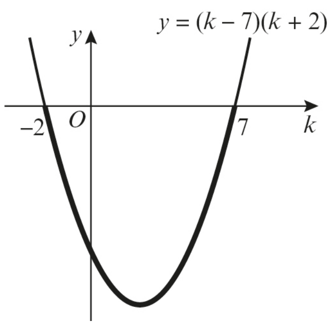
1 $2kx^2 + 5kx + 5k - 3 = 0$
Using the discriminant:
$b^2 - 4ac \geq 0$ for real roots.
$(5k)^2 - 4(2k)(5k - 3) \geq 0$
$25k^2 - 40k^2 + 24k \geq 0$
$-15k^2 + 24k \geq 0$
$3k(-5k + 8) \geq 0$
$3k(-5k + 8) = 0$
$k = 0$ or $k = \frac{8}{5}$



$0 < k \leq \frac{8}{5}$

2 $2x - k = 3x^2 + 2kx + 5$
$3x^2 + 2kx - 2x + 5 + k = 0$
$3x^2 + (2k - 2)x + 5 + k = 0$
If the line and parabola do not intersect then there are no solutions.

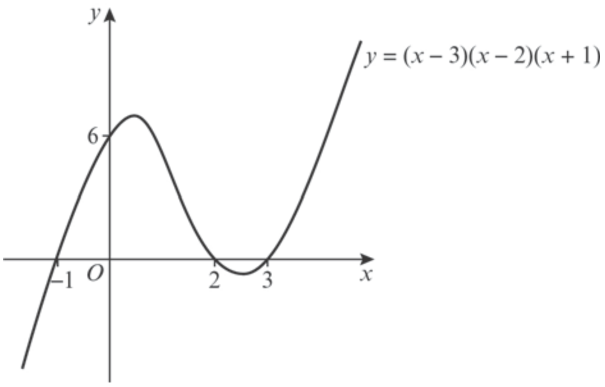
Using the discriminant:
$b^2 - 4ac < 0$
$(2k - 2)^2 - 4(3)(5 + k) < 0$
$4k^2 - 8k + 4 - 60 - 12k < 0$
$4k^2 - 20k - 56 < 0$
$k^2 - 5k - 14 < 0$
$k^2 - 5k - 14 = 0$
$(k - 7)(k + 2) = 0$
$k = 7$ or $k = -2$



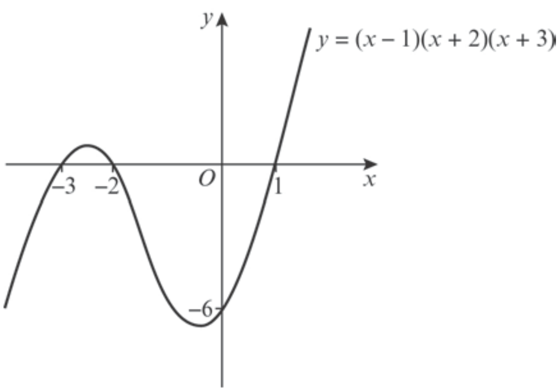
The line and the parabola do not intersect in the interval $-2 < k < 7$

Graphs and transformations 4A

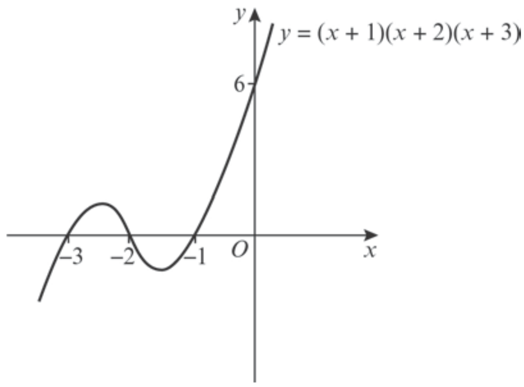
1 a $y = (x - 3)(x - 2)(x + 1)$
$0 = (x - 3)(x - 2)(x + 1)$
So $x = 3$, $x = 2$ or $x = -1$
So the curve crosses the x-axis at $(3, 0)$, $(2, 0)$ and $(-1, 0)$.
When $x = 0$, $y = (-3) \times (-2) \times 1 = 6$
So the curve crosses the y-axis at $(0, 6)$.
$x \to \infty, y \to \infty$
$x \to -\infty, y \to -\infty$



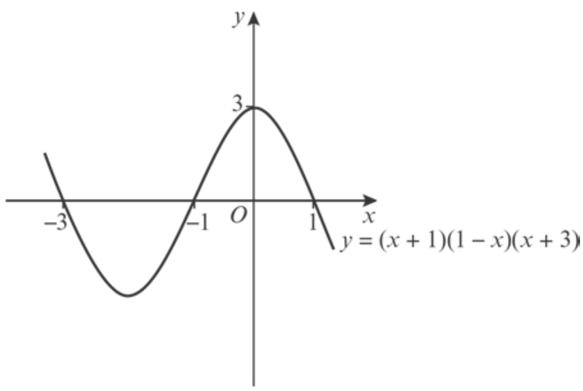
b $y = (x - 1)(x + 2)(x + 3)$
$0 = (x - 1)(x + 2)(x + 3)$
So $x = 1$, $x = -2$ or $x = -3$
So the curve crosses the x-axis at $(1, 0)$, $(-2, 0)$ and $(-3, 0)$.
When $x = 0$, $y = (-1) \times 2 \times 3 = -6$
So the curve crosses the y-axis at $(0, -6)$.
$x \to \infty, y \to \infty$
$x \to -\infty, y \to -\infty$



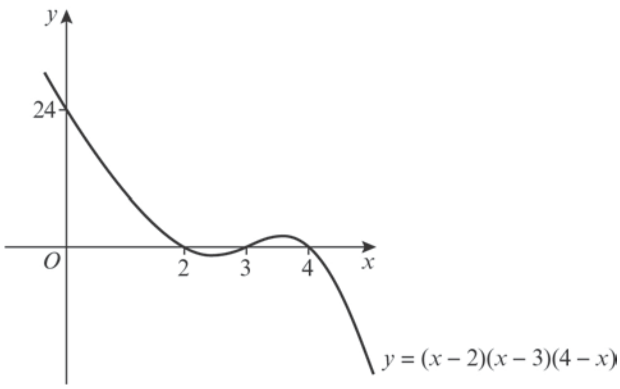
c $y = (x + 1)(x + 2)(x + 3)$
$0 = (x + 1)(x + 2)(x + 3)$
So $x = -1$, $x = -2$ or $x = -3$
So the curve crosses the x-axis at $(-1, 0)$, $(-2, 0)$ and $(-3, 0)$.
When $x = 0$, $y = 1 \times 2 \times 3 = 6$
So the curve crosses the y-axis at $(0, 6)$.
$x \to \infty, y \to \infty$
$x \to -\infty, y \to -\infty$



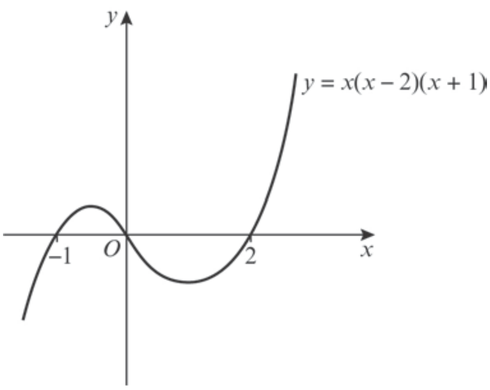
d $y = (x + 1)(1 - x)(x + 3)$
$0 = (x + 1)(1 - x)(x + 3)$
So $x = -1$, $x = 1$ or $x = -3$
So the curve crosses the x-axis at $(-1, 0)$, $(1, 0)$ and $(-3, 0)$.
When $x = 0$, $y = 1 \times 1 \times 3 = 3$
So the curve crosses the y-axis at $(0, 3)$.
$x \to \infty, y \to -\infty$
$x \to -\infty, y \to \infty$



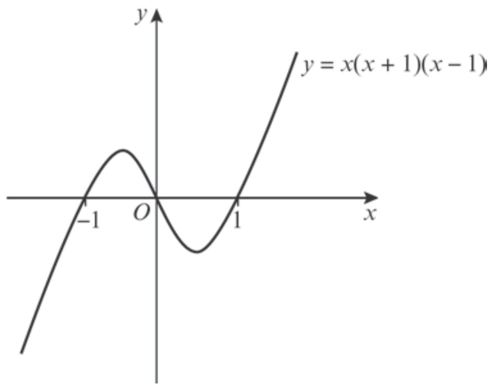
1 **e** $y = (x - 2)(x - 3)(4 - x)$
$0 = (x - 2)(x - 3)(4 - x)$
So $x = 2$, $x = 3$ or $x = 4$
So the curve crosses the x-axis at (2, 0), (4, 0) and (4, 0).
When $x = 0$, $y = (-2) \times (-3) \times 4 = 24$
So the curve crosses the y-axis at (0, 24).
$x \to \infty, y \to -\infty$
$x \to -\infty, y \to \infty$



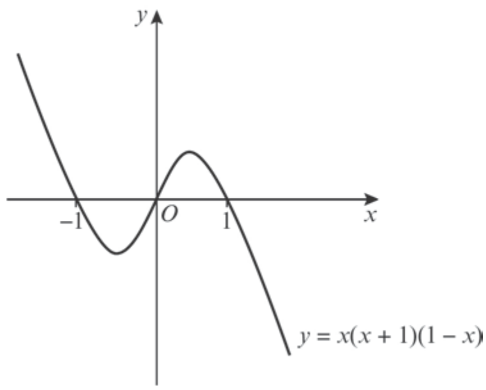
f $y = x(x - 2)(x + 1)$
$0 = x(x - 2)(x + 1)$
So $x = 0$, $x = 2$ or $x = -1$
So the curve crosses the x-axis at (0, 0), (2, 0) and (−1, 0).
$x \to \infty, y \to \infty$
$x \to -\infty, y \to -\infty$



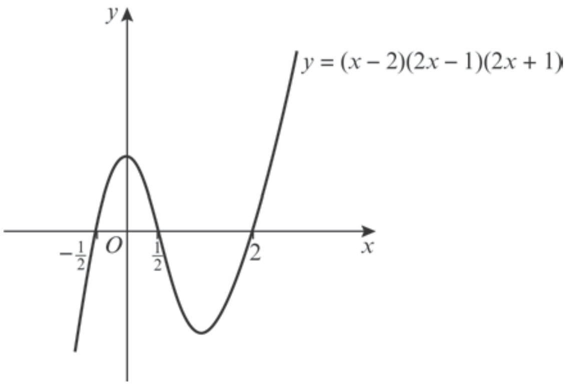
g $y = x(x + 1)(x - 1)$
$0 = x(x + 1)(x - 1)$
So $x = 0$, $x = -1$ or $x = 1$
So the curve crosses the x-axis at (0, 0), (−1, 0) and (1, 0).
$x \to \infty, y \to \infty$
$x \to -\infty, y \to -\infty$



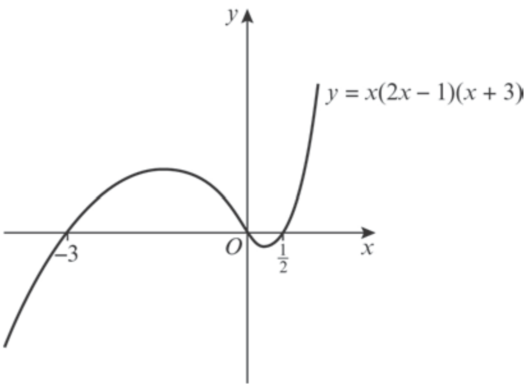
h $y = x(x + 1)(1 - x)$
$0 = x(x + 1)(1 - x)$
So $x = 0$, $x = -1$ or $x = 1$
So the curve crosses the x-axis at (0, 0), (−1, 0) and (1, 0).
$x \to \infty, y \to -\infty$
$x \to -\infty, y \to \infty$



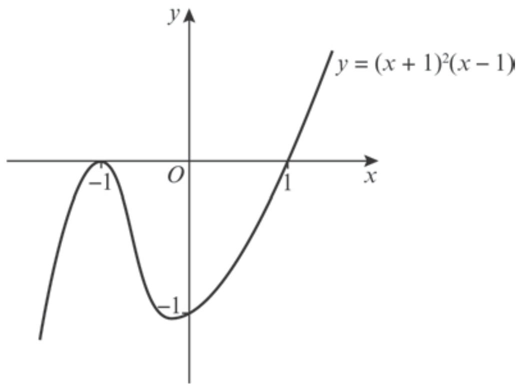
1 i $y = (x - 2)(2x - 1)(2x + 1)$
$0 = (x - 2)(2x - 1)(2x + 1)$
So $x = 2$, $x = \frac{1}{2}$ or $x = -\frac{1}{2}$
So the curve crosses the x-axis at $(2, 0)$, $\left(\frac{1}{2}, 0\right)$ and $\left(-\frac{1}{2}, 0\right)$.
When $x = 0$, $y = (-2) \times (-1) \times 1 = 2$
So the curve crosses the y-axis at $(0, 2)$.
$x \to \infty, y \to \infty$
$x \to -\infty, y \to -\infty$



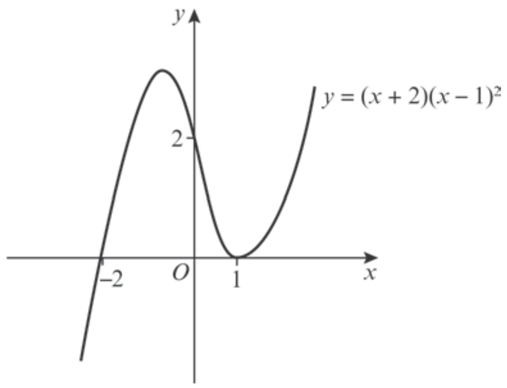
j $y = x(2x - 1)(x + 3)$
$0 = x(2x - 1)(x + 3)$
So $x = 0$, $x = \frac{1}{2}$ or $x = -3$
So the curve crosses the x-axis at $(0, 0)$, $\left(\frac{1}{2}, 0\right)$ and $(-3, 0)$.
$x \to \infty, y \to \infty$
$x \to -\infty, y \to -\infty$



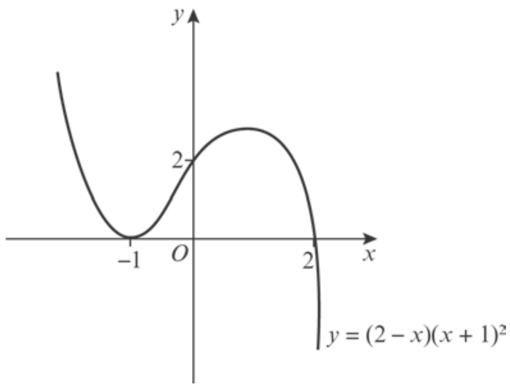
2 a $y = (x + 1)^2(x - 1)$
$0 = (x + 1)^2(x - 1)$
So $x = -1$ or $x = 1$
So the curve crosses the x-axis at $(1, 0)$ and touches the x-axis at $(-1, 0)$.
When $x = 0$, $y = 1^2 \times (-1) = -1$
So the curve crosses the y-axis at $(0, -1)$.
$x \to \infty, y \to \infty$
$x \to -\infty, y \to -\infty$



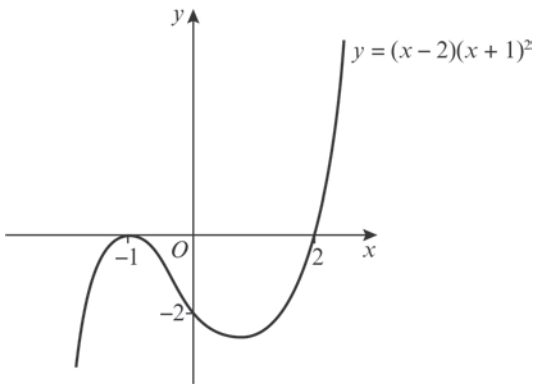
b $y = (x + 2)(x - 1)^2$
$0 = (x + 2)(x - 1)^2$
So $x = -2$ or $x = 1$
So the curve crosses the x-axis at $(-2, 0)$ and touches the x-axis at $(1, 0)$.
When $x = 0$, $y = 2 \times (-1)^2 = 2$
So the curve crosses the y-axis at $(0, 2)$.
$x \to \infty, y \to \infty$
$x \to -\infty, y \to -\infty$



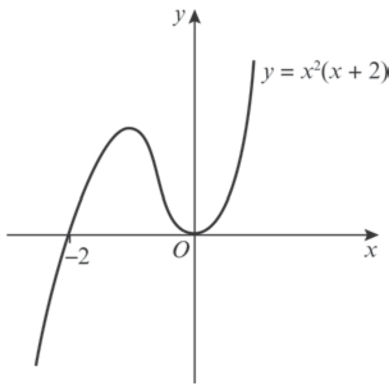
2 c $y = (2 - x)(x + 1)^2$
$0 = (2 - x)(x + 1)^2$
So $x = 2$ or $x = -1$
So the curve crosses the x-axis at $(2, 0)$ and touches the x-axis at $(-1, 0)$.
When $x = 0$, $y = 2 \times 1^2 = 2$
So the curve crosses the y-axis at $(0, 2)$.
$x \to \infty, y \to -\infty$
$x \to -\infty, y \to \infty$



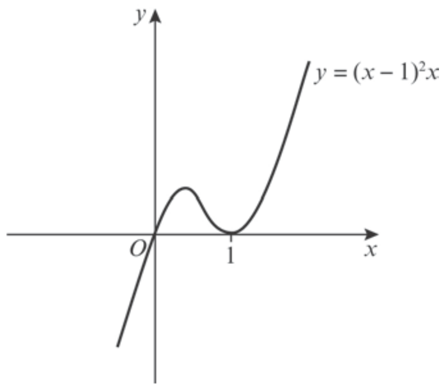
d $y = (x - 2)(x + 1)^2$
$0 = (x - 2)(x + 1)^2$
So $x = 2$ or $x = -1$
So the curve crosses the x-axis at $(2, 0)$ and touches the x-axis at $(-1, 0)$.
When $x = 0$, $y = (-2) \times 1^2 = -2$
So the curve crosses the y-axis at $(0, -2)$.
$x \to \infty, y \to \infty$
$x \to -\infty, y \to -\infty$



e $y = x^2(x + 2)$
$0 = x^2(x + 2)$
So $x = 0$ or $x = -2$
So the curve crosses the x-axis at $(-2, 0)$ and touches the x-axis at $(0, 0)$.
$x \to \infty, y \to \infty$
$x \to -\infty, y \to -\infty$



f $y = (x - 1)^2 x$
$0 = (x - 1)^2 x$
So $x = 1$ or $x = 0$
So the curve crosses the x-axis at $(0, 0)$ and touches the x-axis at $(1, 0)$.
$x \to \infty, y \to \infty$
$x \to -\infty, y \to -\infty$



2 g $y = (1 - x)^2(3 + x)$

$0 = (1 - x)^2(3 + x)$

So $x = 1$ or $x = -3$

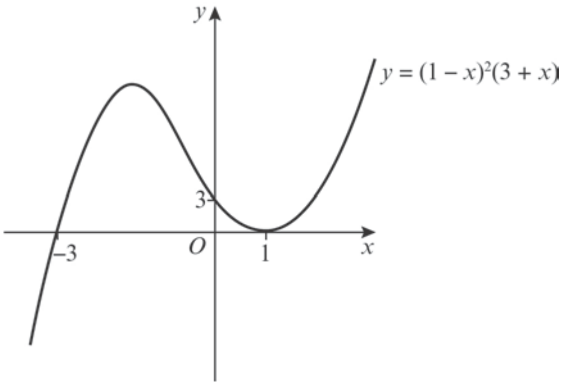
So the curve crosses the x-axis at $(-3, 0)$ and touches the x-axis at $(1, 0)$.

When $x = 0$, $y = 1^2 \times 3 = 3$

So the curve crosses the y-axis at $(0, 3)$.

$x \to \infty, y \to \infty$

$x \to -\infty, y \to -\infty$



h $y = (x - 1)^2(3 - x)$

$0 = (x - 1)^2(3 - x)$

So $x = 1$ or $x = 3$

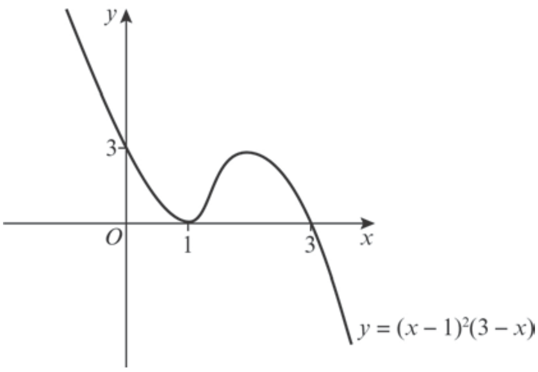
So the curve crosses the x-axis at $(3, 0)$ and touches the x-axis at $(1, 0)$.

When $x = 0$, $y = (-1)^2 \times 3 = 3$

So the curve crosses the y-axis at $(0, 3)$.

$x \to \infty, y \to -\infty$

$x \to -\infty, y \to \infty$



i $y = x^2(2 - x)$

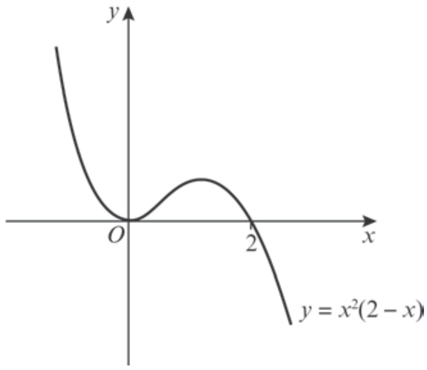
$0 = x^2(2 - x)$

So $x = 0$ or $x = 2$

So the curve crosses the x-axis at $(2, 0)$ and touches the x-axis at $(0, 0)$.

$x \to \infty, y \to -\infty$

$x \to -\infty, y \to \infty$



j $y = x^2(x - 2)$

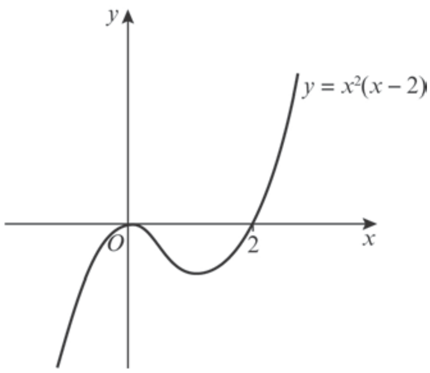
$0 = x^2(x - 2)$

So $x = 0$ or $x = 2$

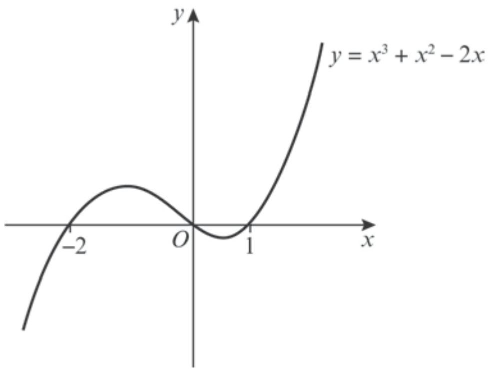
So the curve crosses the x-axis at $(2, 0)$ and touches the x-axis at $(0, 0)$.

$x \to \infty, y \to \infty$

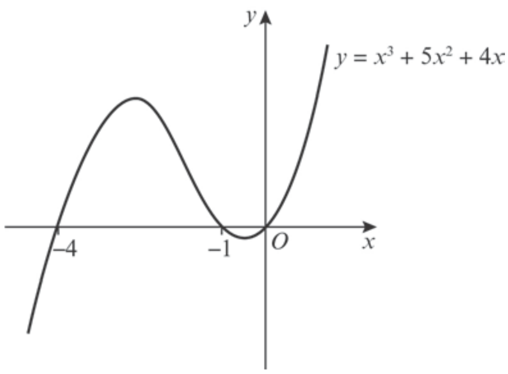
$x \to -\infty, y \to -\infty$



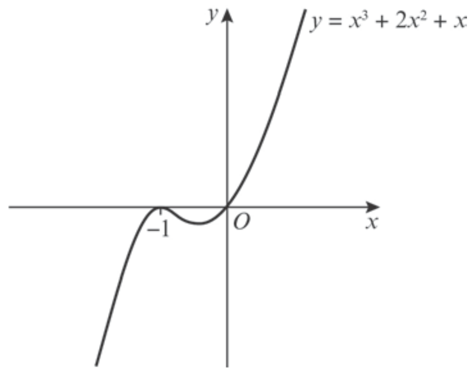
3 a $y = x^3 + x^2 - 2x$
$= x(x^2 + x - 2)$
$= x(x + 2)(x - 1)$
$0 = x(x + 2)(x - 1)$
So $x = 0$, $x = -2$ or $x = 1$
So the curve crosses the x-axis at (0, 0), (−2, 0) and (1, 0).
$x \to \infty, y \to \infty$
$x \to -\infty, y \to -\infty$



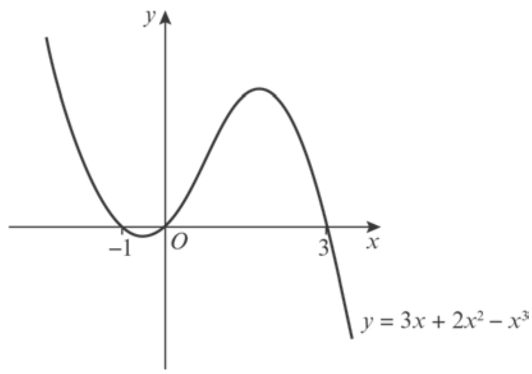
b $y = x^3 + 5x^2 + 4x$
$= x(x^2 + 5x + 4)$
$= x(x + 4)(x + 1)$
$0 = x(x + 4)(x + 1)$
So $x = 0$, $x = -4$ or $x = -1$
So the curve crosses the x-axis at (0, 0), (−4, 0) and (−1, 0).
$x \to \infty, y \to \infty$
$x \to -\infty, y \to -\infty$



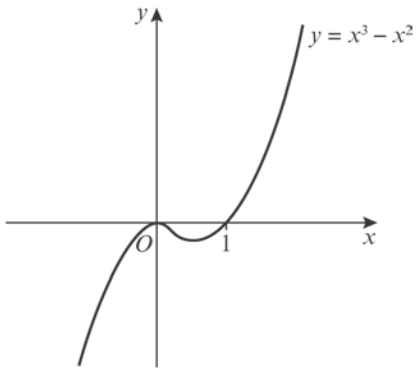
c $y = x^3 + 2x^2 + x$
$= x(x^2 + 2x + 1)$
$= x(x + 1)^2$
$0 = x(x + 1)^2$
So $x = 0$ or $x = -1$
So the curve crosses the x-axis at (0, 0) and touches the x-axis at (−1, 0).
$x \to \infty, y \to \infty$
$x \to -\infty, y \to -\infty$



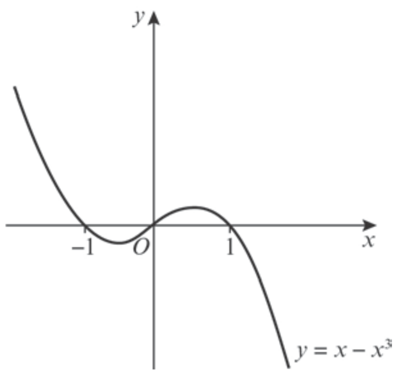
d $y = 3x + 2x^2 - x^3$
$= x(3 + 2x - x^2)$
$= x(3 - x)(1 + x)$
$0 = x(3 - x)(1 + x)$
So $x = 0$, $x = 3$ or $x = -1$
So the curve crosses the x-axis at (0, 0), (3, 0) and (−1, 0).
$x \to \infty, y \to -\infty$
$x \to -\infty, y \to \infty$



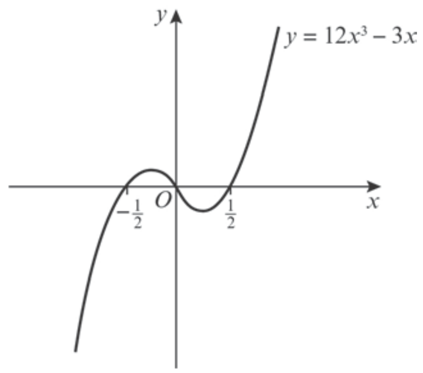
3 e $y = x^3 - x^2$
$= x^2(x - 1)$
$0 = x^2(x - 1)$
So $x = 0$ or $x = 1$
So the curve crosses the x-axis at (1, 0)
and touches the x-axis at 0, 0).
$x \to \infty, y \to \infty$
$x \to -\infty, y \to -\infty$



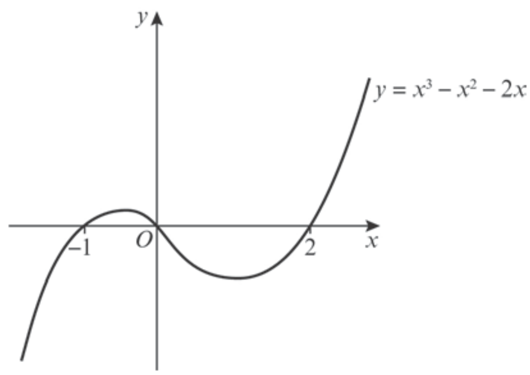
f $y = x - x^3$
$= x(1 - x^2)$
$= x(1 - x)(1 + x)$
$0 = x(1 - x)(1 + x)$
So $x = 0$, $x = 1$ or $x = -1$
So the curve crosses the x-axis at
(0, 0), (1, 0) and (−1, 0).
$x \to \infty, y \to -\infty$
$x \to -\infty, y \to \infty$



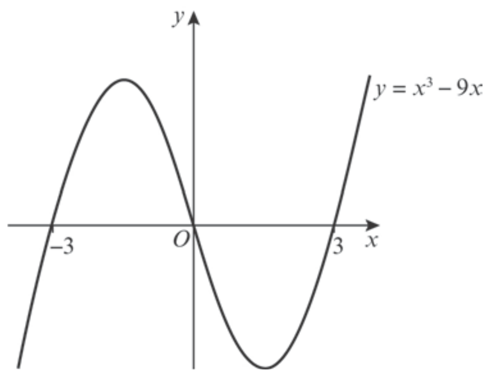
g $y = 12x^3 - 3x$
$= 3x(4x^2 - 1)$
$= 3x(2x - 1)(2x + 1)$
$0 = 3x(2x - 1)(2x + 1)$
So $x = 0$, $x = \frac{1}{2}$ or $x = -\frac{1}{2}$
So the curve crosses the x-axis at
(0, 0), $\left(\frac{1}{2}, 0\right)$ and $\left(-\frac{1}{2}, 0\right)$.
$x \to \infty, y \to \infty$
$x \to -\infty, y \to -\infty$



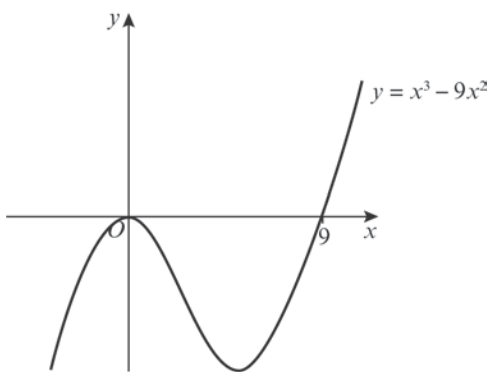
h $y = x^3 - x^2 - 2x$
$= x(x^2 - x - 2)$
$= x(x + 1)(x - 2)$
$0 = x(x + 1)(x - 2)$
So $x = 0$, $x = -1$ or $x = 2$
So the curve crosses the x-axis at
(0, 0), (−1, 0) and (2, 0).
$x \to \infty, y \to \infty$
$x \to -\infty, y \to -\infty$



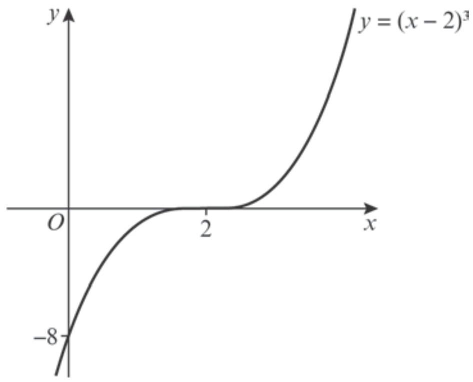
3 i $y = x^3 - 9x$
$= x(x^2 - 9)$
$= x(x - 3)(x + 3)$
$0 = x(x - 3)(x + 3)$
So $x = 0$, $x = 3$ or $x = -3$
So the curve crosses the x-axis at $(0, 0)$, $(3, 0)$ and $(-3, 0)$.
$x \to \infty, y \to \infty$
$x \to -\infty, y \to -\infty$



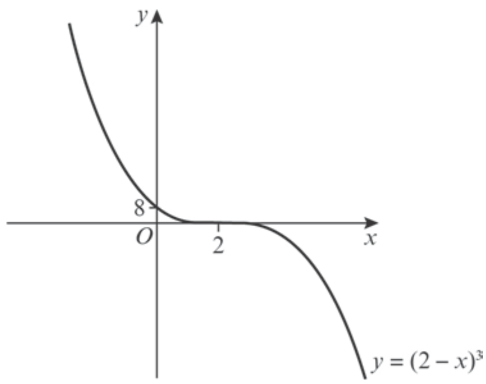
j $y = x^3 - 9x^2$
$= x^2(x - 9)$
$0 = x^2(x - 9)$
So $x = 0$ or $x = 9$
So the curve crosses the x-axis at $(0, 0)$ and touches the x-axis at 0, 0).
$x \to \infty, y \to \infty$
$x \to -\infty, y \to -\infty$



4 a $y = (x - 2)^3$
$0 = (x - 2)^3$
So $x = 2$ and the curve crosses the x-axis at $(2, 0)$ only.
When $x = 0$, $y = (-2)^3 = -8$
So the curve crosses the y-axis at $(0, -8)$.
$x \to \infty, y \to \infty$
$x \to -\infty, y \to -\infty$



b $y = (2 - x)^3$
$0 = (2 - x)^3$
So $x = 2$ and the curve crosses the x-axis at $(2, 0)$ only.
When $x = 0$, $y = 2^3 = 8$
So the curve crosses y-axis at $(0, 8)$.
$x \to \infty, y \to -\infty$
$x \to -\infty, y \to \infty$



4 c $y = (x - 1)^3$

$0 = (x - 1)^3$

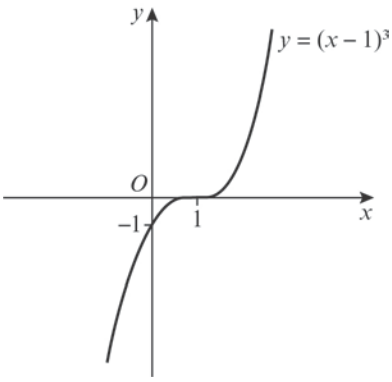
So $x = 1$ and the curve crosses the x-axis at $(1, 0)$ only.

When $x = 0$, $y = (-1)^3 = -1$

So the curve crosses y-axis at $(0, -1)$.

$x \to \infty, y \to \infty$

$x \to -\infty, y \to -\infty$



d $y = (x + 2)^3$

$0 = (x + 2)^3$

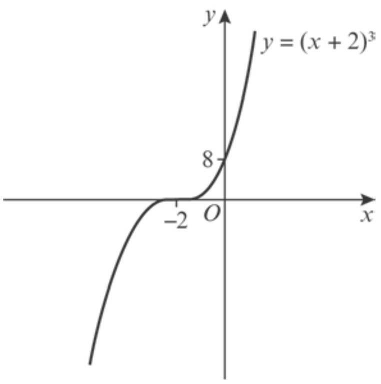
So $x = -2$ and the curve crosses the x-axis at $(-2, 0)$ only.

When $x = 0$, $y = 2^3 = 8$

So the curve crosses y-axis at $(0, 8)$.

$x \to \infty, y \to \infty$

$x \to -\infty, y \to -\infty$



e $y = -(x + 2)^3$

$0 = -(x + 2)^3$

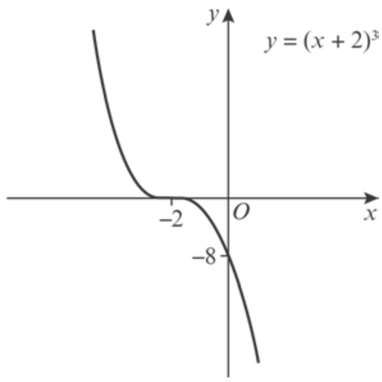
So $x = -2$ and the curve crosses the x-axis at $(-2, 0)$ only.

When $x = 0$, $y = -2^3 = 8$

So the curve crosses the y-axis at $(0, -8)$.

$x \to \infty, y \to -\infty$

$x \to -\infty, y \to \infty$



f $y = (x + 3)^3$

$0 = (x + 3)^3$

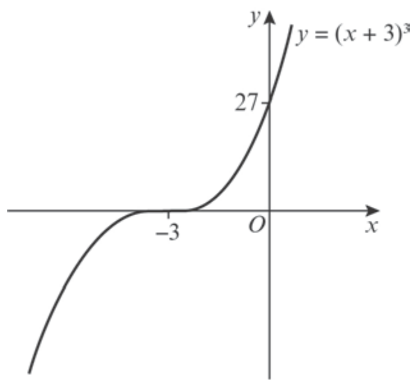
So $x = -3$ and the curve crosses the x-axis at $(-3, 0)$ only.

When $x = 0$, $y = 3^3 = 27$

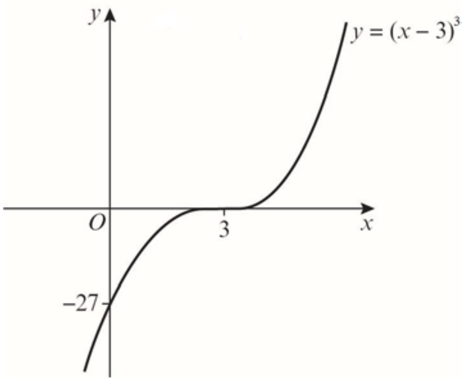
So the curve crosses the y-axis at $(0, 27)$.

$x \to \infty, y \to \infty$

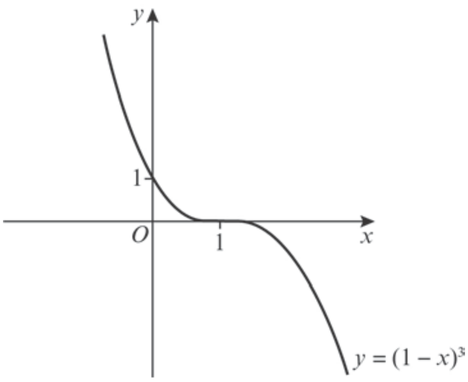
$x \to -\infty, y \to -\infty$



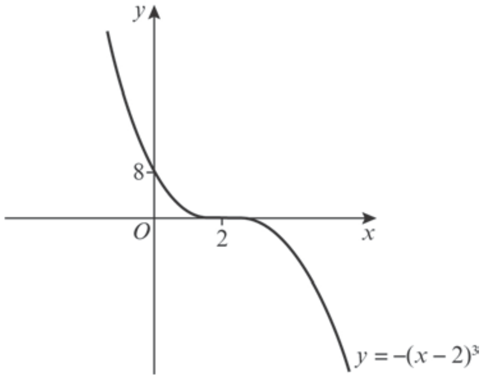
4 g $y = (x-3)^3$
$0 = (x-3)^3$
So $x = 3$ and the curve crosses the x-axis at $(3, 0)$ only.
When $x = 0$, $y = (-3)^3 = -27$
So the curve crosses the y-axis at $(0, -27)$.
$x \to \infty, y \to -\infty$
$x \to -\infty, y \to \infty$



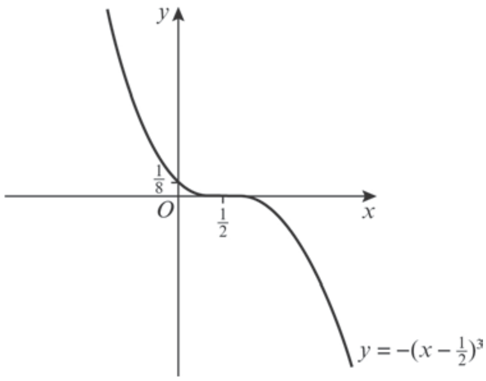
h $y = (1-x)^3$
$0 = (1-x)^3$
So $x = 1$ and the curve crosses the x-axis at $(1, 0)$ only.
When $x = 0$, $y = 1^3 = 1$
So the curve crosses the y-axis at $(0, 1)$.
$x \to \infty, y \to -\infty$
$x \to -\infty, y \to \infty$



i $y = -(x-2)^3$
$0 = -(x-2)^3$
So $x = 2$ and the curve crosses the x-axis at $(2, 0)$ only.
When $x = 0$, $y = -(-2)^3 = 8$
So the curve crosses the y-axis at $(0, 8)$.
$x \to \infty, y \to -\infty$
$x \to -\infty, y \to \infty$



j $y = -\left(x-\frac{1}{2}\right)^3$
$0 = -\left(x-\frac{1}{2}\right)^3$
So $x = \frac{1}{2}$ and the curve crosses the x-axis at $(\frac{1}{2}, 0)$ only.
When $x = 0$, $y = -\left(-\frac{1}{2}\right)^3 = \frac{1}{8}$
So the curve crosses the y-axis at $\left(0, \frac{1}{8}\right)$.
$x \to \infty, y \to -\infty$
$x \to -\infty, y \to \infty$



5 a $y = x^3 + bx^2 + cx + d$
$y = (x + 3)(x + 2)(x - 1)$
$= (x + 3)(x^2 + x - 2)$
$= x^3 + 4x^2 + x - 6$
$b = 4$, $c = 1$, $d = -6$

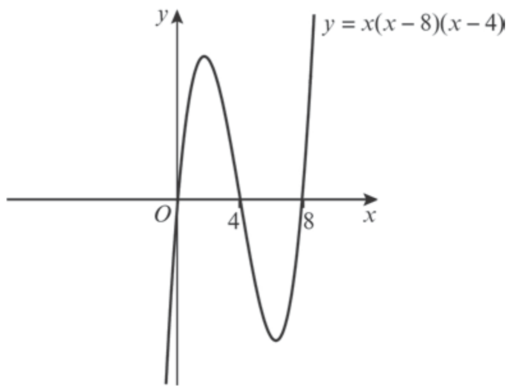
b When $x = 0$, $y = -6$
So the curve crosses the y-axis at $(0, -6)$.

6 $y = ax^3 + bx^2 + cx + d$
$y = a(x + 1)(x - 2)(x - 3)$
$= a(x + 1)(x^2 - 5x + 6)$
$= a(x^3 - 4x^2 + x + 6)$
The curve crosses the y-axis at $(0, 2)$, so when $x = 0$, $y = 2$.
$2 = a(0 - 0 + 0 + 6)$
$a = \frac{1}{3}$
$y = \frac{1}{3}(x^3 - 4x^2 + x + 6)$
$= \frac{1}{3}x^3 - \frac{4}{3}x^2 + \frac{1}{3}x + 2$
$a = \frac{1}{3}$, $b = -\frac{4}{3}$, $c = \frac{1}{3}$, $d = 2$

7 a $f(x) = (x - 10)(x^2 - 2x) + 12x$
$= x^3 - 12x^2 + 20x + 12x$
$= x^3 - 12x^2 + 32x$
$= x(x^2 - 12x + 32)$

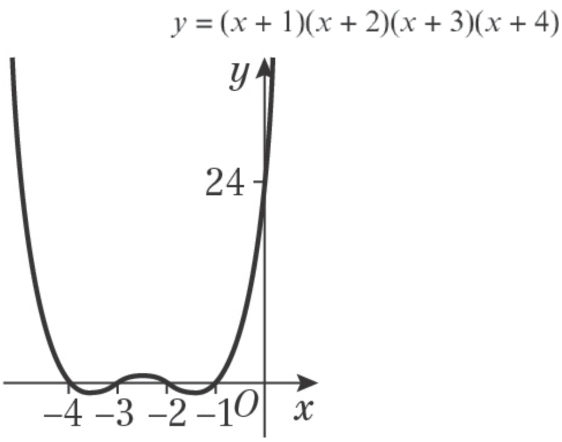
b $f(x) = x(x^2 - 12x + 32)$
$= x(x - 4)(x - 8)$

c $0 = x(x - 4)(x - 8)$
So $x = 0$, $x = 4$ or $x = 8$
So the curve crosses the x-axis at $(0, 0)$, $(4, 0)$ and $(8, 0)$
$x \to \infty, y \to \infty$
$x \to -\infty, y \to -\infty$

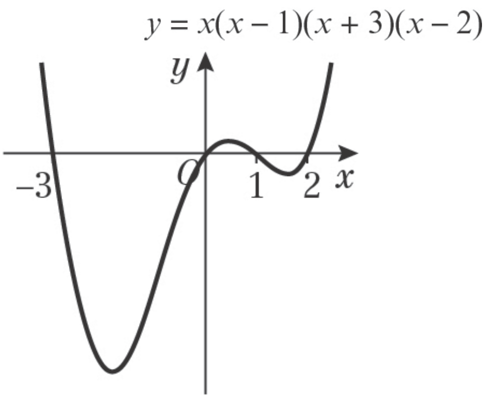


Graphs and transformations 4B

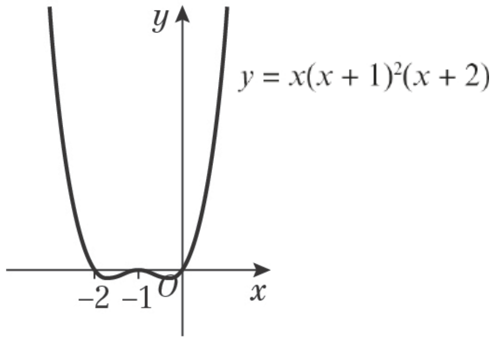
1 a $y = (x + 1)(x + 2)(x + 3)(x + 4)$
$0 = (x + 1)(x + 2)(x + 3)(x + 4)$
So $x = -1$, $x = -2$, $x = -3$ or $x = -4$
The curve crosses the x-axis at $(-1, 0)$, $(-2, 0)$, $(-3, 0)$ and $(-4, 0)$.
When $x = 0$, $y = 1 \times 2 \times 3 \times 4 = 24$
So the curve crosses the y-axis at $(0, 24)$.
$x \to \infty, y \to \infty$
$x \to -\infty, y \to \infty$



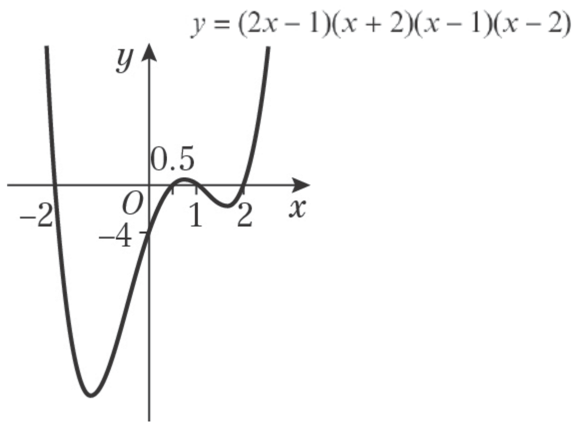
b $y = x(x - 1)(x + 3)(x - 2)$
$0 = x(x - 1)(x + 3)(x - 2)$
So $x = 0$, $x = 1$, $x = -3$ or $x = 2$
The curve crosses the x-axis at $(0, 0)$, $(1, 0)$, $(-3, 0)$ and $(2, 0)$.
$x \to \infty, y \to \infty$
$x \to -\infty, y \to \infty$



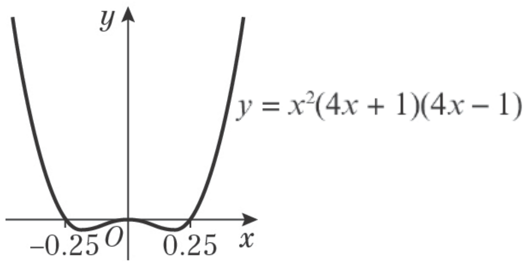
c $y = x(x + 1)^2(x + 2)$
$0 = x(x + 1)^2(x + 2)$
So $x = 0$, $x = -1$ or $x = -2$
The curve crosses the x-axis at $(0, 0)$ and $(-2, 0)$ and touches it at $(-1, 0)$.
$x \to \infty, y \to \infty$
$x \to -\infty, y \to \infty$



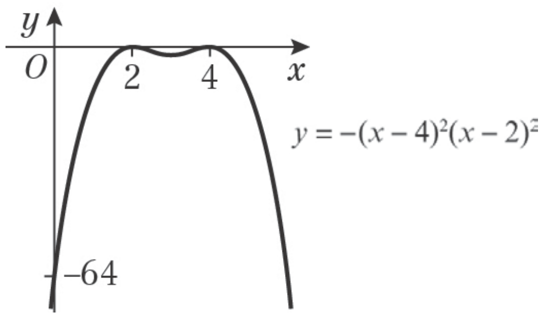
d $y = (2x - 1)(x + 2)(x - 1)(x - 2)$
$0 = (2x - 1)(x + 2)(x - 1)(x - 2)$
So $x = \frac{1}{2}$, $x = -2$, $x = 1$ or $x = 2$
The curve crosses the x-axis at $\left(\frac{1}{2}, 0\right)$, $(-2, 0)$, $(1, 0)$ and $(2, 0)$.
When $x = 0$, $y = (-1) \times 2 \times (-1) \times (-2) = -4$
So the curve crosses the y-axis at $(0, -4)$.
$x \to \infty, y \to \infty$
$x \to -\infty, y \to \infty$



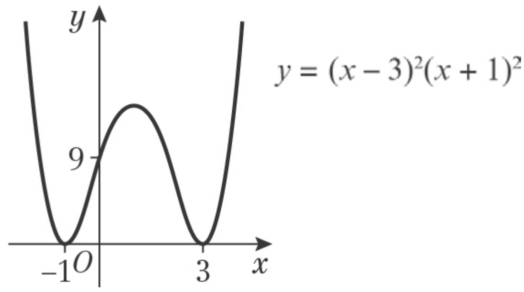
1 e $y = x^2(4x + 1)(4x - 1)$
$0 = x^2(4x + 1)(4x - 1)$
So $x = 0$, $x = -\frac{1}{4}$ or $x = \frac{1}{4}$
The curve crosses the x-axis at $\left(-\frac{1}{4}, 0\right)$ and $\left(\frac{1}{4}, 0\right)$ and touches it at (0, 0).
$x \to \infty, y \to \infty$
$x \to -\infty, y \to \infty$



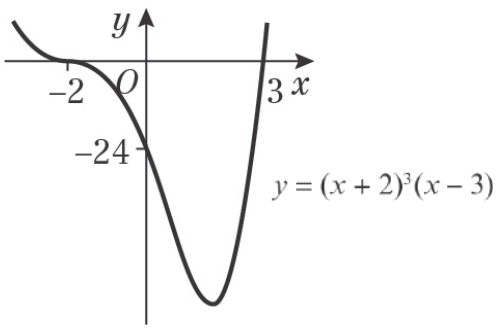
f $y = -(x - 4)^2(x - 2)^2$
$0 = -(x - 4)^2(x - 2)^2$
So $x = 4$ or $x = 2$
The curve touches the x-axis at (4, 0) and (2, 0).
When $x = 0$, $y = -(-4)^2 \times (-2)^2 = -64$
So the curve crosses the y-axis at (0, −64).
$x \to \infty, y \to -\infty$
$x \to -\infty, y \to -\infty$



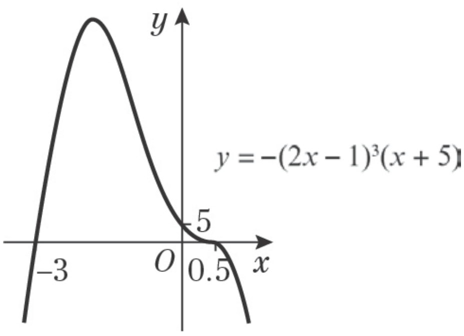
g $y = (x - 3)^2(x + 1)^2$
$0 = (x - 3)^2(x + 1)^2$
So $x = 3$ or $x = -1$
The curve touches the x-axis at (3, 0) and (−1, 0).
When $x = 0$, $y = (-3)^2 \times 1^2 = 9$
So the curve crosses the y-axis at (0, 9).
$x \to \infty, y \to \infty$
$x \to -\infty, y \to \infty$



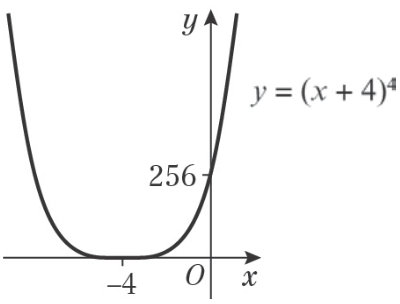
h $y = (x + 2)^3(x - 3)$
$0 = (x + 2)^3(x - 3)$
So $x = -2$ or $x = 3$
The curve crosses the x-axis at (−2, 0) and (3, 0).
When $x = 0$, $y = 2^3 \times (-3) = -24$
So the curve crosses the y-axis at (0, −24).
$x \to \infty, y \to \infty$
$x \to -\infty, y \to \infty$



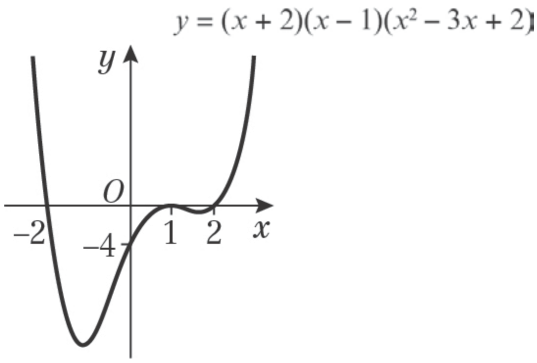
1 i $y = -(2x-1)^3(x+5)$
$0 = -(2x-1)^3(x+5)$
So $x = \frac{1}{2}$ or $x = -5$
The curve crosses the x-axis at $(\frac{1}{2}, 0)$ and $(-5, 0)$.
When $x = 0$, $y = -(-1)^3 \times 5 = 5$
So the curve crosses the y-axis at $(0, 5)$.
$x \to \infty, y \to -\infty$
$x \to -\infty, y \to -\infty$



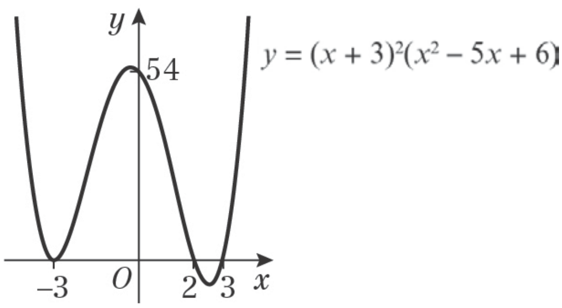
j $y = (x+4)^4$
$0 = (x+4)^4$
So $x = -4$
The curve touches the x-axis at $(-4, 0)$.
When $x = 0$, $y = 4^4 = 256$
So the curve crosses the y-axis at $(0, 256)$.
$x \to \infty, y \to \infty$
$x \to -\infty, y \to \infty$



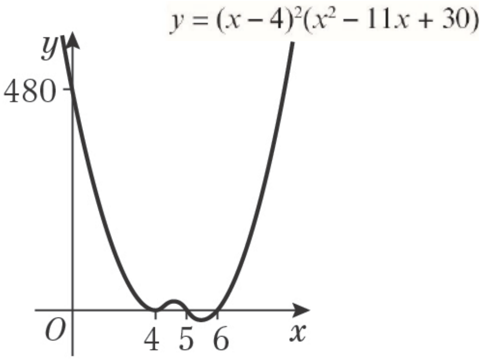
2 a $y = (x+2)(x-1)(x^2-3x+2)$
$= (x+2)(x-1)(x-1)(x-2)$
$= (x+2)(x-1)^2(x-2)$
$0 = (x+2)(x-1)^2(x-2)$
So $x = -2$, $x = 1$ or $x = 2$
The curve crosses the x-axis at $(-2, 0)$ and $(2, 0)$ and touches it at $(1, 0)$.
When $x = 0$, $y = 2 \times (-1)^2 \times (-2) = -4$
So the curve crosses the y-axis at $(0, -4)$.
$x \to \infty, y \to \infty$
$x \to -\infty, y \to \infty$



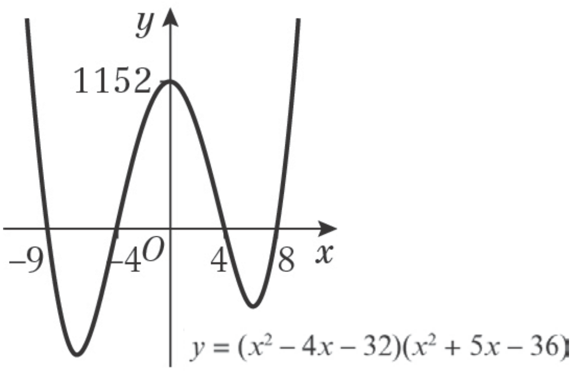
b $y = (x+3)^2(x^2-5x+6)$
$= (x+3)^2(x-2)(x-3)$
$0 = (x+3)^2(x-2)(x-3)$
So $x = -3$, $x = 2$ or $x = 3$
The curve crosses the x-axis at $(2, 0)$ and $(3, 0)$ and touches it at $(-3, 0)$.
When $x = 0$, $y = 3^2 \times (-2) \times (-3) = 54$
So the curve crosses the y-axis at $(0, 54)$.
$x \to \infty, y \to \infty$
$x \to -\infty, y \to \infty$



2 c $y = (x - 4)^2(x^2 - 11x + 30)$
$= (x - 4)^2(x - 5)(x - 6)$
$0 = (x - 4)^2(x - 5)(x - 6)$
So $x = 4$, $x = 5$ or $x = 6$
The curve crosses the x-axis at (5, 0) and (6, 0) and touches it at (4, 0).
When $x = 0$, $y = (-4)^2 \times (-5) \times (-6) = 480$
So the curve crosses the y-axis at (0, 480).
$x \to \infty, y \to \infty$
$x \to -\infty, y \to \infty$



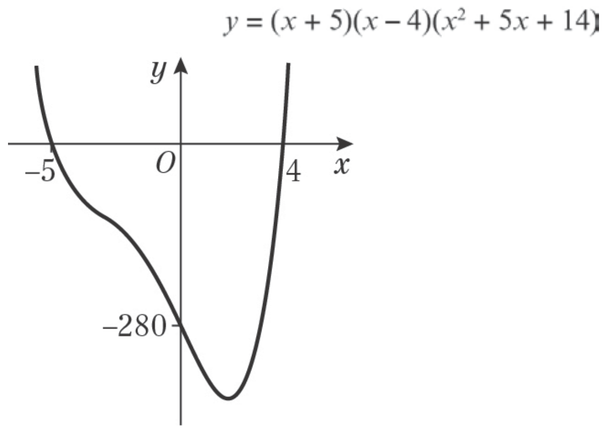
d $y = (x^2 - 4x - 32)(x^2 + 5x - 36)$
$= (x - 8)(x + 4)(x + 9)(x - 4)$
$0 = (x - 8)(x + 4)(x + 9)(x - 4)$
So $x = 8$, $x = -4$, $x = -9$ or $x = 4$
The curve crosses the x-axis at (8, 0), (−4, 0), (−9, 0) and (4, 0).
When $x = 0$, $y = (-8) \times 4 \times 9 \times (-4) = 1152$
So the curve crosses the y-axis at (0, 1152).
$x \to \infty, y \to \infty$
$x \to -\infty, y \to \infty$



3 a $y = x^4 + bx^3 + cx^2 + dx + e$
$y = (x + 2)(x + 1)(x - 2)(x - 3)$
When $x = 0$, $y = 2 \times 1 \times -2 \times -3 = 12$
So the curve crosses the y-axis at point P, which has coordinates (0, 12).

b $y = (x + 2)(x + 1)(x - 2)(x - 3)$
$= (x + 2)(x + 1)(x^2 - 5x + 6)$
$= (x + 2)(x^3 - 4x^2 + x + 6)$
$= x^4 - 2x^3 - 7x^2 + 8x + 12$
$b = -2$, $c = -7$, $d = 8$ and $e = 12$

4 $y = (x + 5)(x - 4)(x^2 + 5x + 14)$
The discriminant of the quadratic factor
$= b^2 - 4ac$
$= 5^2 - 4 \times 1 \times 14$
$= -31$, so there are no real roots.
$0 = (x + 5)(x - 4)(x^2 + 5x + 14)$
$x = -5$, $x = 4$ or $x^2 + 5x + 14 = 0$
The curve crosses the x-axis at (−5, 0) and (4, 0).
When $x = 0$, $y = 5 \times (-4) \times 14 = -280$
So the curve crosses the y-axis at (0, −280).
$x \to \infty, y \to \infty$
$x \to -\infty, y \to \infty$



Challenge

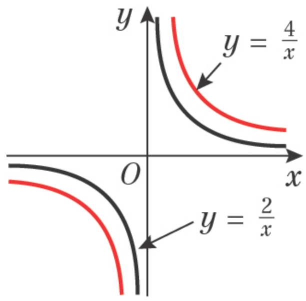
$y = ax^4 + bx^3 + cx^2 + dx + e$
$y = a(x + 1)^2(x - 3)^2$
When $x = 0$, $y = 3$:
$3 = a \times 1^2 \times (-3)^2$
$a = \frac{1}{3}$
$y = \frac{1}{3}(x + 1)^2(x - 3)^2$
$= \frac{1}{3}(x + 1)^2(x^2 - 6x + 9)$
$= \frac{1}{3}(x + 1)(x^3 - 5x^2 + 3x + 9)$
$= \frac{1}{3}(x^4 - 4x^3 - 2x^2 + 12x + 9)$
$= \frac{1}{3}x^4 - \frac{4}{3}x^3 - \frac{2}{3}x^2 + 4x + 3$
$a = \frac{1}{3}$, $b = -\frac{4}{3}$, $c = -\frac{2}{3}$, $d = 4$ and $e = 3$

Graphs and transformations 4C

1 a For $x > 0$, $\dfrac{4}{x} > \dfrac{2}{x}$ (since $4 > 2$)

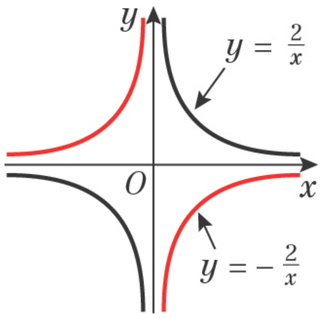
For $x < 0$, $\dfrac{4}{x} < \dfrac{2}{x}$

So $y = \dfrac{4}{x}$ is above $y = \dfrac{2}{x}$ in first quadrant and below in third quadrant.



b For $x > 0$, $y = \dfrac{2}{x} > 0$ and $y = -\dfrac{2}{x} < 0$

For $x < 0$, $y = \dfrac{2}{x} < 0$ and $y = -\dfrac{2}{x} > 0$



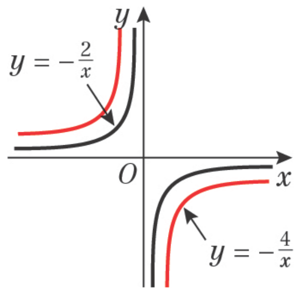
c Graphs are like $y = -\dfrac{1}{x}$ and so exist in second and fourth quadrants.

For $x > 0$, $-\dfrac{4}{x} < -\dfrac{2}{x}$

For $x < 0$, $-\dfrac{4}{x} > -\dfrac{2}{x}$

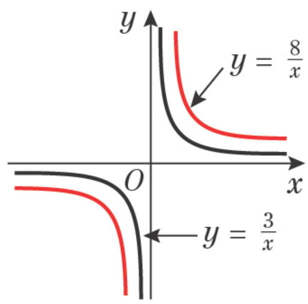
So $y = -\dfrac{4}{x}$ is above $y = -\dfrac{2}{x}$ in second quadrant and below in fourth quadrant.

c



d For $x > 0$, $\dfrac{8}{x} > \dfrac{3}{x}$

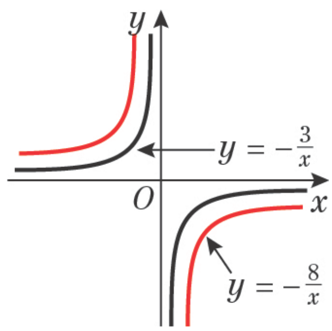
So $y = \dfrac{8}{x}$ is above $y = \dfrac{3}{x}$ in first quadrant and below in third quadrant.



e For $x > 0$, $-\dfrac{8}{x} < -\dfrac{3}{x}$

For $x < 0$, $-\dfrac{8}{x} > -\dfrac{3}{x}$

So $y = -\dfrac{8}{x}$ is above $y = -\dfrac{3}{x}$ in second quadrant and below in fourth quadrant.



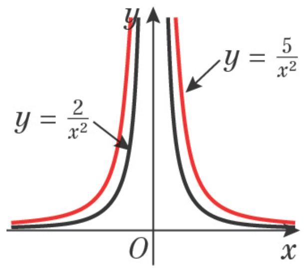
2 a $y = \dfrac{2}{x^2}$ and $y = \dfrac{5}{x^2}$

These are $y = \dfrac{k}{x^2}$ graphs, with $k > 0$.

x^2 is always positive and $k > 0$ so the y-values are all positive.

$\dfrac{5}{x^2} > \dfrac{2}{x^2}$ (since $5 > 2$)

So $y = \dfrac{5}{x^2}$ is above $y = \dfrac{2}{x^2}$.



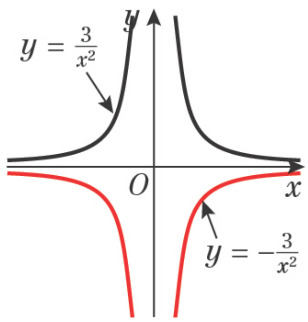
b $y = \dfrac{3}{x^2}$ and $y = -\dfrac{3}{x^2}$

$y = \dfrac{3}{x^2}$ is a $y = \dfrac{k}{x^2}$ graph, with $k > 0$.

x^2 is always positive and $k > 0$ so the y-values are all positive.

$y = -\dfrac{3}{x^2}$ is a $y = \dfrac{k}{x^2}$ graph, with $k < 0$.

x^2 is always positive and $k < 0$ so the y-values are all negative.



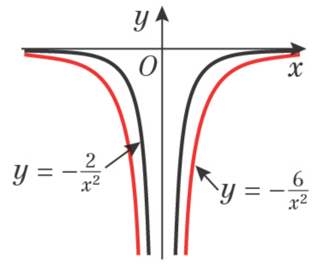
c $y = -\dfrac{2}{x^2}$ and $y = -\dfrac{6}{x^2}$

These are $y = \dfrac{k}{x^2}$ graphs, with $k < 0$.

x^2 is always positive and $k < 0$ so the y-values are all negative.

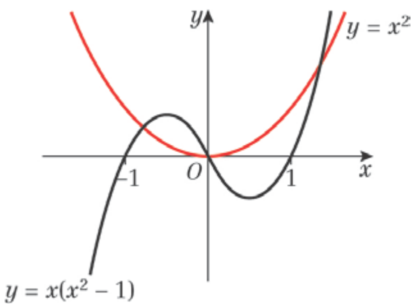
$-\dfrac{6}{x^2} < -\dfrac{2}{x^2}$ (since $-6 > -2$)

So $y = -\dfrac{6}{x^2}$ is below $y = -\dfrac{2}{x^2}$.



Graphs and transformations 4D

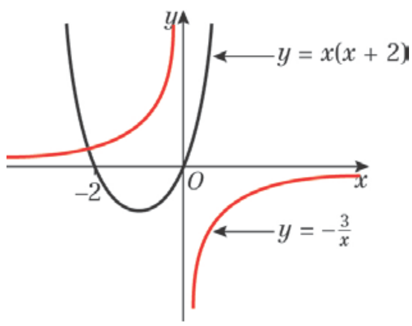
1 a i $y = x^2$ is standard.
$y = x(x^2 - 1)$
$= x(x - 1)(x + 1)$
$0 = x(x - 1)(x + 1)$
So $x = 0$, $x = 1$ or $x = -1$
So the curve crosses the x-axis at $(0, 0)$, $(1, 0)$ and $(-1, 0)$.
$x \to \infty, y \to \infty$
$x \to -\infty, y \to -\infty$



ii Three points of intersection

iii Equation: $x^2 = x(x^2 - 1)$

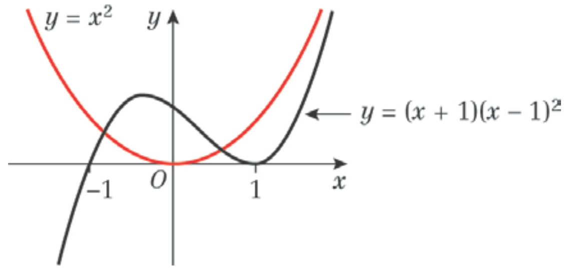
b i $y = x(x + 2)$
As $a = 1$ is positive, the graph has a $\bigvee$ shape and a minimum point.
$0 = x(x + 2)$
So $x = 0$ or $x = -2$
So the curve crosses the x-axis at $(0, 0)$ and $(-2, 0)$.
$y = -\dfrac{3}{x}$ is like $y = -\dfrac{1}{x}$ and so exists in the second and fourth quadrants.



ii One point of intersection

iii Equation: $x(x + 2) = -\dfrac{3}{x}$

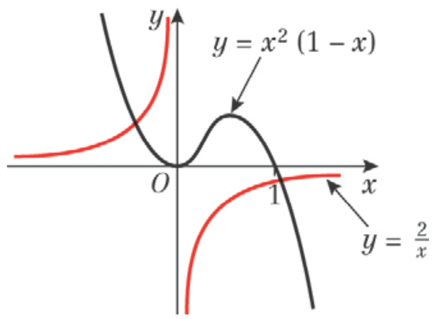
c i $y = x^2$ is standard.
$y = (x + 1)(x - 1)^2$
$0 = (x + 1)(x - 1)^2$
So $x = -1$ or $x = 1$
So the curve crosses the x-axis at $(-1, 0)$ and touches it at $(1, 0)$.
$x \to \infty, y \to +\infty$
$x \to -\infty, y \to -\infty$



ii Three points of intersection

iii Equation: $x^2 = (x + 1)(x - 1)^2$

d i $y = x^2(1 - x)$
$0 = x^2(1 - x)$
So $x = 0$ or $x = 1$
So the curve crosses the x-axis at $(1, 0)$ and touches it at $(0, 0)$.
$x \to \infty, y \to -\infty$
$x \to -\infty, y \to \infty$
$y = -\dfrac{2}{x}$ is like $y = -\dfrac{1}{x}$ and so exists in the second and fourth quadrants.



ii Two points of intersection

iii Equation: $x^2(1 - x) = -\dfrac{2}{x}$

1 e i $y = x(x - 4)$

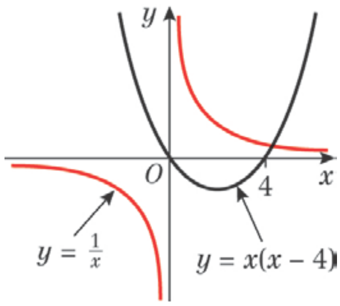
As $a = 1$ is positive, the graph has a $\smile$ shape and a minimum point.

$0 = x(x - 4)$

So $x = 0$ or $x = 4$

So the curve crosses the x-axis at (0, 0) and (4, 0).

$y = \dfrac{1}{x}$ is standard.



ii One point of intersection

iii Equation: $x(x - 4) = \dfrac{1}{x}$

f i $y = x(x - 4)$

As $a = 1$ is positive, the graph has a $\smile$ shape and a minimum point.

$0 = x(x - 4)$

So $x = 0$ or $x = 4$

So the curve crosses the x-axis at (0, 0) and (4, 0).

$y = -\dfrac{1}{x}$ is standard and in the second and fourth quadrants.

When $x = 2$,

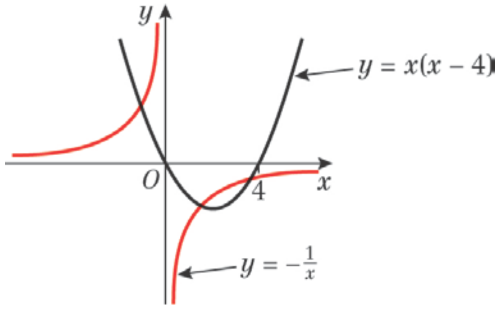
$y = -\dfrac{1}{x}$ gives $y = -\frac{1}{2}$

$y = x(x - 4)$ gives $y = 2(-2) = -4$

So when $x = 2$, $x(x - 4) < -\dfrac{1}{x}$

So $y = -\dfrac{1}{x}$ cuts $y = x(x - 4)$ in the fourth quadrant.

f i



ii Three points of intersection

iii Equation: $x(x - 4) = -\dfrac{1}{x}$

g i $y = x(x - 4)$

As $a = 1$ is positive, the graph has a $\smile$ shape and a minimum point.

$0 = x(x - 4)$

So $x = 0$ or $x = 4$

So the curve crosses the x-axis at (0, 0) and (4, 0).

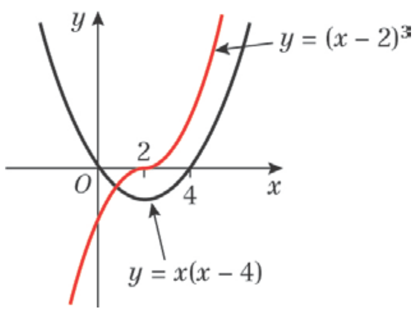
$y = (x - 2)^3$

$0 = (x - 2)^3$

So $x = 2$ and the curve crosses the x-axis at (2, 0) only.

$x \to \infty, y \to \infty$

$x \to -\infty, y \to -\infty$



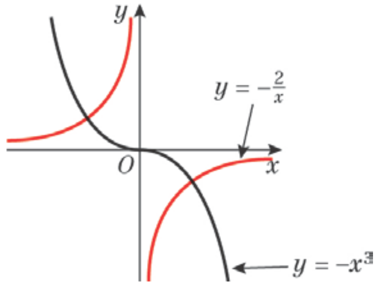
ii One point of intersection

iii $x(x - 4) = (x - 2)^3$

h i $y = -x^3$ is standard.

$y = -\dfrac{2}{x}$ is like $y = -\dfrac{1}{x}$ and in the second and fourth quadrants.

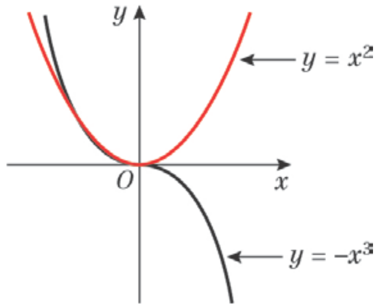
1 h i



ii Two points of intersection

iii $-x^3 = -\dfrac{2}{x}$ or $x^3 = \dfrac{2}{x}$

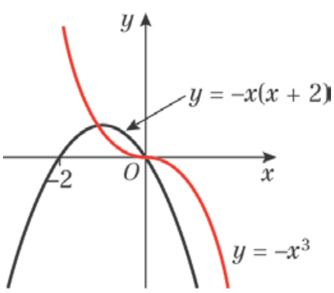
i i $y = -x^3$ is standard.
$y = x^2$ is standard.



ii Two points of intersection
(At (0, 0) the curves actually touch. They intersect in the second quadrant.)

iii $-x^3 = x^2$

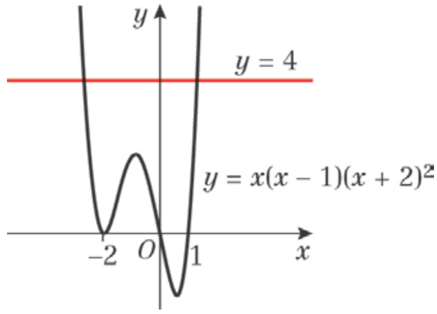
j i $y = -x^3$ is standard.
$y = -x(x + 2)$
As $a = -1$ is negative, the graph has a $\cap$ shape and a maximum point.
$0 = -x(x + 2)$
So $x = 0$ or $x = -2$
So the curve crosses the x-axis at (0, 0) and (−2, 0).



ii Three points of intersection

1 j iii $-x^3 = -x(x + 2)$ or $x^3 = x(x + 2)$

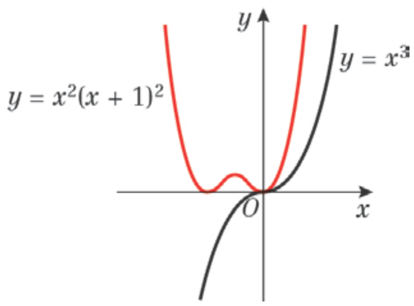
k i $y = 4$
$y = x(x - 1)(x + 2)^2$
$0 = x(x - 1)(x + 2)^2$
So $x = 0$, $x = 1$ or $x = -2$
The curve crosses the x-axis at (0, 0) and (1, 0) and touches it at (−2, 0).
$x \to \infty$, $y \to \infty$
$x \to -\infty$, $y \to -\infty$



ii Two points of intersection

iii $x(x - 1)(x + 2)^2 = 4$

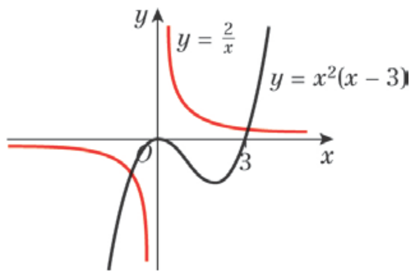
l i $y = x^3$ is standard.
$y = x^2(x + 1)^2$
$0 = x^2(x + 1)^2$
So $x = 0$ or $x = -1$
The curve touches the x-axis at (0, 0) and (−1, 0).
$x \to \infty$, $y \to \infty$
$x \to -\infty$, $y \to \infty$



ii One point of intersection

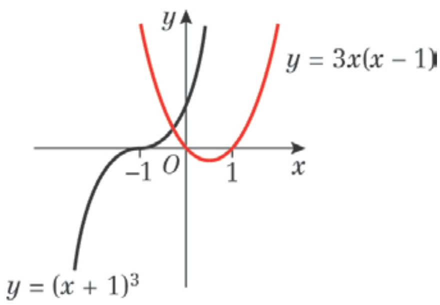
iii $x^3 = x^2(x + 1)^2$

2 a $y = x^2(x - 3)$
$0 = x^2(x - 3)$
So $x = 0$ or $x = 3$
The curve crosses the x-axis at (3, 0) and touches it at (0, 0).
$x \to \infty, y \to \infty$
$x \to -\infty, y \to -\infty$
$y = \frac{2}{x}$ is like $y = \frac{1}{x}$.



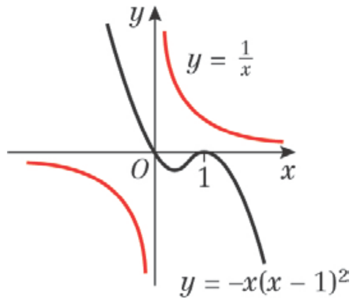
b From the sketch, there are only two points of intersection of the curves. This means there are only two values of x where
$x^2(x - 3) = \frac{2}{x}$
$x^3(x - 3) = 2$
So this equation has two real solutions.

3 a $y = (x + 1)^3$
$0 = (x + 1)^3$
So $x = -1$ and the curve crosses the x-axis at $(-1, 0)$ only.
$x \to \infty, y \to \infty$
$x \to -\infty, y \to -\infty$
$y = 3x(x - 1)$
As $a = 3$ is positive, the graph has a $\bigvee$ shape and a minimum point.
$0 = 3x(x - 1)$
So $x = 0$ or $x = 1$
So the curve crosses the x-axis at (0, 0) and (1, 0).



3 b From the sketch, there is only one point of intersection of the curves. This means there is only one value of x where
$$(x + 1)^3 = 3x(x - 1)$$
$$x^3 + 3x^2 + 3x + 1 = 3x^2 - 1$$
$$x^3 + 6x + 1 = 0$$
So this equation has one real solution.

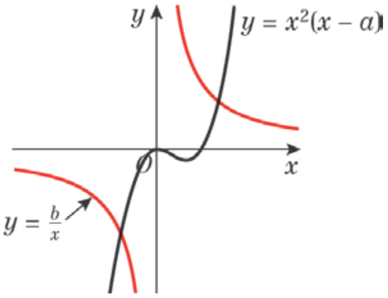
4 a $y = \frac{1}{x}$ is standard.
$y = -x(x - 1)^2$
$0 = -x(x - 1)^2$
So $x = 0$ or $x = 1$
The curve crosses the x-axis at (0, 0) and touches it at (1, 0).
$x \to \infty, y \to -\infty$
$x \to -\infty, y \to \infty$



b From the sketch, there are no points of intersection of the curves. This means there are no values of x where
$$\frac{1}{x} = -x(x - 1)^2$$
$$1 = -x^2(x - 1)^2$$
$$1 + x^2(x - 1)^2 = 0$$
So this equation has no real solutions.

5 a $y = x^2(x - a)$
$0 = x^2(x - a)$
So $x = 0$ or $x = a$
The curve crosses the x-axis at $(a, 0)$ and touches it at (0, 0).
$x \to \infty, y \to \infty$
$x \to -\infty, y \to -\infty$
$y = \frac{b}{x}$ is a $y = \frac{k}{x}$ graph, with $k > 0$.

5 a



b From the sketch, there are two points of intersection of the curves. This means there are two values of x where

$$x^2(x-a) = \frac{b}{x}$$
$$x^3(x-a) = b$$
$$x^4 - ax^3 - b = 0$$

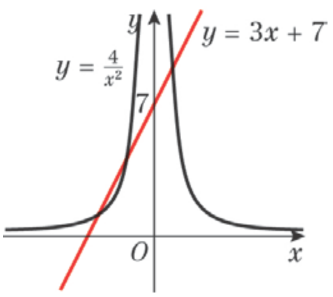
So this equation has two real solutions.

6 a $y = \dfrac{4}{x^2}$ is a $y = \dfrac{k}{x^2}$ graph, with $k > 0$.

x^2 is always positive and $k > 0$ so the y-values are all positive.

$y = 3x + 7$
$0 = 3x + 7$
So $x = -\dfrac{7}{3}$

$y = 3x + 7$ is a straight line crossing the x-axis at $\left(-\frac{7}{3}, 0\right)$.



b There are three points of intersection, so there are three real solutions to the equation

$$\frac{4}{x^2} = 3x + 7$$

c $(x + 1)(x + 2)(3x - 2) = 0$
$(x + 1)(3x^2 + 4x - 4) = 0$
$3x^3 + 7x^2 - 4 = 0$
$3x^3 + 7x^2 = 4$
$x^2(3x + 7) = 4$
$3x + 7 = \dfrac{4}{x^2}$

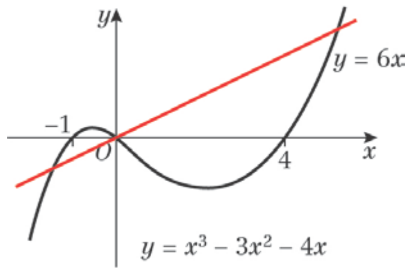
6 d $(x + 1)(x + 2)(3x - 2) = 0$

So $x = -1$, $x = -2$ or $x = \frac{2}{3}$

Using $y = 3x + 7$:
when $x = -1$, $y = 3(-1) + 7 = 4$
when $x = -2$, $y = 3(-2) + 7 = 1$
when $x = \frac{2}{3}$, $y = 3\left(\frac{2}{3}\right) + 7 = 9$

So the points of intersection are $(-1, 4)$, $(-2, 1)$ and $\left(\frac{2}{3}, 9\right)$.

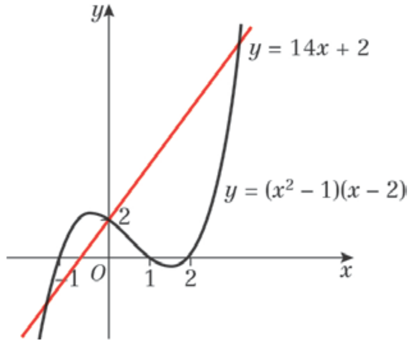
7 a $y = x^3 - 3x^2 - 4x$
$= x(x^2 - 3x - 4)$
$0 = x(x - 4)(x + 1)$
So $x = 0$, $x = 4$ or $x = -1$
The curve crosses the x-axis at $(0, 0)$, $(4, 0)$ and $(-1, 0)$.
$x \to \infty, y \to \infty$
$x \to -\infty, y \to -\infty$
$y = 6x$ is a straight line through $(0, 0)$.



b $x^3 - 3x^2 - 4x = 6x$
$x^3 - 3x^2 - 10x = 0$
$x(x^2 - 3x - 10) = 0$
$x(x - 5)(x + 2) = 0$
So $x = 0$, $x = 5$ or $x = -2$
Using $y = 6x$:
when $x = 0$, $y = 0$
when $x = 5$, $y = 30$
when $x = -2$, $y = -12$
So the points of intersection are $(0, 0)$, $(5, 30)$ and $(-2, -12)$.

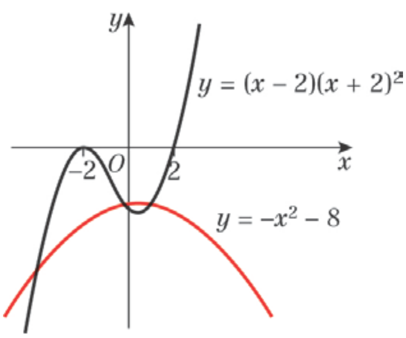
8 a $y = (x^2 - 1)(x - 2)$
$= (x - 1)(x + 1)(x - 2)$
$0 = (x - 1)(x + 1)(x - 2)$
So $x = 1$, $x = -1$or $x = 2$
The curve crosses the x-axis at $(1, 0)$, $(-1, 0)$ and $(2, 0)$.
When $x = 0$, $y = (-1)^2 \times (-2) = 2$
$x \to \infty, y \to \infty$
$x \to -\infty, y \to -\infty$
$y = 14x + 2$ is a straight line passing through $(0, 2)$ and $(-\frac{1}{7}, 0)$.

8 a



b $(x^2 - 1)(x - 2) = 14x + 2$
$x^3 - 2x^2 - x + 2) = 14x + 2$
$x^3 - 2x^2 - 15x = 0$
$x(x^2 - 2x - 15) = 0$
$x(x - 5)(x + 3) = 0$
$x = 0$, $x = 5$ or $x = -3$
Using $y = 14x + 2$:
when $x = 0$, $y = 2$
when $x = 5$, $y = 14(5) + 2 = 72$
when $x = -3$, $y = 14(-3) + 2 = -40$
So the points of intersection are (0, 2), (5, 72) and (−3, −40).

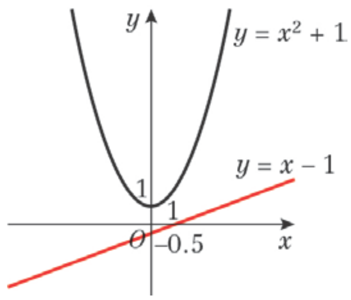
9 a $y = (x - 2)(x + 2)^2$
$0 = (x - 2)(x + 2)^2$
So $x = 2$ or $x = -2$
The curve crosses the x-axis at (2, 0) and touches it at (−2, 0).
$x \to \infty, y \to \infty$
$x \to -\infty, y \to -\infty$
$y = -x^2 - 8$
As $a = -1$ is negative, the graph has a $\frown$ shape and a maximum point at (0, −8).



b $(x + 2)^2(x - 2) = -x^2 - 8$
$(x^2 + 4x + 4)(x - 2) = -x^2 - 8$
$x^3 + 4x^2 + 4x - 2x^2 - 8x - 8 = -x^2 - 8$
$x^3 + 3x^2 - 4x = 0$
$x(x^2 + 3x - 4) = 0$
$x(x - 1)(x + 4) = 0$
So $x = 0$, $x = 1$ or $x = -4$

9 b Using $y = -x^2 - 8$:
when $x = 0$, $y = -0^2 - 8 = -8$
when $x = 1$, $y = -1^2 - 8 = -9$
when $x = -4$, $y = -(-4)^2 - 8 = -24$
So the points of intersection are (0, −8), (1, −9) and (−4, −24).

10 a $y = x^2 + 1$
As $a = 1$ is positive, the graph has a $\frown$ shape and a minimum point at (0, 1).
$2y = x - 1$
$y = \frac{1}{2}x - \frac{1}{2}$
This is a straight line passing through $\left(0, -\frac{1}{2}\right)$ and (1, 0).



b From the sketch, there are no points of intersection of the curves. This means there are no values of x where
$x^2 + 1 = \frac{1}{2}x - \frac{1}{2}$
$2x^2 + 2 = x - 1$
$2x^2 - x + 3 = 0$
So this equation has no real solutions.

c $x^2 + a = \frac{1}{2}x - \frac{1}{2}$
$2x^2 + 2a = x - 1$
$2x^2 - x + 2a + 1 = 0$
Using the discriminant for two real roots,
$b^2 - 4ac > 0$
$(-1)^2 - 4(2)(2a + 1) > 0$
$1 - 16a - 8 > 0$
$-16a - 7 > 0$
$16a < -7$
$a < -\frac{7}{16}$

11 a $y = x^2(x - 1)(x + 1)$
$0 = x^2(x - 1)(x + 1)$
So $x = 0$, $x = 1$ and $x = -1$
The curve crosses the x-axis at (1, 0) and (−1, 0) and touches it at (0, 0).
$x \to \infty, y \to \infty$
$x \to -\infty, y \to \infty$
$y = \frac{1}{3}x^3 + 1$

11 a $0 = \frac{1}{3}x^3 + 1$

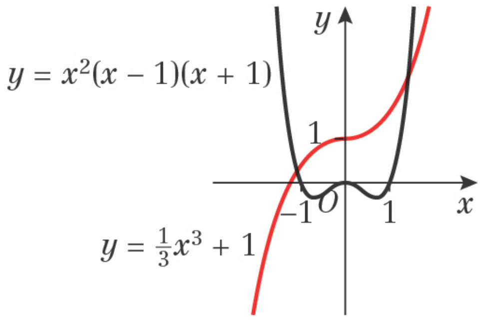
$\frac{1}{3}x^3 = -1$

$x^3 = -3$

$x = -\sqrt[3]{3}$

The curve crosses the x-axis at $\left(-\sqrt[3]{3}, 0\right)$.

When $x = 0$, $y = 1$



b From the sketch, there are two points of intersection of the curves. This means there are two values of x where

$x^2(x-1)(x+1) = \frac{1}{3}x^3 + 1$

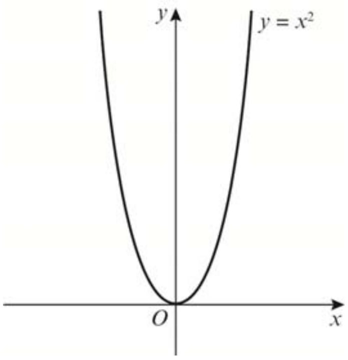
$3x^2(x-1)(x+1) = x^3 + 3$

So this equation has two real solutions.

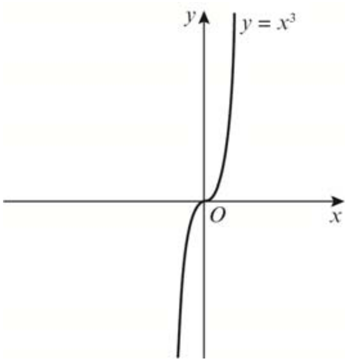
Graphs and transformations 4E

1 Sketches of original graphs:

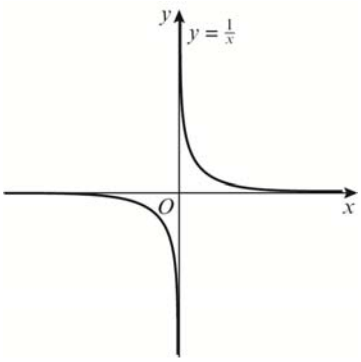
$f(x) = x^2$



$f(x) = x^3$

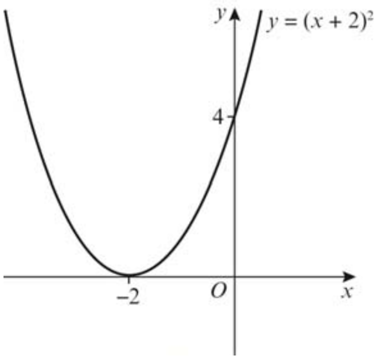


$f(x) = \frac{1}{x}$



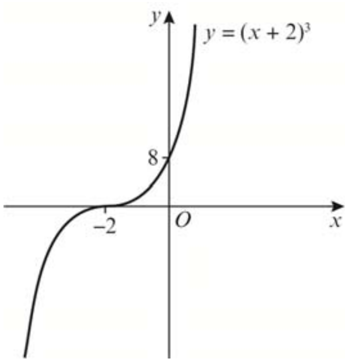
a $f(x + 2)$ is a translation of the graph of $f(x)$ by $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$, or two units to the left.

a i $f(x) = x^2$, $f(x + 2) = (x + 2)^2$



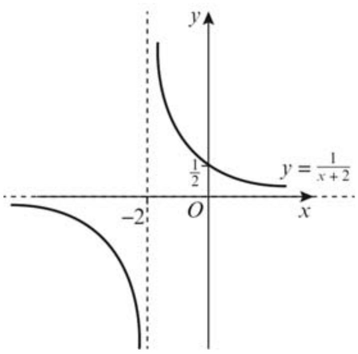
The curve touches the x-axis at $(-2, 0)$ and crosses the y-axis at $(0, 4)$.

ii $f(x) = x^3$, $f(x + 2) = (x + 2)^3$



The curve crosses the x-axis at $(-2, 0)$ and crosses the y-axis at $(0, 8)$.

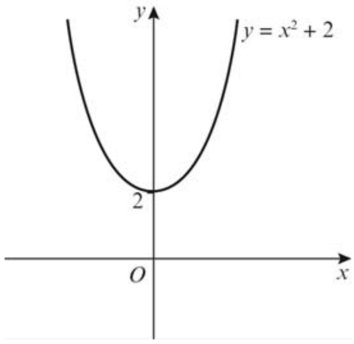
iii $f(x) = \frac{1}{x}$, $f(x + 2) = \frac{1}{x+2}$



The curve crosses the y-axis at $(0, \frac{1}{2})$.
The horizontal asymptote is $y = 0$.
The vertical asymptote is $x = -2$.

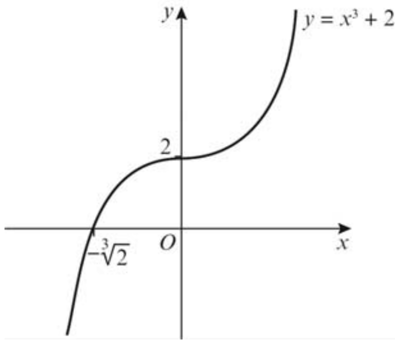
1 b f(x) + 2 is a translation of the graph of f(x) by $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$, or two units up.

i $f(x) = x^2$, $f(x) + 2 = x^2 + 2$



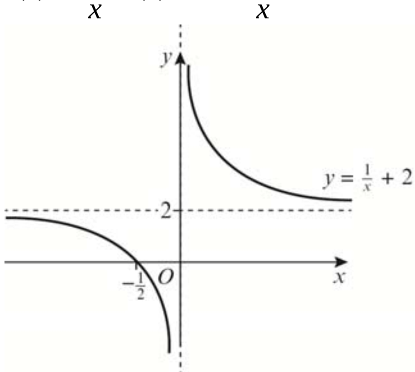
The curve crosses the y-axis at (0, 2).

ii $f(x) = x^3$, $f(x) + 2 = x^3 + 2$



The curve crosses the x-axis at $\left(-\sqrt[3]{2}, 0\right)$ and crosses the y-axis at (0, 2).

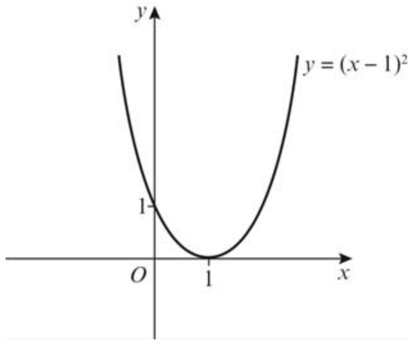
iii $f(x) = \dfrac{1}{x}$, $f(x) + 2 = \dfrac{1}{x} + 2$



The horizontal asymptote is $y = 2$.
The vertical asymptote is $x = 0$.

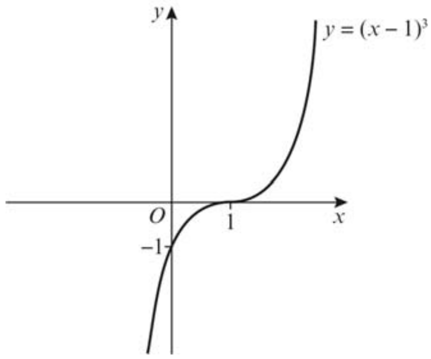
c f(x − 1) is a translation of the graph of f(x) by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, or one unit to the right.

i $f(x) = x^2$, $f(x - 1) = (x - 1)^2$



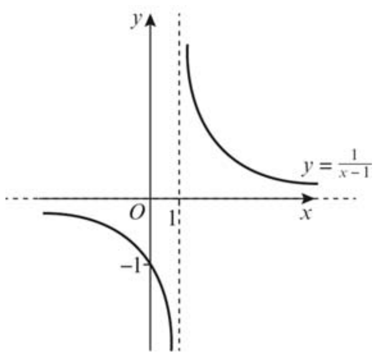
The curve touches the x-axis at (1, 0) and crosses the y-axis at (0, 1).

ii $f(x) = x^3$, $f(x - 1) = (x - 1)^3$



The curve crosses the x-axis at (1, 0) and crosses the y-axis at (0, −1).

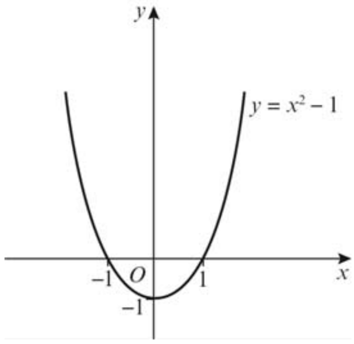
iii $f(x) = \dfrac{1}{x}$, $f(x - 1) = \dfrac{1}{x-1}$



The curve crosses the y-axis at (0, −1).
The horizontal asymptote is $y = 0$.
The vertical asymptote is $x = 1$.

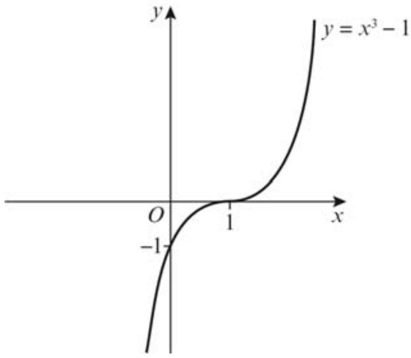
1 d $f(x) - 1$ is a translation of the graph of $f(x)$ by $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$, or one unit down.

i $f(x) = x^2$, $f(x) - 1 = x^2 - 1$



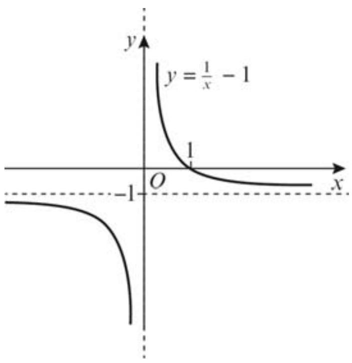
The curve crosses the x-axis at $(-1, 0)$ and $(1, 0)$ and crosses the y-axis at $(0, -1)$.

ii $f(x) = x^3$, $f(x) - 1 = x^3 - 1$



The curve crosses the x-axis at $(1, 0)$ and crosses the y-axis at $(0, -1)$.

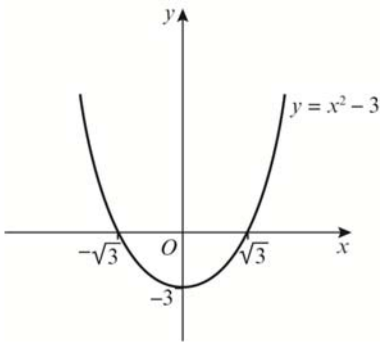
iii $f(x) = \dfrac{1}{x}$, $f(x) - 1 = \dfrac{1}{x} - 1$



1 d iii The curve crosses the x-axis at $(1, 0)$.
The horizontal asymptote is $y = -1$.
The vertical asymptote is $x = 0$.

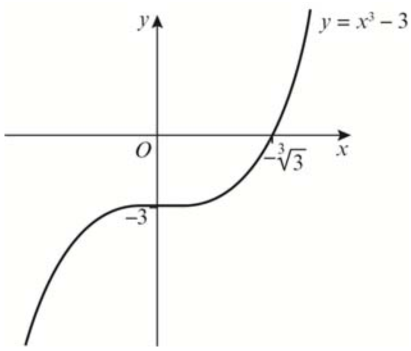
e $f(x) - 3$ is a translation of the graph of $f(x)$ by $\begin{pmatrix} 0 \\ -3 \end{pmatrix}$, or three units down.

i $f(x) = x^2$, $f(x) - 3 = x^2 - 3$



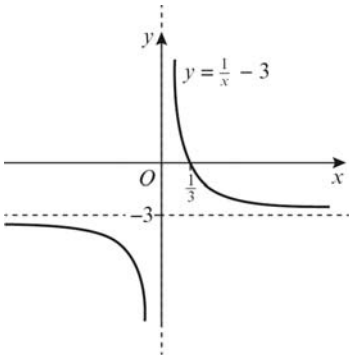
The curve crosses the x-axis at $\left(-\sqrt{3}, 0\right)$ and $\left(\sqrt{3}, 0\right)$ and crosses the y-axis at $(0, -3)$.

ii $f(x) = x^3$, $f(x) - 3 = x^3 - 3$



The curve crosses the x-axis at $\left(-\sqrt[3]{3}, 0\right)$ and crosses the y-axis at $(0, -3)$.

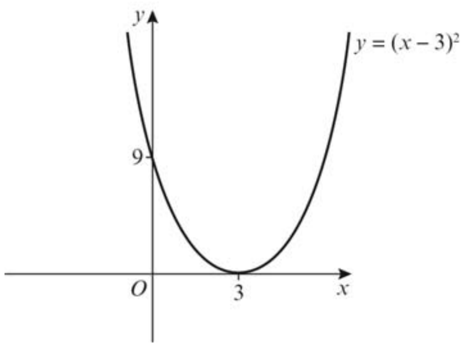
1 e iii $f(x) = \frac{1}{x}$, $f(x) - 3 = \frac{1}{x} - 3$



The curve crosses the x-axis at $\left(\frac{1}{3}, 0\right)$.
The horizontal asymptote is $y = -3$.
The vertical asymptote is $x = 0$.

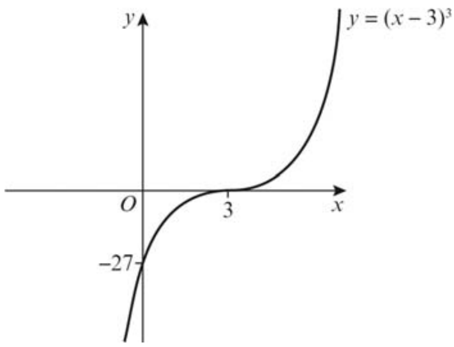
f $f(x - 3)$ is a translation of the graph of $f(x)$ by $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$, or three units to the right.

i $f(x) = x^2$, $f(x - 3) = (x - 3)^2$



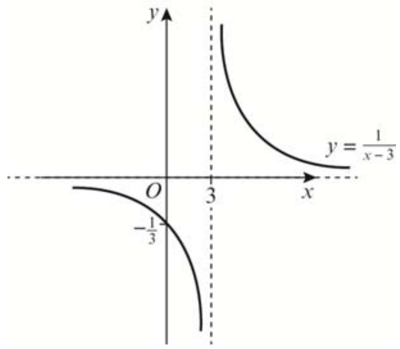
The curve touches the x-axis at (3, 0) and crosses the y-axis at (0, 9).

ii $f(x) = x^3$, $f(x - 3) = (x - 3)^3$



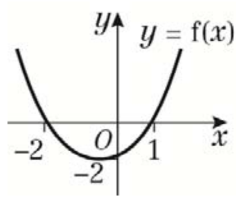
f ii The curve crosses the x-axis at (3,0) and crosses the y-axis at (0, −27).

iii $f(x) = \frac{1}{x}$, $f(x - 3) = \frac{1}{x - 3}$

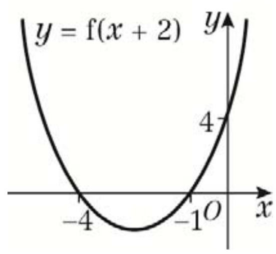


The curve crosses the y-axis at $\left(0, -\frac{1}{3}\right)$.
The horizontal asymptote is $y = 0$.
The vertical asymptote is $x = 3$.

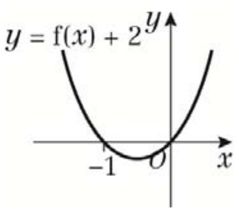
2 a $y = (x - 1)(x + 2)$
As $a = 1$ is positive, the graph has a $\cup$ shape and a minimum point.
$0 = (x - 1)(x + 2)$
So $x = 1$ or $x = -2$
The curve crosses the x-axis at (1, 0) and (−2, 0).
When $x = 0$, $y = (-1) \times 2 = -2$
The curve crosses the y-axis at (0, −2).



b i $f(x + 2)$ is a translation of the graph of $f(x)$ by $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$, or two units to the left.



2 b ii $f(x) + 2$ is a translation of the graph of $f(x)$ by $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$, or two units up.

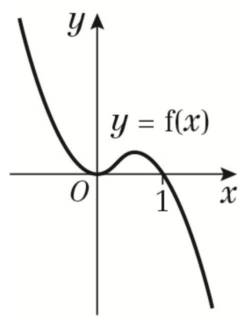


Since axis of symmetry of $f(x)$ is at $x = -\frac{1}{2}$, the same axis of symmetry applies to $f(x) + 2$.
Since one root is at $x = 0$, the other must be symmetric at $x = -1$.

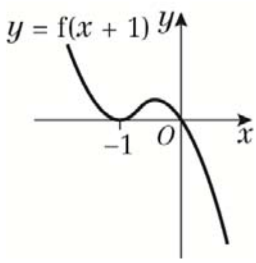
c $y = f(x + 2)$ is
$y = (x + 2 - 1)(x + 2 + 2)$
$= (x + 1)(x + 4)$
When $x = 0$, $y = 4$

$y = f(x) + 2$ is
$y = (x - 1)(x + 2) + 2$
$= x^2 + x - 2 + 2$
$= x^2 + x$
When $x = 0$, $y = 0$

3 a $y = x^2(1 - x)$
$0 = x^2(1 - x)$
So $x = 0$ or $x = 1$
The curve crosses the x-axis at $(1, 0)$ and touches it at $(0, 0)$.
$x \to \infty, y \to -\infty$
$x \to -\infty, y \to \infty$

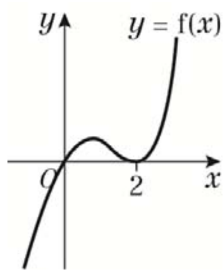


3 b $f(x + 1)$ is a translation of the graph of $f(x)$ by $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$, or one unit to the left.

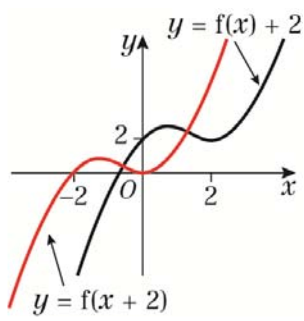


c $f(x + 1) = (x + 1)^2(1 - (x + 1))$
$= -(x + 1)^2x$
When $x = 0$, $y = 0$; the curve passes through $(0, 0)$.

4 a $y = x(x - 2)^2$
$0 = x(x - 2)^2$
So $x = 0$ or $x = 2$
The curve crosses the x-axis at $(0, 0)$ and touches it at $(2, 0)$.
$x \to \infty, y \to \infty$
$x \to -\infty, y \to -\infty$

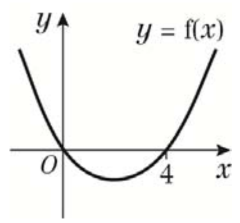


b $f(x) + 2$ is a translation of the graph of $f(x)$ by $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$, or two units up.
$f(x + 2)$ is a translation of the graph of $f(x)$ by $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$, or two units to the left.

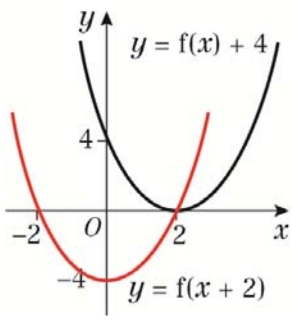


4 c $f(x+2) = (x+2)((x+2)-2)^2$
$= (x+2)x^2$
$(x+2)(x)^2 = 0$
So $x = 0$ and $x = -2$
The graph crosses the axes at (0, 0) and (−2, 0).

5 a $y = x(x-4)$
As $a = 1$ is positive, the graph has a $\cup$ shape and a minimum point.
$0 = x(x-4)$
So $x = 0$ or $x = 4$
The curve crosses the x-axis at (0, 0) and (4, 0).



b $f(x+2)$ is a translation of the graph of $f(x)$ by $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$, or two units to the left.
$f(x)+4$ is a translation of the graph of $f(x)$ by $\begin{pmatrix} 0 \\ 4 \end{pmatrix}$, or four units up.

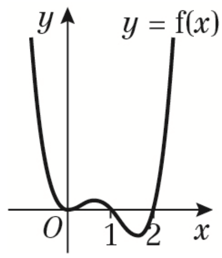


c $f(x+2) = (x+2)((x+2)-4)$
$= (x+2)(x-2)$
$0 = (x+2)(x-2)$
So $x = -2$ or $x = 2$
When $x = 0$, $y = 2 \times (-2) = -4$
So $f(x+2)$ crosses the x-axis at (−2, 0) and (2, 0) and the y-axis at (0, −4).

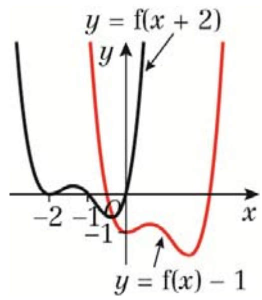
$f(x) + 4 = x(x-4) + 4$
$= x^2 - 4x + 4$
$= (x-2)^2$

5 c $0 = (x-2)^2$
So $x = 2$
When $x = 0$, $y = (-2)^2 = 4$
So $f(x) + 4$ touches the x-axis at (2, 0) and crosses the y-axis at (0, 4).

6 a $y = x^2(x-1)(x-2)$
$0 = x^2(x-1)(x-2)$
So $x = 0$, $x = 1$ or $x = 2$
The curve touches the x-axis at (0, 0) and crosses it at (1, 0) and (2, 0).
$x \to \infty, y \to \infty$
$x \to -\infty, y \to \infty$



b $y = f(x+2)$ is a translation of the graph of $f(x)$ by $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$, or two units to the left.
$y = f(x) - 1$ is a translation of the graph of $f(x)$ by $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$, or one unit down.



7 a $y = f(x-2)$ is a translation of the graph of $f(x)$ by $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$, or two units to the right.
So P translates to (6, −1).

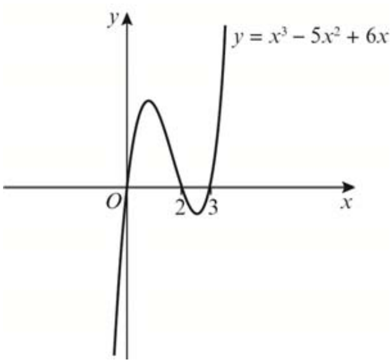
b $y = f(x) + 3$ is a translation of the graph of $f(x)$ by $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$, or three units up.
So P translates to (4, 2).

8 $y = \mathrm{f}(x)$ has asymptotes at $x = 0$ and $y = 0$.
Asymptotes after the translation are at $x = 4$ and $y = 0$, therefore the graph has been translated four units to the right.

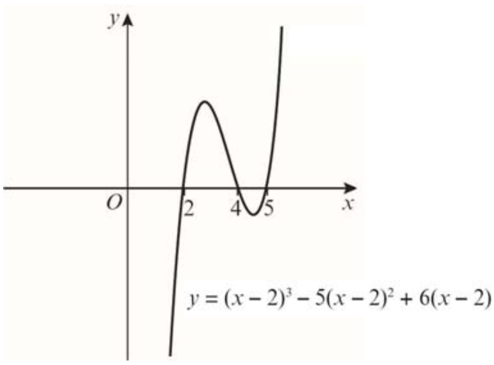
$\mathrm{f}(x) = \dfrac{1}{x}$, $\mathrm{f}(x-4) = \dfrac{1}{x-4}$

$y = \dfrac{1}{x-4}$

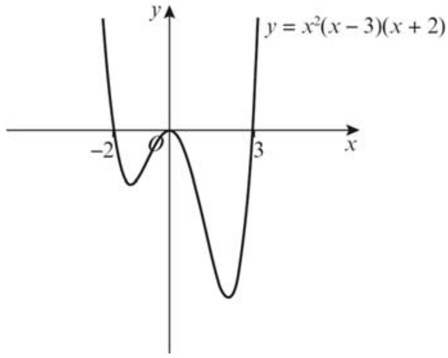
9 a $y = x^3 - 5x^2 + 6x$
$= x(x^2 - 5x + 6)$
$= x(x - 2)(x - 3)$
$0 = x(x - 2)(x - 3)$
So $x = 0$, $x = 2$ or $x = 3$
The curve crosses the x-axis at $(0, 0)$, $(2, 0)$ and $(3, 0)$.
$x \to \infty, y \to \infty$
$x \to -\infty, y \to -\infty$



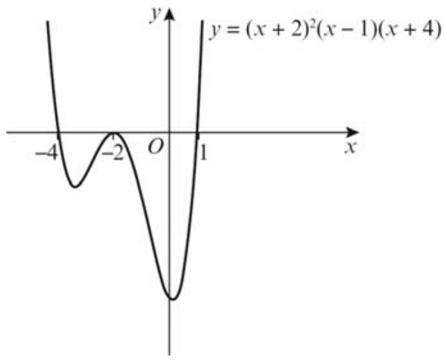
b Let $\mathrm{f}(x) = x^3 - 5x^2 + 6x$
$(x - 2)^3 - 5(x - 2)^2 + 6(x - 2)$ is $\mathrm{f}(x - 2)$, which is a translation of two units to the right.



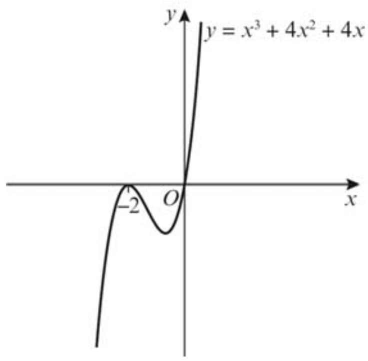
10 a $y = x^2(x - 3)(x + 2)$
$0 = x^2(x - 3)(x + 2)$
So $x = 0$, $x = 3$ or $x = -2$
The curve touches the x-axis at $(0, 0)$ and crosses it at $(3, 0)$ and $(-2, 0)$.
$x \to \infty, y \to \infty$
$x \to -\infty, y \to \infty$



b Let $\mathrm{f}(x) = x^2(x - 3)(x + 2)$
$(x + 2)^2(x - 1)(x + 4)$ is $\mathrm{f}(x + 2)$, which is a translation of two units to the left.

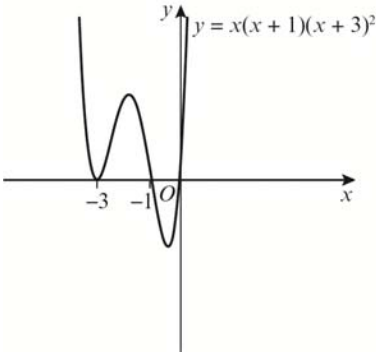


11 a $y = x^3 + 4x^2 + 4x$
$= x(x^2 + 4x + 4)$
$= x(x + 2)^2$
So $x = 0$ or $x = -2$
The curve crosses the x-axis at $(0, 0)$ and touches it at $(-2, 0)$.
$x \to \infty, y \to \infty$
$x \to -\infty, y \to -\infty$



11 b $y = (x + a)^3 + 4(x + a)^2 + 4(x + a)$
$y = x^3 + 4x^2 + 4x$ crosses the x-axis at (0, 0) and (−2, 0).
So for the point (−1, 0) to lie on the curve, the graph must be translated either one unit to the left or one unit to the right.
$a = -1$ or $a = 1$

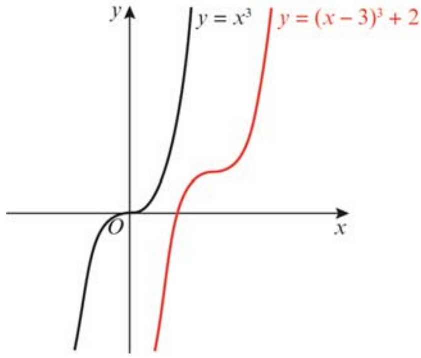
12 a $y = x(x + 1)(x + 3)^2$
$0 = x(x + 1)(x + 3)^2$
So $x = 0$, $x = -1$ or $x = -3$
The curve crosses the x-axis at (0, 0) and (−1, 0) and touches it (−3, 0).
$x \to \infty, y \to \infty$
$x \to -\infty, y \to \infty$



b $y = (x + b)(x + b + 1)(x + b + 3)^2$
$y = x(x + 1)(x + 3)^2$ crosses the x-axis at (0, 0), (−1, 0) and (−3, 0).
So for the point (2, 0) to lie on the curve, the graph must be translated either two units to the right, three units to the right or five units to the right.
$b = -2$, $b = -3$ or $b = -5$

Challenge

1 The graph of $f(x) = x^3$ has a point of inflection at (0, 0).
$y = (x - 3)^3 + 2$ is a translation of three units to the right and two units up.
The point of inflection is (3, 2).



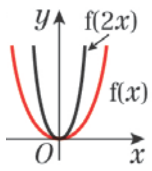
2 a $y = f(x + 2) - 5$ is a translation by $\begin{pmatrix} -2 \\ -5 \end{pmatrix}$,
or two units to the left and five units down.
So the point $Q(-5, -7)$ is transformed to the point (−7, −12).

b The coordinates of the point $Q(-5, -7)$ is transformed to the point (−3, −6).
This is a translation of two units to the right and one unit up.
So $y = f(x - 2) + 1$

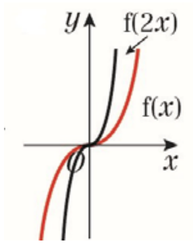
Graphs and transformations 4F

1 a $f(2x)$ is a stretch with scale factor $\frac{1}{2}$ in the x-direction.

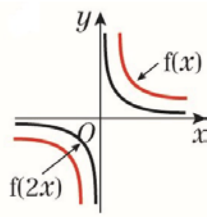
i $f(x) = x^2$, $f(2x) = (2x)^2 = 4x^2$



ii $f(x) = x^3$, $f(2x) = (2x)^3 = 8x^3$

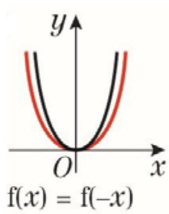


iii $f(x) = \dfrac{1}{x}$, $f(2x) = \dfrac{1}{2x} = \frac{1}{2} \times \dfrac{1}{x}$

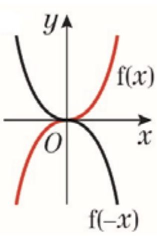


b $f(-x)$ is a reflection in the y-axis (or stretch with scale factor -1 in the x-direction).

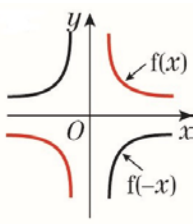
i $f(x) = x^2$, $f(-x) = (-x)^2 = x^2$



ii $f(x) = x^3$, $f(-x) = (-x)^3 = -x^3$

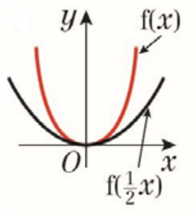


b iii $f(x) = \dfrac{1}{x}$, $f(-x) = \dfrac{1}{-x} = -\dfrac{1}{x}$

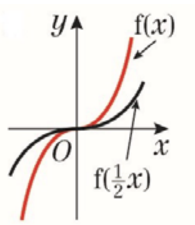


c $f\left(\frac{1}{2}x\right)$ is a stretch with scale factor 2 in the x-direction.

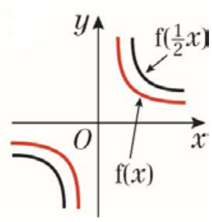
i $f(x) = x^2$, $f\left(\frac{1}{2}x\right) = \left(\frac{1}{2}x\right)^2 = \dfrac{x^2}{4}$



ii $f(x) = x^3$, $f\left(\frac{1}{2}x\right) = \left(\frac{1}{2}x\right)^3 = \dfrac{x^3}{8}$

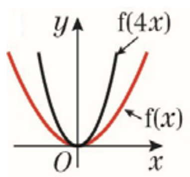


c iii $f(x) = \dfrac{1}{x}$, $f\left(\frac{1}{2}x\right) = \dfrac{1}{\frac{1}{2}x} = \dfrac{2}{x}$

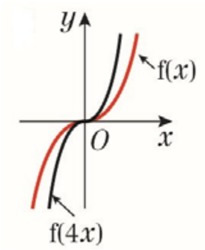


d $f(4x)$ is a stretch with scale factor $\frac{1}{4}$ in the x-direction.

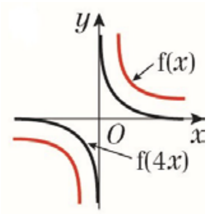
i $f(x) = x^2$, $f(4x) = (4x)^2 = 16x^2$



1 d ii $f(x) = x^3$, $f(4x) = (4x)^3 = 64x^3$

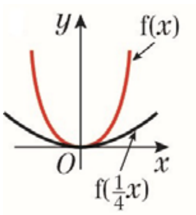


iii $f(x) = \dfrac{1}{x}$, $f(4x) = \dfrac{1}{4x} = \frac{1}{4} \times \dfrac{1}{x}$

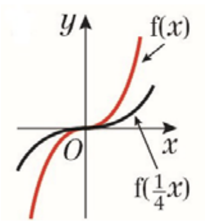


e $f\left(\frac{1}{4}x\right)$ is a stretch with scale factor 4 in the x-direction.

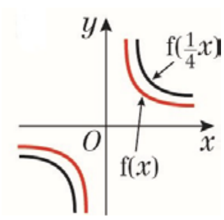
i $f(x) = x^2$, $f\left(\frac{1}{4}x\right) = \left(\frac{1}{4}x\right)^2 = \dfrac{x^2}{16}$



ii $f(x) = x^3$, $f\left(\frac{1}{4}x\right) = \left(\frac{1}{4}x\right)^3 = \dfrac{x^3}{64}$

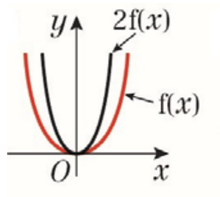


e iii $f(x) = \frac{1}{4}$, $f\left(\frac{1}{4}x\right) = \dfrac{1}{\frac{1}{4}x} = \dfrac{4}{x}$

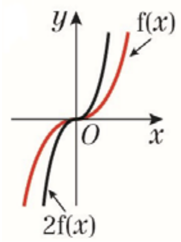


f $2f(x)$ is a stretch with scale factor 2 in the y-direction.

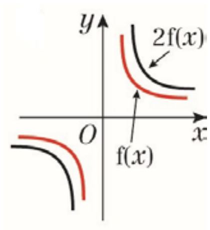
i $f(x) = x^2$, $2f(x) = 2x^2$



ii $f(x) = x^3$, $2f(x) = 2x^3$

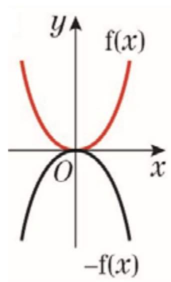


iii $f(x) = \dfrac{1}{x}$, $2f(x) = 2 \times \dfrac{1}{x} = \dfrac{2}{x}$

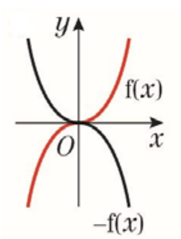


g $-f(x)$ is a reflection in the x-axis (or stretch with scale factor -1 in the y-direction).

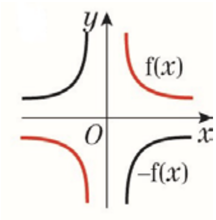
i $f(x) = x^2$, $-f(x) = -x^2$



ii $f(x) = x^3$, $-f(x) = -x^3$

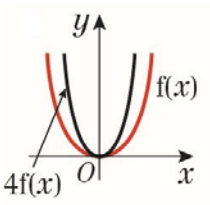


1 g iii $f(x) = \dfrac{1}{x}$, $-f(x) = -\dfrac{1}{x}$

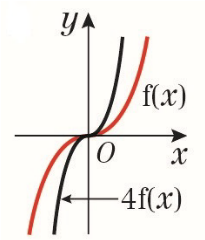


h $4f(x)$ is a stretch with scale factor 4 in the y-direction.

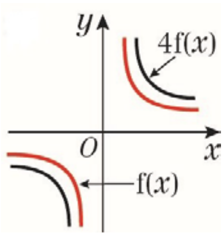
i $f(x) = x^2$, $4f(x) \to y = 4x^2$



ii $f(x) = x^3$, $4f(x) = 4x^3$

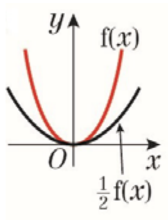


iii $f(x) = \dfrac{1}{x}$, $4f(x) = 4\times\dfrac{1}{x} = \dfrac{4}{x}$

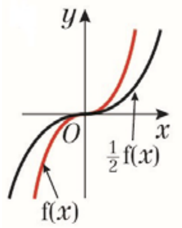


i i $\frac{1}{2}f(x)$ is a stretch with scale factor $\frac{1}{2}$ in the y-direction.

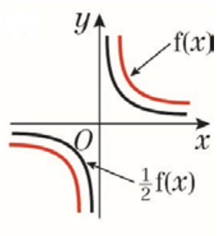
$f(x) = x^2$, $\frac{1}{2}f(x) = \frac{1}{2}x^2$



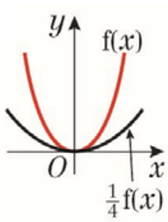
i ii $f(x) = x^3$, $\frac{1}{2}f(x) = \frac{1}{2}x^3$



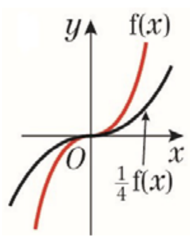
iii $f(x) = \dfrac{1}{x}$, $\frac{1}{2}f(x) = \frac{1}{2}\times\dfrac{1}{x} = \dfrac{1}{2x}$



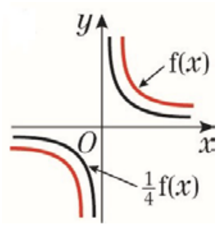
j i $\frac{1}{4}f(x)$ is a stretch with scale factor $\frac{1}{4}$ in the y-direction.
$f(x) = x^2$, $\frac{1}{4}f(x) = \frac{1}{4}x^2$



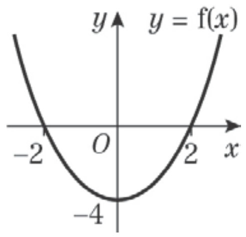
ii $f(x) = x^3$, $\frac{1}{4}f(x) = \frac{1}{4}x^3$



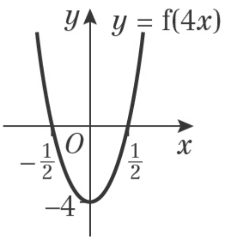
iii $f(x) = \dfrac{1}{x}$, $\frac{1}{4}f(x) = \frac{1}{4}\times\dfrac{1}{x} = \dfrac{1}{4x}$



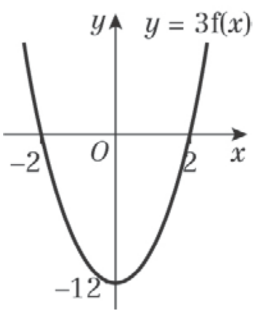
2 a $y = x^2 - 4$
$= (x - 2)(x + 2)$
As $a = 1$ is positive, the graph has a $\cup$ shape and a minimum point.
$0 = (x - 2)(x + 2)$
So $x = 2$ or $x = -2$
The curve crosses the x-axis at $(2, 0)$ and $(-2, 0)$.
When $x = 0$, $y = (-2) \times 2 = -4$
The curve crosses the y-axis at $(0, -4)$.



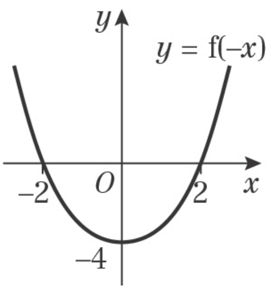
b $f(4x)$ is a stretch with scale factor $\frac{1}{4}$ in the x-direction.



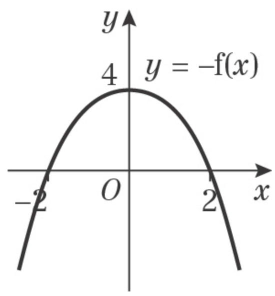
$\frac{1}{3}y = f(x)$
$y = 3f(x)$
$3f(x)$ is a stretch with scale factor 3 in the y-direction.



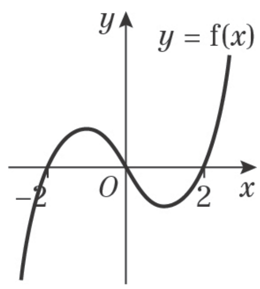
$f(-x)$ is a reflection in the y-axis.



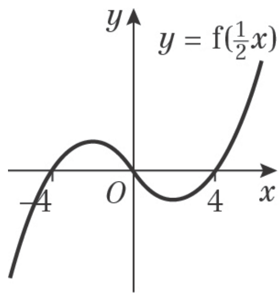
2 b $-f(x)$ is a reflection in the x-axis.



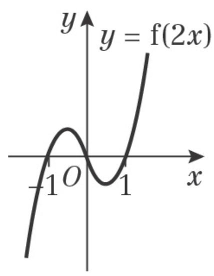
3 a $y = (x - 2)(x + 2)x$
$0 = (x - 2)(x + 2)x$
So $x = 2$, $x = -2$ or $x = 0$
The curve crosses the x-axis at $(2, 0)$, $(-2, 0)$ and $(0, 0)$.
$x \to \infty, y \to \infty$
$x \to -\infty, y \to -\infty$



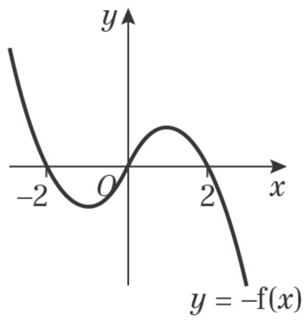
b $f\left(\frac{1}{2}x\right)$ is a stretch with scale factor 2 in the x-direction.



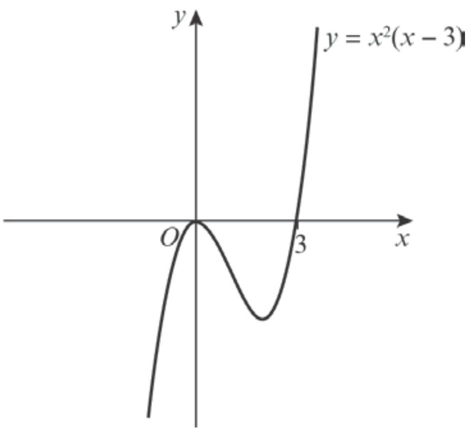
$f(2x)$ is a stretch with scale factor $\frac{1}{2}$ in the x-direction.



3 b $-\mathrm{f}(x)$ is a reflection in the x-axis.

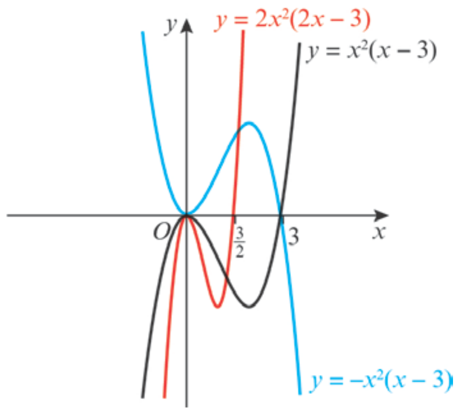


4 a $y = x^2(x - 3)$
$0 = x^2(x - 3)$
So $x = 0$ or $x = 3$
The curve touches the x-axis at (0, 0) and crosses it at (3, 0).
$x \to \infty, y \to \infty$
$x \to -\infty, y \to -\infty$

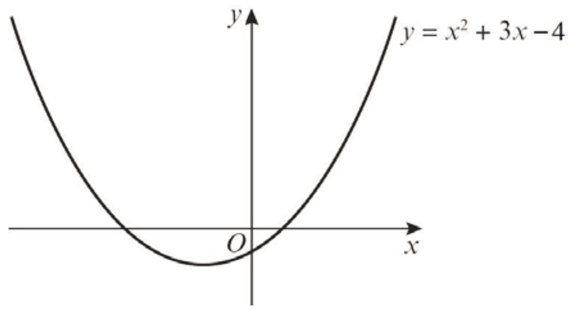


b i $\mathrm{f}(x) = x^2(x - 3)$, so $y = (2x)^2(2x - 3)$ is $\mathrm{f}(2x)$, which is a stretch with scale factor $\frac{1}{2}$ in the x-direction.

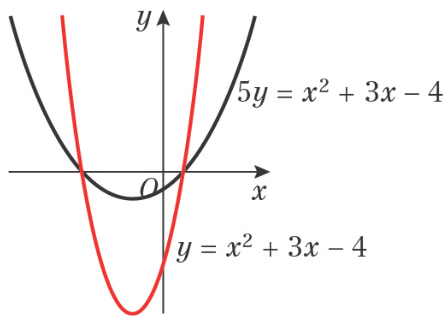
ii $y = -x^2(x - 3)$ is $-\mathrm{f}(x)$, which is a reflection in the x-axis.



5 a $y = x^2 + 3x - 4$
$= (x + 4)(x - 1)$
As $a = 1$ is positive, the graph has a $\cup$ shape and a minimum point.
$0 = (x + 4)(x - 1)$
So $x = -4$ or $x = 1$
The curve crosses the x-axis at $(-4, 0)$ and $(1, 0)$.
When $x = 0$, $y = 4 \times (-1) = -4$
The curve crosses the y-axis at $(0, -4)$.

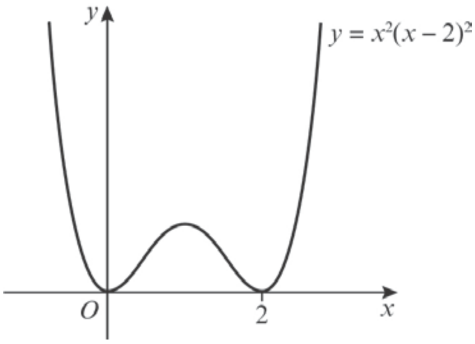


b $5y = x^2 + 3x - 4$
$y = \frac{1}{5}(x^2 + 3x - 4)$
$\mathrm{f}(x) = x^2 + 3x - 4$, so $y = \frac{1}{5}(x^2 + 3x - 4)$ is $\frac{1}{5}\mathrm{f}(x)$, which is a stretch with scale factor $\frac{1}{5}$ in the y-direction.



6 a $y = x^2(x - 2)^2$
$0 = x^2(x - 2)^2$
So $x = 0$ or $x = 2$
The curve touches the x-axis at (0, 0) and (2, 0).
$x \to \infty, y \to \infty$
$x \to -\infty, y \to \infty$

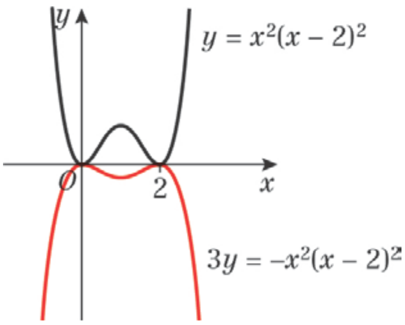
6 a



b $3y = -x^2(x-2)^2$

$y = -\frac{1}{3}x^2(x-2)^2$

$f(x) = x^2(x-2)^2$, so $y = -\frac{1}{3}x^2(x-2)^2$ is $\left(\frac{1}{2}x\right) - \frac{1}{3}f(x)$, which is a stretch with scale factor $\frac{1}{3}$ in the y-direction and a reflection in the x-axis.



7 a $y = f(2x)$ is a stretch with scale factor $\frac{1}{2}$ in the x-direction, so all x-coordinates are halved.

$P(2, -3)$ is transformed to the point $(1, -3)$.

b $y = 4f(x)$ is a stretch with scale factor 4 in the y-direction, so all y-coordinates are multiplied by four.

$P(2, -3)$ is transformed to the point $(2, -12)$.

8 $f(\frac{1}{2}x)$ is a stretch with scale factor 2 in the x-direction, so all x-coordinates are doubled.

$Q(-2, 8)$ is transformed to the point $(-4, 8)$.

9 a $y = (x-2)(x-3)^2$

$0 = (x-2)(x-3)^2$

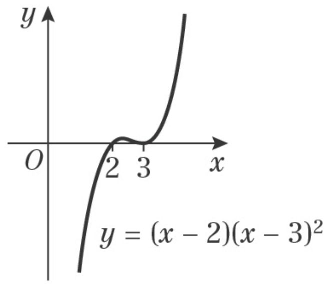
So $x = 2$ or $x = 3$

The curve crosses the x-axis at $(2, 0)$ and touches it at $(3, 0)$.

When $x = 0$, $y = (-2) \times (-3)^2 = -18$

9 a $x \to \infty, y \to \infty$

$x \to -\infty, y \to -\infty$



b $f(x) = (x-2)(x-3)^2$

$y = (ax-2)(ax-3)^2$ is the graph of $y = f(ax)$, which is a stretch with scale factor $\dfrac{1}{a}$ in the x-direction, so all x-coordinates are multiplied by $\dfrac{1}{a}$.

For the coordinate $(2, 0)$ to be transformed to $(1, 0)$, multiply the x-coordinate by $\frac{1}{2}$, giving $a = 2$.

For the coordinate $(3, 0)$ to be transformed to $(1, 0)$, multiply the x-coordinate by $\frac{1}{3}$, giving $a = 3$, $a = 2$ or $a = 3$

Challenge

1 $y = \frac{1}{3}f(2x)$ is a stretch with scale factor $\frac{1}{3}$ in the y-direction, so multiply the y-coordinate by $\frac{1}{3}$ and a stretch with scale factor $\frac{1}{2}$ in the x-direction, so multiply the x-coordinate by $\frac{1}{2}$.

$R(4, -6)$ is transformed to $(2, -2)$.

2 $S(-4, 7)$ is transformed to $S'(-8, 1.75)$.

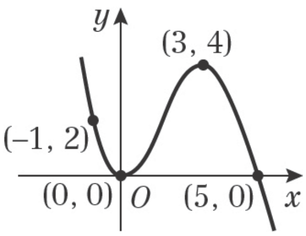
The x-coordinate has doubled, which is a stretch of scale factor 2 in the x-direction.

The y-coordinate has been divided by 4, which is a stretch of scale factor $\frac{1}{4}$ in the y-direction.

The transformation is $y = \frac{1}{4}f\left(\frac{1}{2}x\right)$.

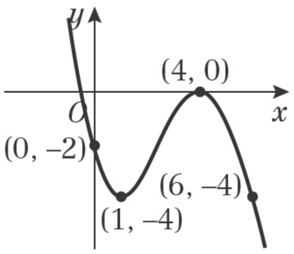
Graphs and transformations 4G

1 a f$(x + 1)$ is a translation by $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$, or one unit to the left.



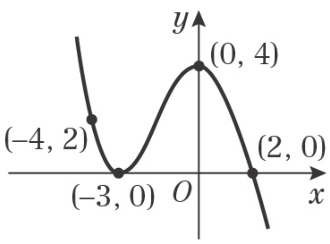
$A'(-1, 2)$, $B'(0, 0)$, $C'(3, 4)$, $D'(5, 0)$

b f$(x) - 4$ is a translation by $\begin{pmatrix} 0 \\ -4 \end{pmatrix}$, or four units down.



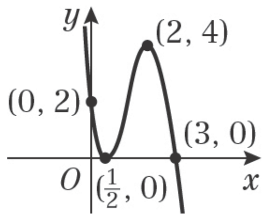
$A'(0, -2)$, $B'(1, -4)$, $C'(4, 0)$, $D'(6, -4)$

c f$(x + 4)$ is a translation by $\begin{pmatrix} -4 \\ 0 \end{pmatrix}$, or four units to the left.



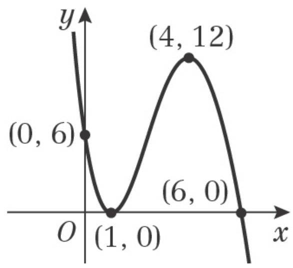
$A'(-4, 2)$, $B'(-3, 0)$, $C'(0, 4)$, $D'(2, 0)$

d f$(2x)$ is a stretch with scale factor $\frac{1}{2}$ in the x-direction.



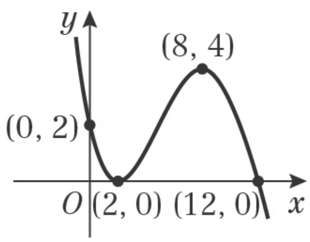
$A'(0, 2)$, $B'\left(\frac{1}{2}, 0\right)$, $C'(2, 4)$, $D'(3, 0)$

e 3f(x) is a stretch with scale factor 3 in the y-direction.



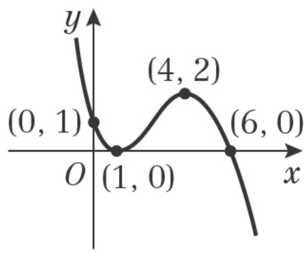
$A'(0, 6)$, $B'(1, 0)$, $C'(4, 12)$, $D'(6, 0)$

f f$\left(\frac{1}{2}x\right)$ is a stretch with scale factor 2 in the x-direction.



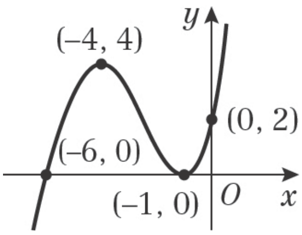
$A'(0, 2)$, $B'(2, 0)$, $C'(8, 4)$, $D'(12, 0)$

g $\frac{1}{2}$f(x) is a stretch with scale factor $\frac{1}{2}$ in the y-direction.



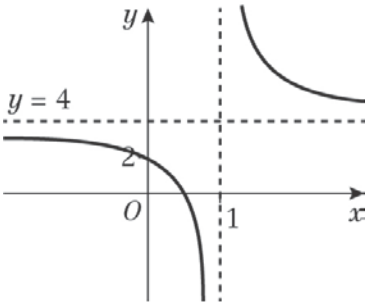
$A'(0, 1)$, $B'(1, 0)$, $C'(4, 2)$, $D'(6, 0)$

1 h f($-x$) is a reflection in the y-axis



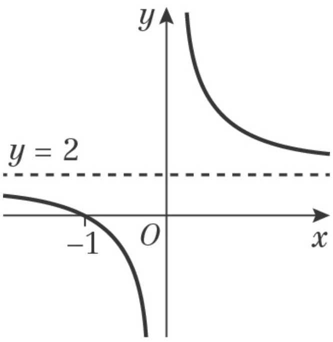
$A'(0, 2)$, $B'(-1, 0)$, $C'(-4, 4)$, $D'(-6, 0)$

2 a f(x) + 2 is a translation by $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$, or two units up.



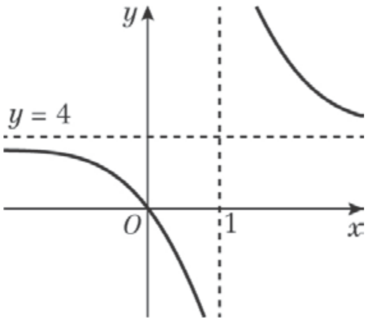
The curve crosses the x-axis at (0, 2) and the y-axis at (a, 0), where $0 < a < 1$.
The horizontal asymptote is $y = 4$.
The vertical asymptote is $x = 1$.

b f(x + 1) is a translation by $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$, or one unit to the left.



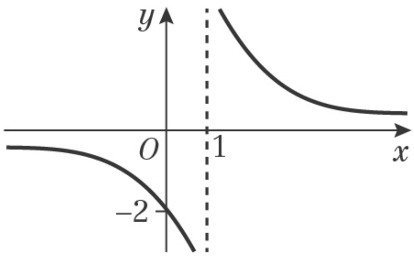
The curve crosses the x-axis at $(-1, 0)$.
The horizontal asymptote is $y = 2$.
The vertical asymptote is $x = 0$.

2 c 2f(x) is a stretch with scale factor 2 in the y-direction.



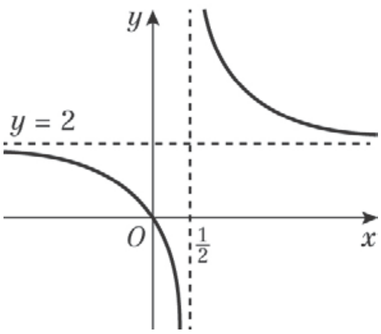
The curve crosses the axes at (0, 0).
The horizontal asymptote is $y = 4$.
The vertical asymptote is $x = 1$.

d f(x) − 2 is a translation by $\begin{pmatrix} 0 \\ -2 \end{pmatrix}$, or two units down.



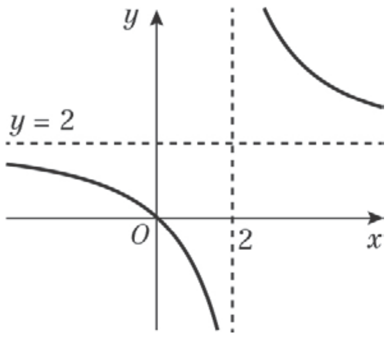
The curve crosses the y-axis at $(0, -2)$.
The horizontal asymptote is $y = 0$.
The vertical asymptote is $x = 1$.

e f($2x$) is a stretch with scale factor $\frac{1}{2}$ in the x-direction.



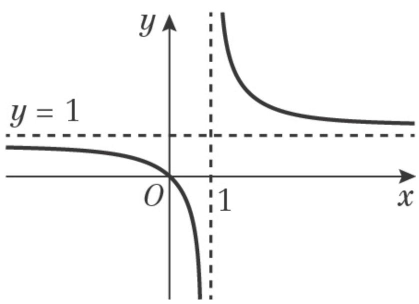
The curve crosses the axes at (0, 0).
The horizontal asymptote is $y = 2$.
The vertical asymptote is $x = \frac{1}{2}$.

2 f $\mathrm{f}\left(\frac{1}{2}x\right)$ is a stretch with scale factor 2 in the x-direction.



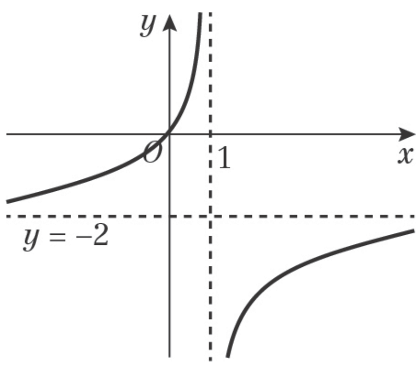
The curve crosses the axes at (0, 0).
The horizontal asymptote is $y = 2$.
The vertical asymptote is $x = 2$.

g $\frac{1}{2}\mathrm{f}(x)$ is a stretch with scale factor $\frac{1}{2}$ in the y-direction.



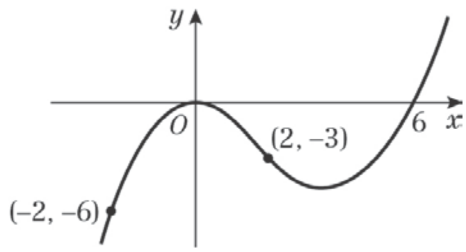
The curve crosses the axes at (0, 0).
The horizontal asymptote is $y = 1$.
The vertical asymptote is $x = 1$.

h $-\mathrm{f}(x)$ is a reflection in the x-axis.



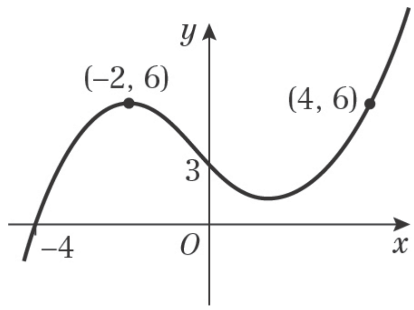
The curve crosses the axes at (0, 0).
The horizontal asymptote is $y = -2$.
The vertical asymptote is $x = 1$.

3 a $\mathrm{f}(x-2)$ is a translation by $\begin{pmatrix}2\\0\end{pmatrix}$, or two units to the right.



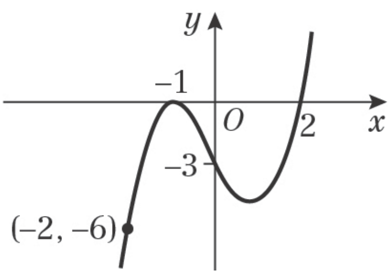
$A'(-2, -6)$, $B'(0, 0)$, $C'(2, -3)$, $D'(6, 0)$

b $\mathrm{f}(x) + 6$ is a translation by $\begin{pmatrix}0\\6\end{pmatrix}$, or six units up.



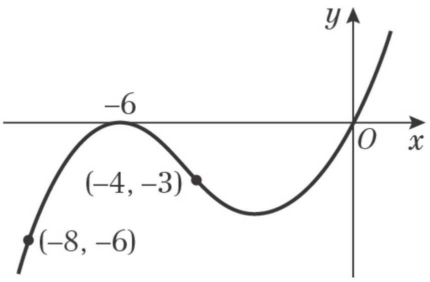
$A'(-4, 0)$, $B'(-2, 6)$, $C'(0, 3)$, $D'(4, 6)$

c $\mathrm{f}(2x)$ is a stretch with scale factor $\frac{1}{2}$ in the x-direction.



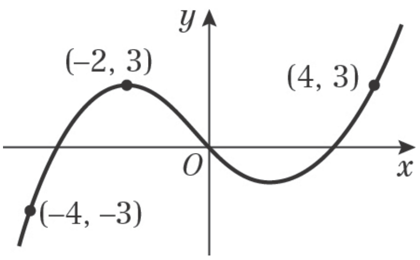
$A'(-2, -6)$, $B'(-1, 0)$, $C'(0, -3)$, $D'(2, 0)$

3 **d** $f(x+4)$ is a translation by $\begin{pmatrix} -4 \\ 0 \end{pmatrix}$, or four units to the left.



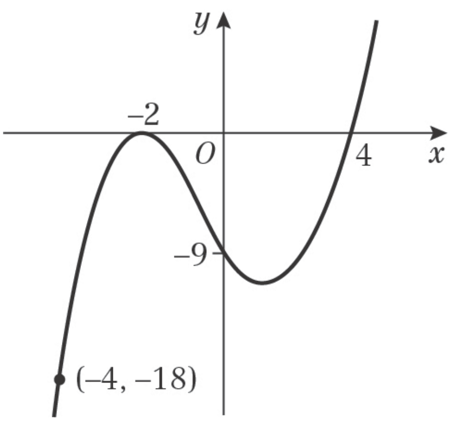
$A'(-8, -6)$, $B'(-6, 0)$, $C'(-4, -3)$, $D'(0, 0)$

e $f(x) + 3$ is a translation by $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$, or three units up.



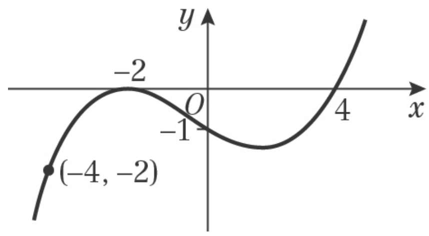
$A'(-4, -3)$, $B'(-2, 3)$, $C'(0, 0)$, $D'(4, 3)$

f $3f(x)$ is a stretch with scale factor 3 in the y-direction.



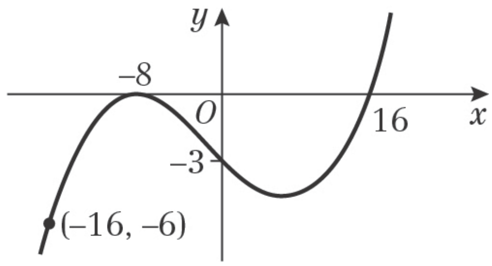
$A'(-4, -18)$, $B'(-2, 0)$, $C'(0, -9)$, $D'(4, 0)$

g $\frac{1}{3}f(x)$ is a stretch with scale factor $\frac{1}{3}$ in the y-direction.



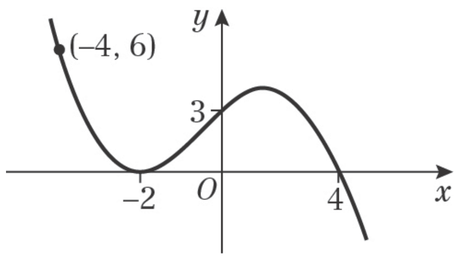
$A'(-4, -2)$, $B'(-2, 0)$, $C'(0, -1)$, $D'(4, 0)$

h $f\left(\frac{1}{4}x\right)$ is a stretch with scale factor 4 in the x-direction.



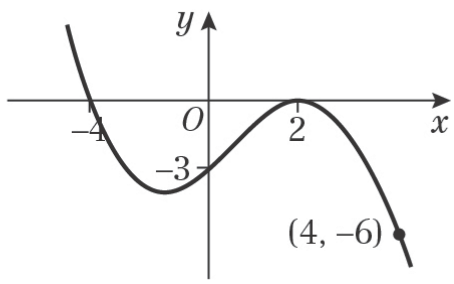
$A'(-16, -6)$, $B'(-8, 0)$, $C'(0, -3)$, $D'(16, 0)$

i $-f(x)$ is a reflection in the x-axis.



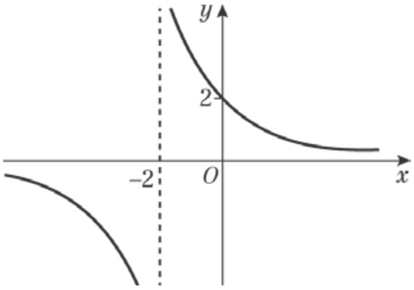
$A'(-4, 6)$, $B'(-2, 0)$, $C'(0, 3)$, $D'(4, 0)$

j $f(-x)$ is a reflection in the y-axis.



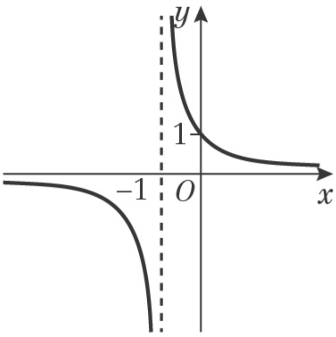
$A'(4, -6)$, $B'(2, 0)$, $C'(0, -3)$, $D'(-4, 0)$

4 a i $2f(x)$ is a stretch with scale factor 2 in the y-direction.



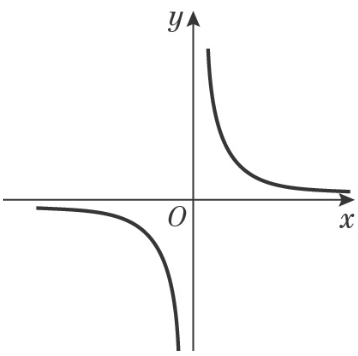
The curve crosses the y-axis at (0, 2).
The horizontal asymptote is $y = 0$.
The vertical asymptote is $x = -2$.

ii $f(2x)$ is a stretch with scale factor $\frac{1}{2}$ in the x-direction.



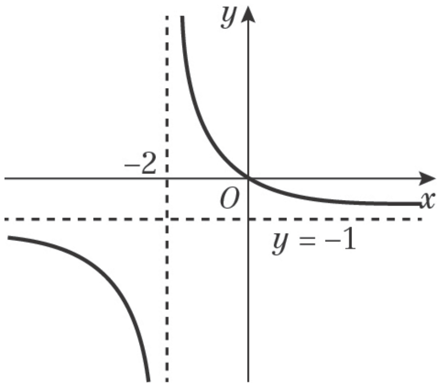
The curve crosses the y-axis at (0, 1).
The horizontal asymptote is $y = 0$.
The vertical asymptote is $x = -1$.

iii $f(x - 2)$ is a translation by $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$, or two units to the right.



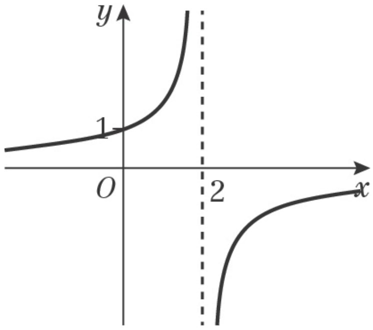
There are no intersections with the axes.
The horizontal asymptote is $y = 0$.
The vertical asymptote is $x = 0$.

iv $f(x) - 1$ is a translation by $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$, or one unit down.



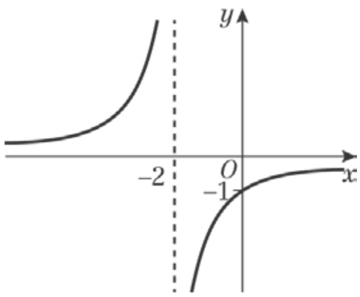
The curve crosses the axes at (0, 0).
The horizontal asymptote is $y = -1$.
The vertical asymptote is $x = -2$.

v $f(-x)$ is a reflection in the y-axis.



The curve crosses the y-axis at (0, 1).
The horizontal asymptote is $y = 0$.
The vertical asymptote is $x = 2$.

vi $-f(x)$ is a reflection in the x-axis.



The curve crosses the y-axis at (0, −1).
The horizontal asymptote is $y = 0$.
The vertical asymptote is $x = -2$.

4 b The shape of the curve is like $y = \dfrac{k}{x}$, $k > 0$.

$x = -2$ asymptote suggests denominator is zero when $x = -2$, so denominator is $x + 2$.
Also, $f(0) = 1$ means the numerator must be 2.

$$f(x) = \frac{2}{x+2}$$

5 a $P(2, 1)$ is mapped to $Q(4, 1)$.
The x-coordinate has doubled, which is a stretch with scale factor 2 in the x-direction.

$y = f\left(\frac{1}{2}x\right)$

$a = \frac{1}{2}$

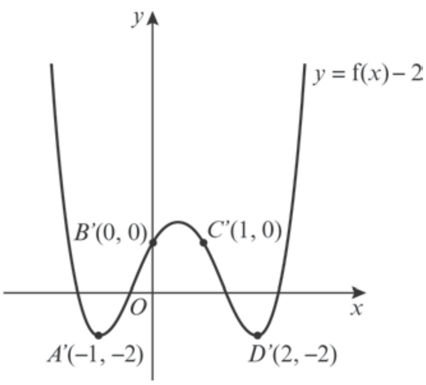
b i $f(x - 4)$ is a translation by $\begin{pmatrix}4\\0\end{pmatrix}$, or four units to the right.
So P is mapped to $(6, 1)$.

ii $3f(x)$ is a stretch with scale factor 3 in the y-direction.
So P is mapped to $(2, 3)$.

iii $\frac{1}{2}f(x) - 4$ is a stretch with scale factor $\frac{1}{2}$ in the y-direction and then a translation by $\begin{pmatrix}0\\-4\end{pmatrix}$, or four units down.

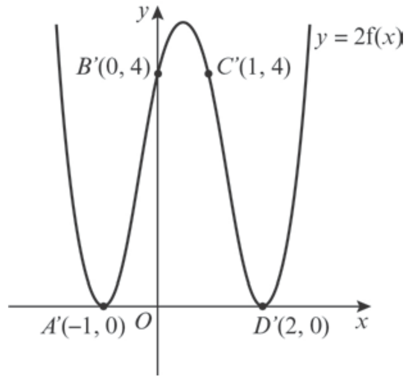
So P is mapped to $\left(2, -3\frac{1}{2}\right)$

6 a $y + 2 = f(x)$
$y = f(x) - 2$, which is a translation by $\begin{pmatrix}0\\-2\end{pmatrix}$, or two units down.



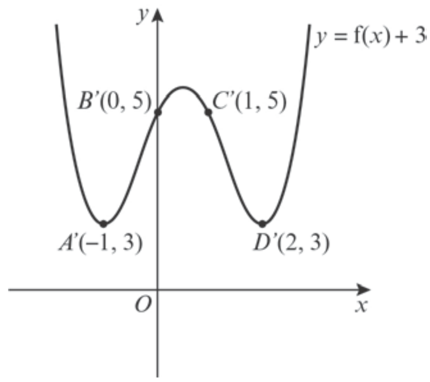
$A'(-1, -2)$, $B'(0, 0)$, $C'(1, 0)$, $D'(2, -2)$

6 b $\frac{1}{2}y = f(x)$
$y = 2f(x)$, which is a stretch with scale factor 2 in the y-direction.



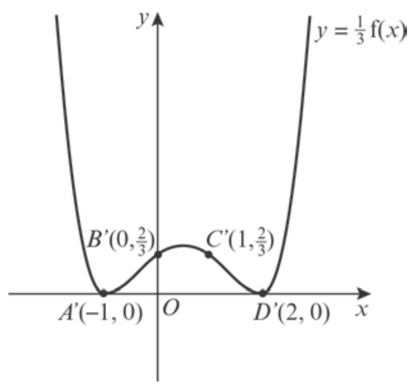
$A'(-1, 0)$, $B'(0, 4)$, $C'(1, 4)$, $D'(2, 0)$

c $y - 3 = f(x)$
$y = f(x) + 3$, which is a translation by $\begin{pmatrix}0\\3\end{pmatrix}$, or three units up.



$A'(-1, 3)$, $B'(0, 5)$, $C'(1, 5)$, $D'(2, 3)$

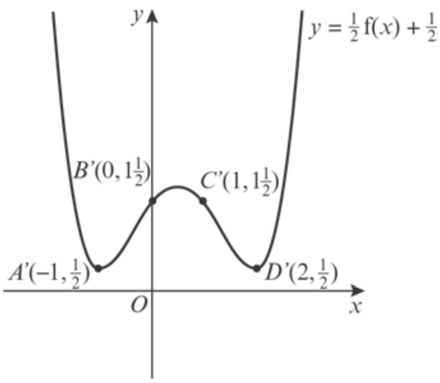
d $3y = f(x)$
$y = \frac{1}{3}f(x)$, which is a stretch with scale factor $\frac{1}{3}$ in the y-direction.



$A'(-1, 0)$, $B'\left(0, \frac{2}{3}\right)$, $C'\left(1, \frac{2}{3}\right)$, $D'(2, 0)$

6 e $2y - 1 = \text{f}(x)$

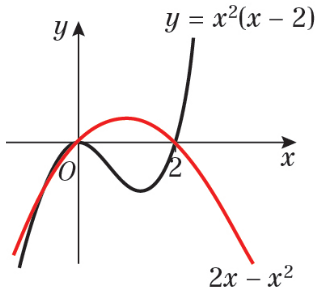
$y = \frac{1}{2}\text{f}(x) + \frac{1}{2}$, which is a stretch with scale factor $\frac{1}{2}$ in the y-direction, then a translation by $\begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}$, or $\frac{1}{2}$ unit up.



$A'\left(-1, \frac{1}{2}\right)$, $B'\left(0, 1\frac{1}{2}\right)$, $C'\left(1, 1\frac{1}{2}\right)$,
$D'\left(2, \frac{1}{2}\right)$

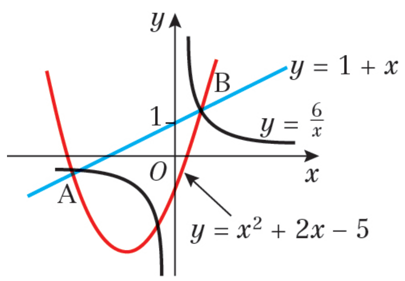
Graphs and transformations, Mixed Exercise 4

1 a $y = x^2(x - 2)$
$0 = x^2(x - 2)$
So $x = 0$ or $x = 2$
The curve crosses the x-axis at (2, 0) and touches it at (0, 0).
$x \to \infty, y \to \infty$
$x \to -\infty, y \to -\infty$
$y = 2x - x^2$
$\quad = x(2 - x)$
As $a = -1$ is negative, the graph has a ∩ shape and a maximum point.
$0 = x(2 - x)$
So $x = 0$ or $x = 2$
The curve crosses the x-axis at (0, 0) and (2, 0).



b $x^2(x - 2) = x(2 - x)$
$x^2(x - 2) - x(2 - x) = 0$
$x^2(x - 2) + x(x - 2) = 0$
$x(x - 2)(x + 1) = 0$
So $x = 0$, $x = 2$ or $x = -1$
Using $y = x(2 - x)$:
when $x = 0$, $y = 0 \times 2 = 0$
when $x = 2$, $y = 2 \times 0 = 0$
when $x = 0$, $y = (-1) \times 3 = -3$
The points of intersection are (0, 0), (2, 0) and (−1, −3).

2 a $y = \dfrac{6}{x}$ is like $y = \dfrac{1}{x}$.
$y = 1 + x$ is a straight line.

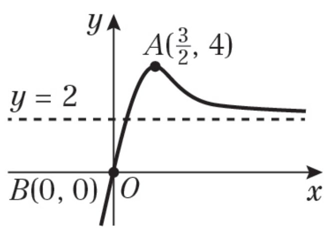


b $\dfrac{6}{x} = 1 + x$
$6 = x + x^2$
$0 = x^2 + x - 6$
$0 = (x + 3)(x - 2)$
So $x = 2$ or $x = -3$
Using $y = 1 + x$:
when $x = 2$, $y = 1 + 2 = 3$
when $x = -3$, $y = 1 - 3 = -2$
So A is (−3, −2) and B is (2, 3).

c Substituting the points A and B into $y = x^2 + px + q$:
A: $-2 = 9 - 3p + q$ **(1)**
B: $3 = 4 + 2p + q$ **(2)**
(1) − **(2)**:
$-5 = 5 - 5p$
$p = 2$
Substituting in **(1)**:
$-2 = 9 - 6 + q$
$q = -5$

d $y = x^2 + 2x - 5$
As $a = 1$ is positive, the graph has a ∪ shape and a minimum point.
$y = (x + 1)^2 - 6$
So the minimum is at (−1, −6).

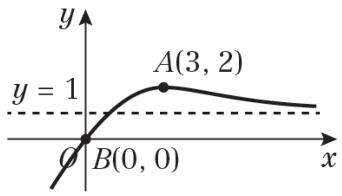
3 a f(2x) is a stretch with scale factor $\frac{1}{2}$ in the x-direction.



$A'\left(\frac{3}{2}, 4\right)$, $B'(0, 0)$
The asymptote is $y = 2$.

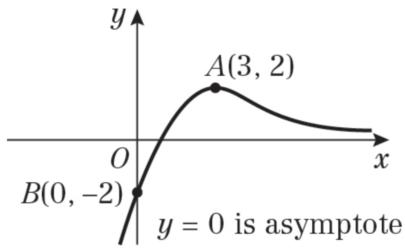
b $\frac{1}{2}$f(x) is a stretch with scale factor $\frac{1}{2}$ in the y-direction.

3 b



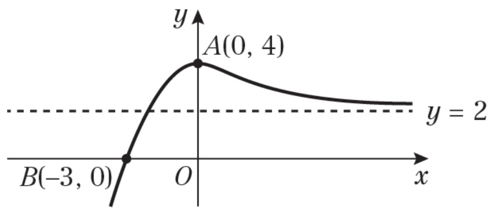
$A'(3, 2)$, $B'(0, 0)$
The asymptote is $y = 1$.

c $f(x) - 2$ is a translation by $\begin{pmatrix} 0 \\ -2 \end{pmatrix}$, or two units down.



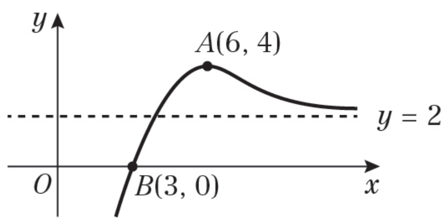
$A'(3, 2)$, $B'(0, -2)$
The asymptote is $y = 0$.

d $f(x + 3)$ is a translation by $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$, or three units to the left.



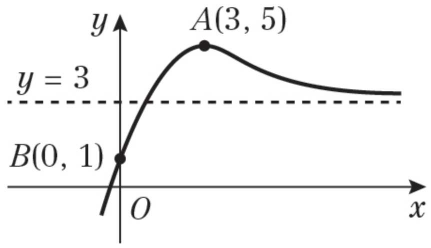
$A'(0, 4)$, $B'(-3, 0)$
The asymptote is $y = 2$.

e $f(x - 3)$ is a translation by $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$, or three units to the right.



e $A'(6, 4)$, $B'(3, 0)$
The asymptote is $y = 2$.

f $f(x) + 1$ is a translation by $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, or one unit up.



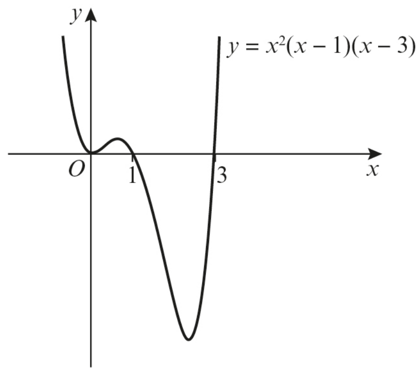
$A'(3, 5)$, $B'(0, 1)$
The asymptote is $y = 3$.

4

$$2 = 5 + 2x - x^2$$
$$x^2 - 2x - 3 = 0$$
$$(x - 3)(x + 1) = 0$$

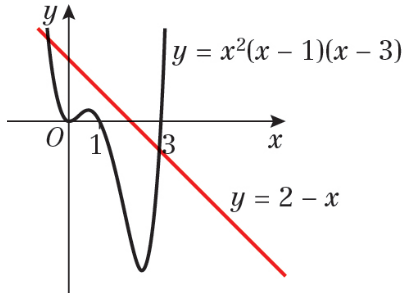
So $x = -1$ or $x = 3$

5 a $y = x^2(x - 1)(x - 3)$
$0 = x^2(x - 1)(x - 3)$
So $x = 0$, $x = 1$ or $x = 3$
The curve touches the x-axis at $(0, 0)$ and crosses it at $(1, 0)$ and $(3, 0)$.
$x \to \infty, y \to \infty$
$x \to -\infty, y \to \infty$



b $y = 2 - x$ is a straight line.
It crosses the x-axis at $(2, 0)$ and the y-axis at $(0, 2)$.

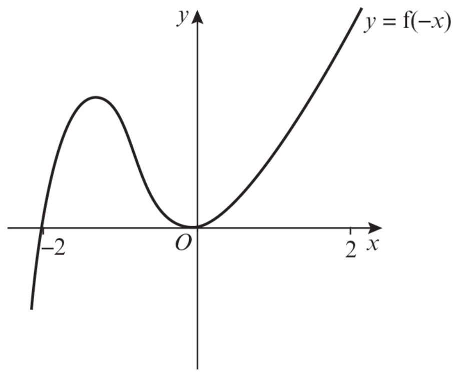
5 b



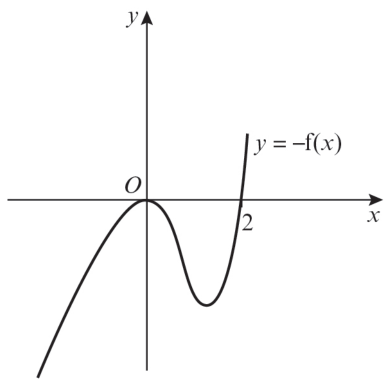
c As there are two points of intersection, $x^2(x - 1)(x - 3) = 2 - x$ has two real solutions.

d $y = \mathrm{f}(x) + 2$ is a translation by $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$, or two units up.
So $y = \mathrm{f}(x) + 2$ crosses the y-axis at $(0, 2)$.

6 a $\mathrm{f}(-x)$ is a reflection in the y-axis.

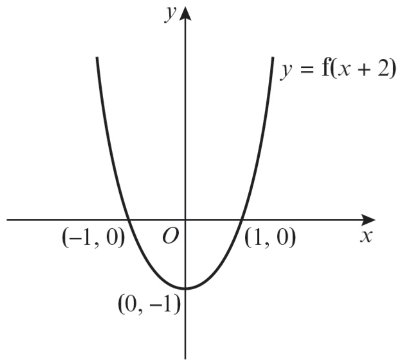


b $-\mathrm{f}(x)$ is a reflection in the x-axis.

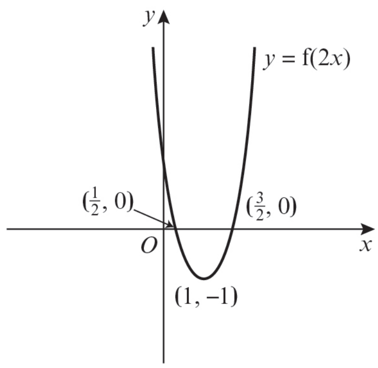


7 a Let $y = a(x - p)(x - q)$
Since $(1, 0)$ and $(3, 0)$ are on the curve then $p = 1$ and $q = 3$.
So $y = a(x - 1)(x - 3)$
Using $(2, -1)$:
$-1 = a(1)(-1)$
$a = 1$
So $y = (x - 1)(x - 3) = x^2 - 4x + 3$

b i $\mathrm{f}(x + 2) = (x + 1)(x - 1)$, or a translation by $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$, or two units to the left.



ii $\mathrm{f}(2x) = (2x - 1)(2x - 3)$, or a stretch with scale factor $\frac{1}{2}$ in the x-direction.



8 a $\mathrm{f}(x) = (x - 1)(x - 2)(x + 1)$
When $x = 0$, $y = (-1) \times (-2) \times 1 = 2$
So the curve crosses the y-axis at $(0, 2)$.

b $y = a\mathrm{f}(x)$ is a stretch with scale factor a in the y-direction.
The y-coordinate has multiplied by -2, therefore $y = -2\mathrm{f}(x)$.
$a = -2$

8 c $f(x) = (x-1)(x-2)(x+1)$
$0 = (x-1)(x-2)(x+1)$
So $x = 1$, $x = 2$ or $x = -1$
The curve crosses the x-axis at (1, 0), (2, 0) and (−1, 0).
$y = f(x + b)$ is a translation b units to the left.
For the point (0, 0) to lie on the translated curve, either the point (1, 0), (2, 0) or (−1, 0) has translated to the point (1, 0).
For the coordinate (1, 0) to be translated to (0, 0), $b = 1$.
For the coordinate (2, 0) to be translated to (0, 0), $b = 2$.
For the coordinate (−1, 0) to be translated to (0, 0), $b = -1$.
$b = -1$, $b = 1$ or $b = 2$

9 a i $y = f(3x)$ is a stretch with scale factor $\frac{1}{3}$ in the x-direction. Find $\frac{1}{3}$ of the x-coordinate.
P is transformed to $\left(\frac{4}{3}, 3\right)$.

ii $\frac{1}{2}y = f(x)$
$y = 2f(x)$, which is a stretch with scale factor 2 in the y-direction.
P is transformed to (4, 6).

iii $y = f(x - 5)$ is a translation by $\begin{pmatrix}5\\0\end{pmatrix}$, or five units to the right.
P is transformed to (9, 3).

iv $-y = f(x)$
$y = -f(x)$, which is a reflection of the curve in the x-axis.
(4, −3)

v $2(y + 2) = f(x)$
$y = \frac{1}{2}f(x) - 2$, which is a stretch with scale factor $\frac{1}{2}$ in the y-direction and then a translation by $\begin{pmatrix}0\\-2\end{pmatrix}$, or two units down.
P is transformed to $\left(4, -\frac{1}{2}\right)$.

9 b $P(4, 3)$ is transformed to (2, 3).
Either the x-coordinate has halved, which is a stretch with scale factor $\frac{1}{2}$ in the x-direction, or it has had 2 subtracted from it, which is a translation by $\begin{pmatrix}-2\\0\end{pmatrix}$, or two units to the left.

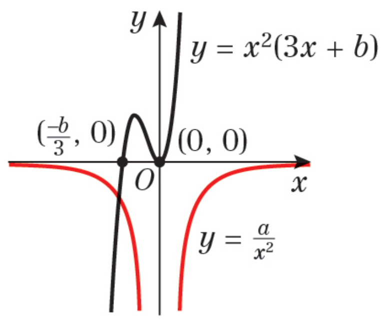
So the transformation is $y = f(2x)$ or $y = f(x + 2)$.

c i $P(4, 3)$ is translated to the point (8, 6).
The x-coordinate of P has 4 added to it and the y-coordinate has 3 added to it.
$y = f(x - 4) + 3$

ii $P(4, 3)$ is stretched to the point (8, 6).
The x-coordinate of P has doubled and the y-coordinate has doubled.
$y = 2f\left(\frac{1}{2}x\right)$

10 a $y = -\dfrac{a}{x^2}$ is a $y = \dfrac{k}{x^2}$ graph with $k < 0$.
x^2 is always positive and $k < 0$, so the y-values are all negative.

$y = x^2(3x + b)$
$0 = x^2(3x + b)$
So $x = 0$ or $x = -\dfrac{b}{3}$
The curve crosses the x-axis at $\left(-\dfrac{b}{3}, 0\right)$ and touches it at (0, 0).
$x \to \infty$, $y \to \infty$
$x \to -\infty$, $y \to -\infty$



10 b From the sketch, there is only one point of intersection of the curves. This means there is only one value of x where

$$-\frac{a}{x^2} = x^2(3x + b)$$
$$-a = x^4(3x + b)$$

$x^4(3x + b) + a = 0$

So this equation has one real solution.

11 a $x^3 - 6x^2 + 9x$

$= x(x^2 - 6x + 9)$

$= x(x - 3)^2$

b $y = x(x - 3)^2$

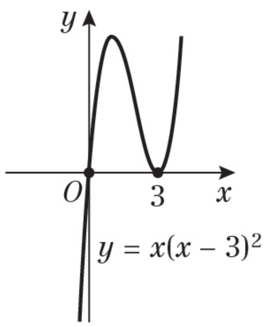
$0 = x(x - 3)^2$

So $x = 0$ or $x = 3$

The curve crosses the x-axis at (0, 0) and touches it at (3, 0).

$x \to \infty, y \to \infty$

$x \to -\infty, y \to -\infty$



c $y = (x - k)^3 - 6(x - k)^2 + 9(x - k)$ is a translation of the curve $y = x^3 - 6x^2 + 9x$ by $\begin{pmatrix} k \\ 0 \end{pmatrix}$, or k units to the right.

For the point (−4, 0) to lie on the translated curve, either the point (0, 0) or (3, 0) has translated to the point (−4, 0).

For the coordinate (0, 0) to be translated to (−4, 0), $k = -4$.

For the coordinate (3, 0) to be translated to (−4, 0), $k = -7$.

$k = -4$ or $k = -7$

12 a $y = x(x - 2)^2$

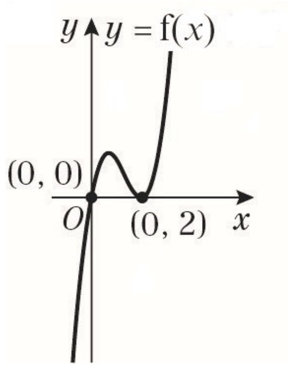
$0 = x(x - 2)^2$

So $x = 0$ or $x = 2$

The curve crosses the x-axis at (0, 0) and touches it at (2, 0).

$x \to \infty, y \to \infty$

$x \to -\infty, y \to -\infty$



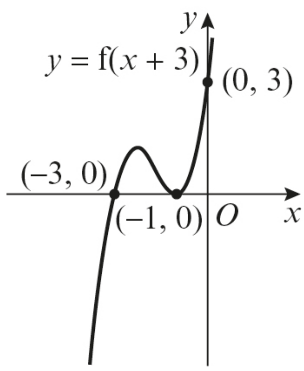
b $y = f(x + 3)$ is a translation by vector $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$ of $y = f(x)$, or three units to the left.

So the curve crosses the x-axis at (−3, 0) and touches it at (−1, 0).

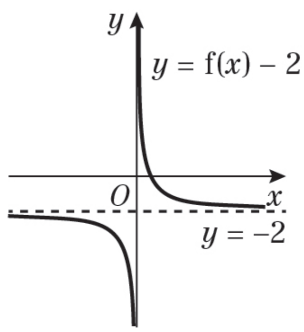
When $x = 3$, $f(x) = 3(3 - 2)^2$

$= 3$

So $f(x + 3)$ crosses the y-axis at (0, 3).



13 a $y = f(x) - 2$ is a translation by $\begin{pmatrix} 0 \\ -2 \end{pmatrix}$, or two units down.



13 a The horizontal asymptote is $y = -2$.
The vertical asymptote is $x = 0$.

b From the sketch, the curve crosses the x-axis.

$y = f(x) - 2$

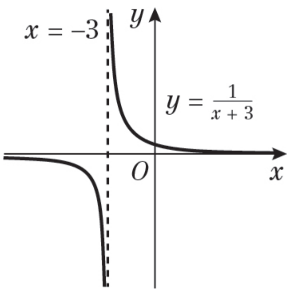
$= \frac{1}{x} - 2$

$0 = \frac{1}{x} - 2$

$x = \frac{1}{2}$

So the curve cuts the x-axis at $\left(\frac{1}{2}, 0\right)$.

c $y = f(x + 3)$ is a translation by $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$, or three units to the left.



d The horizontal asymptote is $y = 0$.
The vertical asymptote is $x = -3$.

$y = f(x + 3)$

$= \frac{1}{x+3}$

When $x = 0$, $y = \frac{1}{3}$

So the curve cuts the y-axis at $\left(0, \frac{1}{3}\right)$.

Challenge

$R(6, -4)$

$y = f(x + c) - d$ is a translation by $\begin{pmatrix} -c \\ 0 \end{pmatrix}$, or c units to the left and a translation by $\begin{pmatrix} 0 \\ -d \end{pmatrix}$, or d units down.

So R is transformed to $(6 - c, -4 - d)$.

Review Exercise 1

1 a $8^{\frac{1}{3}}$

Use $a^{\frac{1}{m}} = \sqrt[m]{a}$, so $a^{\frac{1}{3}} = \sqrt[3]{a}$

$= \sqrt[3]{8}$

$= 2$

b $8^{-\frac{2}{3}} = \dfrac{1}{8^{\frac{2}{3}}}$ $\left(\text{Use } a^{-m} = \dfrac{1}{a^m}\right)$

First find $8^{\frac{2}{3}}$ $a^{\frac{n}{m}} = \sqrt[m]{(a^n)}$ or $\left(\sqrt[m]{a}\right)^n$

$8^{\frac{2}{3}} = \left(\sqrt[3]{8}\right)^2$

$8^{\frac{2}{3}} = 2^2 = 4$

$8^{-\frac{2}{3}} = \dfrac{1}{8^{\frac{2}{3}}}$

$= \frac{1}{4}$

2 a $125^{\frac{4}{3}}$

$a^{\frac{n}{m}} = \sqrt[m]{(a^n)}$ or $\left(\sqrt[m]{a}\right)^n$

$= \left(\sqrt[3]{125}\right)^4$

$= 5^4$

$= 625$

b $24x^2 \div 18x^{\frac{4}{3}}$

(Use $a^m \div a^n = a^{m-n}$)

$= \dfrac{24x^2}{18x^{\frac{4}{3}}} = \dfrac{4x^2}{3x^{\frac{4}{3}}}$ cancelling by 6

$= \dfrac{4x^{\frac{2}{3}}}{3}$ because $2 - \dfrac{4}{3} = \dfrac{2}{3}$

3 a $\sqrt{80}$

Use $\sqrt{(bc)} = \sqrt{b}\sqrt{c}$

$= \sqrt{16} \times \sqrt{5}$

$= 4\sqrt{5}$ $(a = 4)$

b $\left(4-\sqrt{5}\right)^2 = \left(4-\sqrt{5}\right)\left(4-\sqrt{5}\right)$

$= 4\left(4-\sqrt{5}\right) - \sqrt{5}\left(4-\sqrt{5}\right)$

$= 16 - 4\sqrt{5} - 4\sqrt{5} + 5$

$= 21 - 8\sqrt{5}$

$(b = 21$ and $c = -8)$

4 a $\left(4+\sqrt{3}\right)\left(4-\sqrt{3}\right)$

$= 4\left(4-\sqrt{3}\right) + \sqrt{3}\left(4-\sqrt{3}\right)$

$= 16 - 4\sqrt{3} + 4\sqrt{3} - 3$

$= 13$

b $\dfrac{26}{4+\sqrt{3}} \times \dfrac{4-\sqrt{3}}{4-\sqrt{3}} = \dfrac{26\left(4-\sqrt{3}\right)}{\left(4+\sqrt{3}\right)\left(4-\sqrt{3}\right)}$

$= \dfrac{26\left(4-\sqrt{3}\right)}{13}$

$= 2\left(4-\sqrt{3}\right)$

$= 8 - 2\sqrt{3}$

$(a = 8$ and $b = -2)$

5 a mean $= \dfrac{1-\sqrt{k}+2+5\sqrt{k}+2\sqrt{k}}{3}$

$= \dfrac{3+6\sqrt{k}}{3}$

$= 1 + 2\sqrt{k}$

5 b range $= 2+5\sqrt{k}-(1-\sqrt{k})$
$= 1+6\sqrt{k}$

6 a $y^{-1}=\left(\frac{1}{25}x^4\right)^{-1}$
$=\dfrac{1}{\frac{1}{25}x^4}$
$=\dfrac{25}{x^4}$
$=25x^{-4}$

b $5y^{\frac{1}{2}}=5\left(\frac{1}{25}x^4\right)^{\frac{1}{2}}$
$=5\left(\frac{1}{5}x^2\right)$
$=x^2$

7 Area $=\frac{1}{2}h(a+b)$
$=\frac{1}{2}\left(2\sqrt{2}\right)\left(3+\sqrt{2}+5+3\sqrt{2}\right)$
$=\sqrt{2}\left(8+4\sqrt{2}\right)$
$=8\sqrt{2}+8$

The area of the trapezium is
$8+8\sqrt{2}$ cm^2.

8 $\dfrac{p+q}{p-q}=\dfrac{(3-2\sqrt{2})+(2-\sqrt{2})}{(3-2\sqrt{2})-(2-\sqrt{2})}$
$=\dfrac{5-3\sqrt{2}}{1-\sqrt{2}}$
$=\dfrac{(5-3\sqrt{2})}{(1-\sqrt{2})}\times\dfrac{(1+\sqrt{2})}{(1+\sqrt{2})}$
$=\dfrac{5+5\sqrt{2}-3\sqrt{2}-6}{1+\sqrt{2}-\sqrt{2}-2}$
$=\dfrac{-1+2\sqrt{2}}{-1}$
$=1-2\sqrt{2}\ (m=1, n=-2)$

9 a $x^2-10x+16=(x-8)(x-2)$

b Let $x=8^y$
$8^{2y}-10(8^y)+16=(8^y-8)(8^y-2)=0$
So $8^y=8$ or $8^y=2$
$y=1$ or $y=\frac{1}{3}$

10 a $x^2-8x=(x-4)^2-16$
Complete the square for $x^2-8x-29$
$x^2-8x-29=(x-4)^2-16-29$
$=(x-4)^2-45$
$(a=-4 \text{ and } b=-45)$

b $x^2-8x-29=0$
$(x-4)^2-45=0$
Use the result from part **a**:
$(x-4)^2=45$
Take the square root of both sides:
$x-4=\pm\sqrt{45}$
$x=4\pm\sqrt{45}$
$\sqrt{45}=\sqrt{9}\times\sqrt{5}=3\sqrt{5}$
since $\sqrt{(ab)}=\sqrt{a}\sqrt{b}$
Roots are $4\pm3\sqrt{5}$
$(c=4 \text{ and } d=\pm3)$

11 $f(a)=a(a-2)$ and $g(a)=a+5$
$a(a-2)=a+5$
$a^2-2a-a-5=0$
$a^2-3a-5=0$
Using the quadratic formula:
$a=\dfrac{3\pm\sqrt{(-3)^2-4(1)(-5)}}{2(1)}$
$=\dfrac{3\pm\sqrt{29}}{2}$
$=4.19$ or -1.19
As $a>0$, $a=4.19$ (3 s.f.)

12 a The height of the athlete's shoulder above the ground is 1.7 m

b $1.7 + 10t - 5t^2 = 0$
Using the quadratic formula when $a = -5$, $b = 10$ and $c = 1.7$

$$t = \frac{-10 \pm \sqrt{(10)^2 - 4(-5)(1.7)}}{2(-5)}$$
$$= \frac{-10 \pm \sqrt{134}}{-10}$$
$$= -0.16 \text{ or } 2.16$$
As $t > 0$, $t = 2.16$ s (3 s.f.)

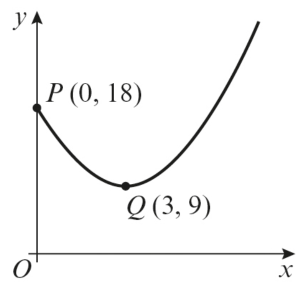
c $1.7 + 10t - 5t^2 = 1.7 - 5(t^2 - 2t)$
$= 1.7 - 5((t - 1)^2 - 1)$
$= 1.7 - 5(t - 1)^2 + 5$
$= 6.7 - 5(t - 1)^2$
$A = 6.7$, $B = 5$ and $C = 1$

d Maximum when $(t - 1) = 0$, $t = 1$s and maximum height = 6.7 m

13 a $\mathrm{f}(x) = x^2 - 6x + 18$
$x^2 - 6x = (x-3)^2 - 9$
Complete the square for $x^2 - 6x + 18$
$x^2 - 6x + 18 = (x-3)^2 - 9 + 18$
$= (x-3)^2 + 9$
$a = 3$ and $b = 9$

b $y = x^2 - 6x + 18$
$y = (x-3)^2 + 9$
$(x-3)^2 \geq 0$
Squaring a number cannot give a negative result.
The minimum value of $(x - 3)^2$ is 0, when $x = 3$.
So the minimum value of y is $0 + 9 = 0$, when $x = 3$.
Q is the point (3, 9).
The curve crosses the y-axis where $x = 0$.

13 b For $x = 0$, $y = 18$
P is the point (0, 18).
The graph of $y = x^2 - 6x + 18$ is a $\cup$ shape.
For $y = ax^2 + bx + c$, if $a > 0$, the shape is $\cup$.



Use the information about P and Q to sketch the curve $x \geq 0$, so the part where $x < 0$ is not needed.

c $y = (x-3)^2 + 9$
Put $y = 41$ into the equation of C.
$41 = (x-3)^2 + 9$
Subtract 9 from both sides.
$32 = (x-3)^2$
$(x-3)^2 = 32$
Take the square root of both sides.
$x - 3 = \pm\sqrt{32}$
$x = 3 \pm \sqrt{32}$
$\sqrt{32} = \sqrt{16} \times \sqrt{2} = 4\sqrt{2}$
using $\sqrt{(ab)} = \sqrt{a}\sqrt{b}$
$x = 3 \pm 4\sqrt{2}$
x-coordinate of R is $3 + 4\sqrt{2}$
The other value is $3 - 4\sqrt{2}$ which is less than 0, so is not needed.

14 a Using the discriminant
$b^2 - 4ac = 0$ for equal roots
$(2\sqrt{2})^2 - 4(1)(k) = 0$
$8 - 4k = 0$
$k = 2$

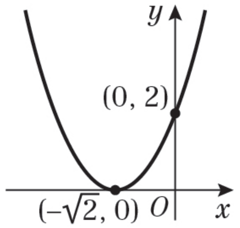
14 b $y = x^2 + 2\sqrt{2}x + 2$

$= (x + \sqrt{2})^2$

When $y = 0$, $(x + \sqrt{2})^2 = 0$

$x = -\sqrt{2}$

When $x = 0$, $y = 2$



15 a $g(x) = x^9 - 7x^6 - 8x^3$

$= x^3(x^6 - 7x^3 - 8)$

To factorise $x^6 - 7x^3 - 8$, let $y = x^3$

$y^2 - 7y - 8 = (y + 1)(y - 8)$

So $g(x) = x^3(x^3 + 1)(x^3 - 8)$

$a = 1$, $b = -8$

b $g(x) = x^3(x^3 + 1)(x^3 - 8) = 0$

$x^3 = 0$, $x^3 = -1$ or $x^3 = 8$

$x = -1$, $x = 0$ or $x = 2$

16 a $x^2 + 10x + 36$

$x^2 + 10x = (x + 5)^2 - 25$

Complete the square for $x^2 + 10x + 36$

$x^2 + 10x + 36 = (x - 5)^2 - 25 + 36$

$= (x + 5)^2 + 11$

$a = 5$ and $b = 11$

b $x^2 + 10x + 36 = 0$

$(x + 5)^2 + 11 = 0$

'Hence' implied in part **a** must be used

$(x + 5)^2 = -11$

A real number squared cannot be negative. There are no real roots.

c $x^2 + 10x + k = 0$

$a = 1$, $b = 10$, $c = k$

16 c For equal roots, $b^2 = 4ac$

$10^2 = 4 \times 1 \times k$

$4k = 100$

$k = 25$

d The graph of $x^2 + 10x + 25$ is a $\cup$ shape.

For $y = ax^2 + bx + c$, if $a > 0$, the shape is $\cup$.

$x = 0$: $y = 0 + 0 + 25 = 25$

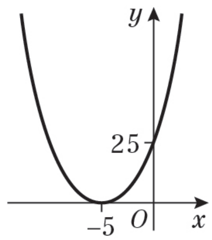
Meets y-axis at (0, 25).

$y = 0: x^2 + 10x + 25 = 0$

$(x + 5)(x + 5) = 0$

$x = -5$

Meets x-axis at (−5, 0).



The graph meets the x-axis at just one point, so it 'touches' the x-axis.

17 a $x^2 + 2x + 3$

$x^2 + 2x = (x + 1)^2 - 1$

Complete the square for $x^2 + 2x + 3$

$x^2 + 2x + 3 = (x + 1)^2 - 1 + 3$

$= (x + 1)^2 + 2$

$a = 1$ and $b = 2$

b The graph of $y = x^2 + 2x + 3$ is a $\cup$ shape.

For $y = ax^2 + bx + c$, if $a > 0$, the shape is $\cup$.

$x = 0 : y = 0 + 0 + 3$

Put $x = 0$ to find the intersection with the y-axis:

Meets y-axis at (0, 3).

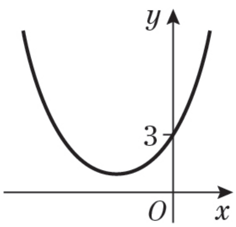
17 b Put $y = 0$ to find the intersection with the x-axis:

$y = 0 : x^2 + 2x + 3 = 0$

$(x + 1)^2 + 2 = 0$

$(x + 1)^2 = -2$

A real number squared cannot be negative, therefore, no real roots, so not intersection with the x-axis.



c $x^2 + 2x + 3$

$a = 1, b = 2, c = 3$

$b^2 - 4ac = 2^2 - 4 \times 1 \times 3$

$= -8$

Since the discriminant is negative, the equation has no real roots, so the graph does not cross the x-axis.

d $x^2 + kx + 3 = 0$

$a = 1, b = k, c = 3$

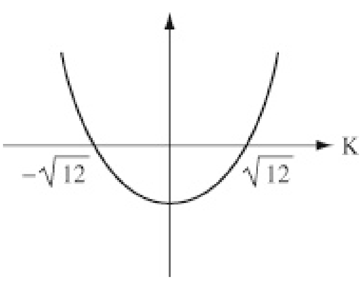
For no real roots, $b^2 < 4ac$

$k^2 < 12$

$k^2 - 12 < 0$

$\left(k+\sqrt{12}\right)\left(k-\sqrt{12}\right)<0$

This is a quadratic inequality with critical values $-\sqrt{12}$ and $\sqrt{12}$.



Critical values:

$k = -\sqrt{12}$, $k = \sqrt{12}$

$-\sqrt{12} < k < \sqrt{12}$

17 d The surds can be simplified using $\sqrt{(ab)} = \sqrt{a}\sqrt{b}$

$\sqrt{12} = \sqrt{4} \times \sqrt{3} = 2\sqrt{3}$

$\left(-2\sqrt{3} < k < 2\sqrt{3}\right)$

18 a $2x^2 - x(x-4) = 8$

$2x^2 - x^2 + 4x = 8$

$x^2 + 4x - 8 = 0$

b $x^2 + 4x - 8 = 0$

$x^2 + 4x = (x+2)^2 - 4$

$(x+2)^2 - 4 - 8 = 0$

$(x+2)^2 = 12$

$x + 2 = \pm\sqrt{12}$

$x = -2 \pm \sqrt{12}$

$\sqrt{12} = \sqrt{4} \times \sqrt{3} = 2\sqrt{3}$

$x = -2 \pm 2\sqrt{3}$

$a = -2$ and $b = 2$

Using $y = x - 4$:

$y = \left(-2 \pm 2\sqrt{3}\right) - 4$

$= -6 \pm 2\sqrt{3}$

Solution: $x = -2 \pm 2\sqrt{3}$

$y = -6 \pm 2\sqrt{3}$

19 a $3(2x+1) > 5 - 2x$

$6x + 3 > 5 - 2x$

$6x + 2x + 3 > 5$

$8x > 2$

$x > \frac{1}{4}$

b $2x^2 - 7x + 3 = 0$

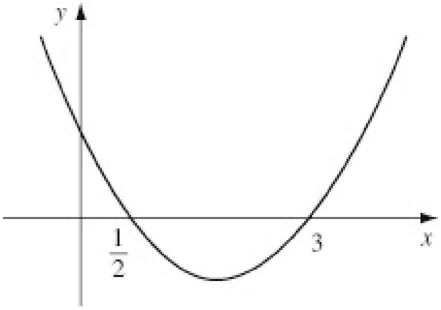
$(2x - 1)(x - 3) = 0$

$(2x - 1) = 0$

$(x - 3) = 0$

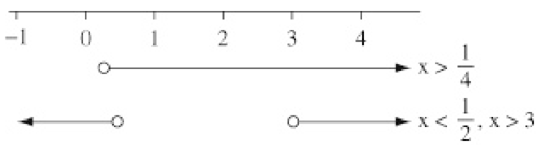
$x = \frac{1}{2}$ or $x = 3$

19 b



$2x^2 - 7x + 3 > 0$ where
$x < \frac{1}{2}$ or $x > 3$

c



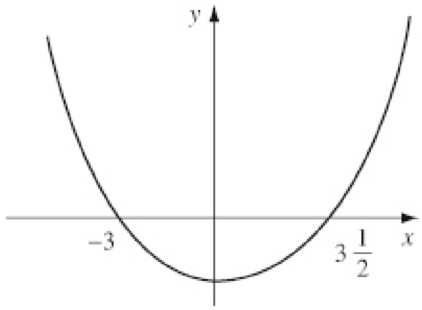
$\frac{1}{4} < x < \frac{1}{2}$ or $x > 3$

20 $-2(x+1) = x^2 - 5x + 2$
$-2x - 2 = x^2 - 5x + 2$
$x^2 - 3x + 4 = 0$
Using the discriminant
$b^2 - 4ac = (-3)^2 - 4(1)(4) = -7$
As $b^2 - 4ac < 0$, there are no real roots.
Hence there is no value of x for which $\mathrm{p}(x) = \mathrm{q}(x)$.

21 a $y = 5 - 2x$
$2x^2 - 3x - (5 - 2x) = 16$
$2x^2 - 3x - 5 + 2x = 16$
$2x^2 - x - 21 = 0$
$(2x - 7)(x + 3) = 0$
$x = 3\frac{1}{2}, \ x = -3$
$x = 3\frac{1}{2}: \ y = 5 - 7 = -2$
$x = -3: y = 5 + 6 = 11$

Solution $x = 3\frac{1}{2}, \ y = -2$
and $x = -3, \ y = 11$

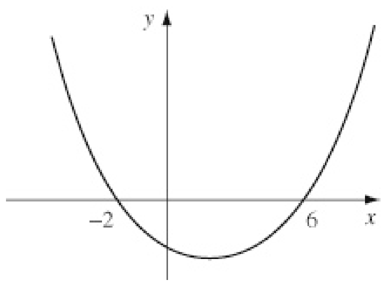
21 b The equation in part **a** could be written as $y = 5 - 2x$ and $y = 2x^2 - 3x - 16$.
Therefore, the solution to $2x^2 - 3x - 16 = 5 - 2x$ are the same as for part **a**.
These are the critical values for $2x^2 - 3x - 16 > 5 - 2x$:
$x = 3\frac{1}{2}$ and $x = -3$.
$2x^2 - 3x - 16 > 5 - 2x$
$(2x^2 - 3x - 16 - 5 + 2x > 0)$
$2x^2 - x - 21 > 0$



$x < -3$ or $x > 3\frac{1}{2}$

22 a $x^2 + kx + (k+3) = 0$
$a = 1, \ b = k, \ c = k + 3$
$b^2 > 4ac$
$k^2 > 4(k+3)$
$k^2 > 4k + 12$
$k^2 - 4k - 12 > 0$

b $k^2 - 4k - 12 = 0$
$(k+2)(k-6) = 0$
$k = -2, \ k = 6$

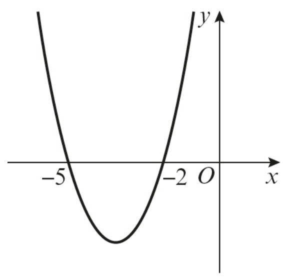


$k^2 - 4k - 12 > 0$ where $k < -2$ or $k > 6$

23 $\dfrac{6}{x+5} < 2$

Multiply both sides by $(x + 5)^2$
$6(x + 5) < 2(x + 5)^2$
$6x + 30 < 2x^2 + 20x + 50$
$2x^2 + 14x + 20 > 0$

Solve the quadratic to find the critical values.
$2x^2 + 14x + 20 = 0$
$2(x^2 + 7x + 10) = 0$
$2(x + 5)(x + 2) = 0$
$x = -5$ or $x = -2$



The solution is $x < -5$ or $x > -2$.

24 a $9 - x^2 = 0$
$(3 + x)(3 - x) = 0$
$x = -3$ or $x = 3$
When $x = 0$, $y = 9$

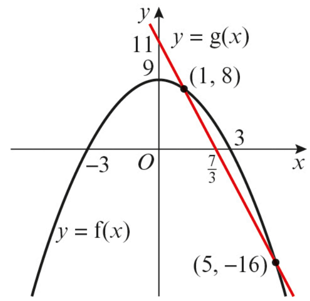
To work out the points of intersection, solve the equations simultaneously.
$9 - x^2 = 14 - 6x$
$x^2 - 6x + 5 = 0$
$(x - 5)(x - 1) = 0$
$x = 1$ or $x = 5$

When $x = 1$, $y = 8$
When $x = 5$, $y = -16$

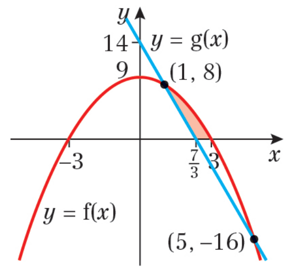
Let $14 - 6x = 0$
$x = \dfrac{14}{6} = \dfrac{7}{3}$

The line crosses the x-axis at $\left(\dfrac{7}{3}, 0\right)$.

24 a

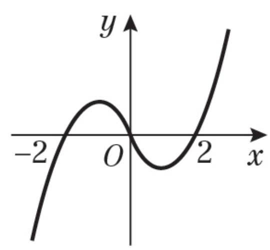


b



25 a $x^3 - 4x = x(x^2 - 4)$
$= x(x+2)(x-2)$

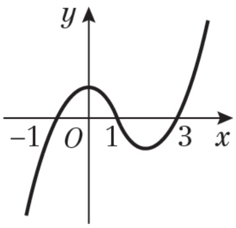
b Curve crosses the x-axis where $y = 0$
$x(x+2)(x-2) = 0$
$x = 0,\ x = -2,\ x = 2$
When $x = 0,\ y = 0$
When $x \to \infty,\ y \to \infty$
When $x \to -\infty,\ y \to -\infty$



Crosses the y-axis at (0, 0).
Crosses the x-axis at (−2, 0), (2, 0).

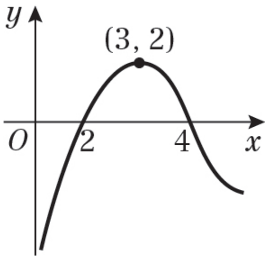
c $y = x^3 - 4x$
$y = (x-1)^3 - 4(x-1)$
This is a translation of +1 in the x-direction.

25 c



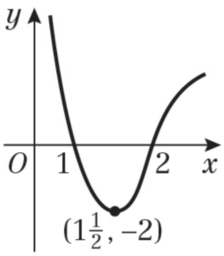
Crosses the x-axis at (−1, 0), (1, 0) and (3, 0).

26 a



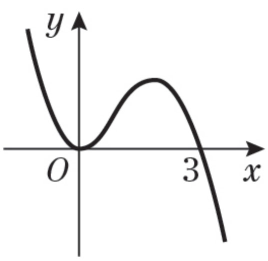
Crosses the x-axis at (2, 0), (4, 0).
Image of P is (3, 2).

26 b



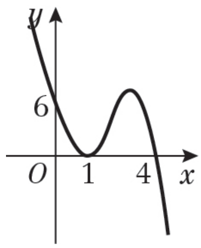
Crosses the x-axis at (1, 0), (2, 0).
Image of P is $\left(1\frac{1}{2}, -2\right)$.

27 a



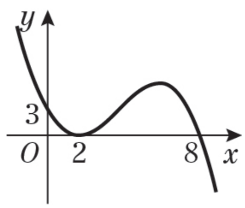
Meets the y-axis at (0, 0).
Meets the x-axis at (0, 0), (3, 0).

27 b



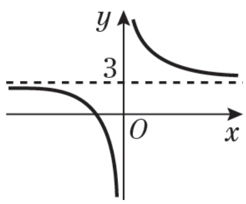
Meets the y-axis at (0, 6).
Meets the x-axis at (1, 0), (4, 0).

c



Meets the y-axis at (0, 3).
Meets the x-axis at (2, 0), (8, 0).

28 a



$y = 3$ is an asymptote.
$x = 0$ is an asymptote.

b The graph does not cross the y-axis (see sketch in part **a**).
Crosses the x-axis where $y = 0$:

$$\frac{1}{x} + 3 = 0$$

$$\frac{1}{x} = -3$$

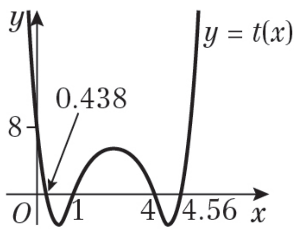
$$x = -\tfrac{1}{3}, \left(-\tfrac{1}{3}, 0\right)$$

29 a $(x^2 - 5x + 2)(x^2 - 5x + 4) = 0$
For $x^2 - 5x + 2 = 0$

Using the quadratic formula:
$$x = \frac{5 \pm \sqrt{(-5)^2 - 4(1)(2)}}{2(1)},$$
$$= \frac{5 \pm \sqrt{17}}{2}$$
$x = 4.56$ or $x = 0.438$

For $x^2 - 5x + 4 = 0$
$(x - 1)(x - 4) = 0$
$x = 1$ or $x = 4$
$x = 0.438, 1, 4$ or 4.56

b When $x = 0, y = 8$



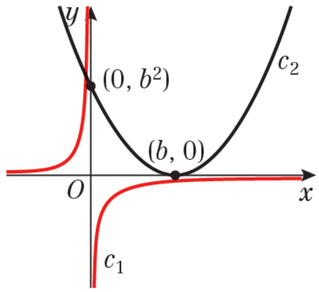
30 a $y = -\text{f}(x)$ is a reflection in the x-axis of $y = \text{f}(x)$, so P is transformed to $(6, 8)$.

b $y = \text{f}(x - 3)$ is a translation 3 units to the right of $y = \text{f}(x)$, so P is transformed to $(9, -8)$.

c $2y = \text{f}(x)$ is $y = \frac{1}{2}\text{f}(x)$ which is a vertical stretch scale factor $\frac{1}{2}$ of $y = \text{f}(x)$, so P is transformed to $(6, -4)$.

31 a $y = -\dfrac{a}{x}$ is the curve $y = \dfrac{k}{x}, k < 0$
$y = (x - b)^2$ is a translation, b units to the right of the curve $y = x^2$
When $x = 0, y = b^2$
When $y = 0, x = b$

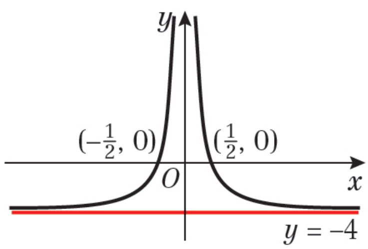
31 a



b The graphs intersect at 1 point, so have 1 point of intersection.

32 a $y = \dfrac{1}{x^2} - 4$ is a translation $\begin{pmatrix} 0 \\ -4 \end{pmatrix}$ of

$y = \dfrac{1}{x^2}$



b When $y = \dfrac{1}{(x+k)^2} - 4$ passes through the origin, $x = 0$ and $y = 0$.

So $\dfrac{1}{k^2} - 4 = 0$
$$\frac{1}{k^2} = 4$$
$$k = \pm\frac{1}{2}$$

Challenge

1 a $x^2 - 10x + 9 = 0$
$(x - 1)(x - 9) = 0$
$x = 1$ or $x = 9$

b $3^{x-2}(3^x - 10) = -1$
$3^{2x-2} - 10 \times 3^{x-2} + 1 = 0$
Multiply by 3^2
$3^{2x} - 10 \times 3^x + 9 = 0$
Let $y = 3^x$
$y^2 - 10y + 9 = 0$
Using your answers from part **a**
$y = 1$ or 9
$3^x = 1$ or $3^x = 9$
$x = 0$ or 2

2 Let x and y be the length and width of the rectangle.
Area $= xy = 6$
Perimeter $= 2x + 2y = 8\sqrt{2}$
$2y = 8\sqrt{2} - 2x$
$y = 4\sqrt{2} - x$
Solving simultaneously
$x\left(4\sqrt{2} - x\right) = 6$
$x^2 - 4\sqrt{2}x + 6 = 0$

Using the quadratic formula

$$x = \frac{4\sqrt{2} \pm \sqrt{(4\sqrt{2})^2 - 4(1)(6)}}{2(1)}$$

$$= \frac{4\sqrt{2} \pm \sqrt{8}}{2}$$

$$= \frac{4\sqrt{2} \pm 2\sqrt{2}}{2}$$

$x = \sqrt{2}$ or $x = 3\sqrt{2}$

When $x = \sqrt{2}$, $y = 3\sqrt{2}$
When $x = 3\sqrt{2}$, $y = \sqrt{2}$

The dimensions of the rectangle are $\sqrt{2}$ cm and $3\sqrt{2}$ cm.

3 Solving simultaneously
$3x^3 + x^2 - x = 2x(x - 1)(x + 1)$
$3x^3 + x^2 - x = 2x(x^2 - 1)$
$3x^3 + x^2 - x = 2x^3 - 2x$
$x^3 + x^2 + x = 0$
$x(x^2 + x + 1) = 0$

The discriminant of $x^2 + x + 1$
$b^2 - 4ac = 1^2 - 4(1)(1) = -3$
$-3 < 0$ so there are no real solutions for $x^2 + x + 1$

The only solution is when $x = 0$ at $(0, 0)$.

4 $f(x) = (x^2 + x - 20)(x^2 + x - 2)$
$\quad = (x + 5)(x - 4)(x + 2)(x - 1)$
when $f(x) = 0$
$x = -5, -2, 1$ or 4

$g(x - k) = (x - k + 5)(x - k - 4)$
$\qquad (x - k + 2)(x - k - 1)$
When $k = 3$,
$g(x - 3) = (x + 2)(x - 7)(x - 1)(x - 4)$

$(x + 2)$, $(x - 1)$ and $(x - 4)$ match
When $k = -3$, $g(x + 3)$
$\quad = (x + 8)(x - 1)(x + 5)(x + 2)$

$(x - 1)$, $(x + 5)$ and $(x + 2)$ match
So $k = -3$ or 3.

5 $16m^2 = 54\sqrt{m}$

$$\frac{m^2}{\sqrt{m}} = \frac{54}{16}$$

$$m^{\frac{3}{2}} = \frac{27}{8}$$

$$\left(m^{\frac{3}{2}}\right)^{\frac{2}{3}} = \left(\frac{27}{8}\right)^{\frac{2}{3}}$$

$$m = \frac{9}{4}$$

5 When $m = 0$, $16m^2 = 0$ and $54\sqrt{m}$

So, real solutions to $16m^2 = 54\sqrt{m}$

are $m = \frac{9}{4}$ and $m = 0$

Straight line graphs 5A

1 a $(x_1, y_1) = (4, 2)$, $(x_2, y_2) = (6, 3)$

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{3-2}{6-4} = \frac{1}{2}$$

b $(x_1, y_1) = (-1, 3)$, $(x_2, y_2) = (5, 4)$

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{4-3}{5-(-1)} = \frac{1}{6}$$

c $(x_1, y_1) = (-4, 5)$, $(x_2, y_2) = (1, 2)$

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{2-5}{1-(-4)} = -\frac{3}{5}$$

d $(x_1, y_1) = (2, -3)$, $(x_2, y_2) = (6, 5)$

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{5-(-3)}{6-2} = \frac{8}{4} = 2$$

e $(x_1, y_1) = (-3, 4)$, $(x_2, y_2) = (7, -6)$

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{-6-4}{7-(-3)} = \frac{-10}{10} = -1$$

f $(x_1, y_1) = (-12, 3)$, $(x_2, y_2) = (-2, 8)$

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{8-3}{-2-(-12)} = \frac{1}{2}$$

g $(x_1, y_1) = (-2, -4)$, $(x_2, y_2) = (10, 2)$

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{2-(-4)}{10-(-2)} = \frac{6}{12} = \frac{1}{2}$$

h $(x_1, y_1) = \left(\frac{1}{2}, 2\right), (x_2, y_2) = \left(\frac{3}{4}, 4\right)$

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{4-2}{\frac{3}{4}-\frac{1}{2}} = \frac{2}{\frac{1}{4}} = 8$$

i $(x_1, y_1) = \left(\frac{1}{4}, \frac{1}{2}\right), (x_2, y_2) = \left(\frac{1}{2}, \frac{2}{3}\right)$

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{\frac{2}{3}-\frac{1}{2}}{\frac{1}{2}-\frac{1}{4}} = \frac{\frac{1}{6}}{\frac{1}{4}} = \frac{2}{3}$$

j $(x_1, y_1) = (-2.4, 9.6)$, $(x_2, y_2) = (0, 0)$

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{0-9.6}{0-(-2.4)} = \frac{-9.6}{2.4} = -4$$

k $(x_1, y_1) = (1.3, -2.2)$,
$(x_2, y_2) = (8.8, -4.7)$

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{-4.7-(-2.2)}{8.8-1.3}$$

2 k $\dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{-2.5}{7.5}$

$= -\dfrac{1}{3}$

l $(x_1, y_1) = (0, 5a)$, $(x_2, y_2) = (10a, 0)$

$\dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{0 - 5a}{10a - 0}$

$= \dfrac{-5a}{10a}$

$= \dfrac{-5}{10}$

$= -\dfrac{1}{2}$

m $(x_1, y_1) = (3b, -2b)$, $(x_2, y_2) = (7b, 2b)$

$\dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{2b - (-2b)}{7b - 3b}$

$= \dfrac{4b}{4b}$

$= 1$

n $(x_1, y_1) = (p, p^2)$, $(x_2, y_2) = (q, q^2)$

$\dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{q^2 - p^2}{q - p}$

$= \dfrac{(q - p)(q + p)}{q - p}$

$= q + p$

2 $(x_1, y_1) = (3, -5)$, $(x_2, y_2) = (6, a)$

$\dfrac{y_2 - y_1}{x_2 - x_1} = 4$

So $\dfrac{a - (-5)}{6 - 3} = 4$

$\Rightarrow \dfrac{a + 5}{3} = 4$

$\Rightarrow a + 5 = 12$

$\Rightarrow a = 7$

3 $(x_1, y_1) = (5, b)$, $(x_2, y_2) = (8, 3)$

$\dfrac{3 - b}{8 - 5} = -3$

$\dfrac{3 - b}{3} = -3$

$3 - b = -9$

$b = 12$

4 $(x_1, y_1) = (c, 4)$, $(x_2, y_2) = (7, 6)$

$\dfrac{6 - 4}{7 - c} = \dfrac{3}{4}$

$\dfrac{2}{7 - c} = \dfrac{3}{4}$

$2 = \dfrac{3}{4}(7 - c)$

$8 = 3(7 - c)$

$8 = 21 - 3c$

$-13 = -3c$

$c = \dfrac{-13}{-3}$

$= \dfrac{13}{3}$

$= 4\dfrac{1}{3}$

5 $(x_1, y_1) = (-1, 2d)$, $(x_2, y_2) = (1, 4)$

$\dfrac{4 - 2d}{1 - (-1)} = -\dfrac{1}{4}$

$\dfrac{4 - 2d}{2} = -\dfrac{1}{4}$

$2 - d = -\dfrac{1}{4}$

$2\dfrac{1}{4} - d = 0$

$d = 2\frac{1}{4}$

6 $(x_1, y_1) = (-3, -2)$, $(x_2, y_2) = (2e, 5)$

$$\frac{5-(-2)}{2e-(-3)} = 2$$
$$\frac{7}{2e+3} = 2$$
$$7 = 2(2e+3)$$
$$7 = 4e+6$$
$$4e = 1$$
$$e = \tfrac{1}{4}$$

7 $(x_1, y_1) = (7, 2)$, $(x_2, y_2) = (f, 3f)$

$$\frac{3f-2}{f-7} = 4$$
$$3f-2 = 4(f-7)$$
$$3f-2 = 4f-28$$
$$-2 = f-28$$
$$28-2 = f$$
$$f = 26$$

8 $(x_1, y_1) = (3, -4)$, $(x_2, y_2) = (-g, 2g)$

$$\frac{2g-(-4)}{-g-3} = -3$$
$$\frac{2g+4}{-g-3} = -3$$
$$2g+4 = -3(-g-3)$$
$$2g+4 = 3g+9$$
$$4 = g+9$$
$$g = -5$$

9 The gradient of AB is:

$$\frac{4-3}{4-2} = \frac{1}{2}$$

The gradient of AC is:

$$\frac{7-3}{10-2} = \frac{4}{8}$$
$$= \frac{1}{2}$$

The gradients are equal and there is a point in common between the two line segments so the points can be joined by a straight line.

10 If the points $A(-2a, 5a)$, $B(0, 4a)$ and $C(6a, a)$ are collinear, then they all lie on the same straight line.

The gradient of AB is:

$$\frac{4a-5a}{0-(-2a)} = \frac{-a}{2a}$$
$$= -\frac{1}{2}$$

The gradient of AC is:

$$\frac{a-5a}{6a-(-2a)} = \frac{-4a}{8a}$$
$$= -\frac{1}{2}$$

The gradients are both $-\frac{1}{2}$ and there is a point in common between the two line segments so the points are collinear.

Straight line graphs 5B

1 a The gradient = –2

b The gradient = –1

c The gradient = 3

d The gradient = $\frac{1}{3}$

e The gradient = $-\frac{2}{3}$

f The gradient = $\frac{5}{4}$

g
$$\begin{aligned} 2x-4y+5 &= \\ 2x+5 &= 4y \\ 4y &= 2x+5 \\ y &= \tfrac{2}{4}x+\tfrac{5}{4} \\ y &= \tfrac{1}{2}x+\tfrac{5}{4} \end{aligned}$$
The gradient $= \frac{1}{2}$

h
$$\begin{aligned} 10x-5y+1 &= 0 \\ 10x+1 &= 5y \\ 5y &= 10x+1 \\ y &= \tfrac{10}{5}x+\tfrac{1}{5} \\ y &= 2x+\tfrac{1}{5} \end{aligned}$$
The gradient $= 2$

i
$$\begin{aligned} -x+2y-4 &= 0 \\ 2y-4 &= x \\ 2y &= x+4 \\ y &= \tfrac{1}{2}x+2 \end{aligned}$$
The gradient $= \frac{1}{2}$

j
$$\begin{aligned} -3x+6y+7 &= 0 \\ 6y &= 3x-7 \\ y &= \tfrac{3}{6}x-\tfrac{7}{6} \\ y &= \tfrac{1}{2}x+\tfrac{7}{6} \end{aligned}$$
The gradient $= \frac{1}{2}$

k
$$\begin{aligned} 4x+2y-9 &= 0 \\ 2y-9 &= -4x \\ 2y &= -4x+9 \\ y &= -\frac{4}{2}x+\frac{9}{2} \\ y &= -2x+\frac{9}{2} \end{aligned}$$
The gradient $= -2$

l
$$\begin{aligned} 9x+6y+2 &= 0 \\ 6y+2 &= -9x \\ 6y &= -9x-2 \\ y &= -\tfrac{9}{6}x-\tfrac{2}{6} \\ y &= -\tfrac{3}{2}x-\tfrac{1}{3} \end{aligned}$$
The gradient $= -\frac{3}{2}$

2 a $c = 4$

b $c = -5$

c $c = -\frac{2}{3}$

d
$$\begin{aligned} y &= -3x \\ y &= -3x+0 \\ c &= 0 \end{aligned}$$

e $c = \frac{7}{5}$

f
$$\begin{aligned} y &= 2-7x \\ y &= -7x+2 \\ c &= 2 \end{aligned}$$

g
$$\begin{aligned} 3x-4y+8 &= 0 \\ 3x+8 &= 4y \\ 4y &= 3x+8 \\ y &= \tfrac{3}{4}x+\tfrac{8}{4} \\ y &= \tfrac{3}{4}x+2 \\ c &= 2 \end{aligned}$$

2 h $4x-5y-10=0$
$$4x-10=5y$$
$$5y=4x-10$$
$$y=\tfrac{4}{5}x-\tfrac{10}{5}$$
$$y=\tfrac{4}{5}x-2$$
$$c=-2$$

i $-2x+y-9=0$
$$y-9=2x$$
$$y=2x+9$$
$$c=9$$

j $7x+4y+12=0$
$$4y+12=-7x$$
$$4y=-7x-12$$
$$y=-\tfrac{7}{4}x-\tfrac{12}{4}$$
$$y=-\tfrac{7}{4}x-3$$
$$c=-3$$

k $7x-2y+3=0$
$$7x+3=2y$$
$$2y=7x+3$$
$$y=\tfrac{7}{2}x+\tfrac{3}{2}$$
$$c=\tfrac{3}{2}$$

l $-5x+4y+2=0$
$$4y+2=5x$$
$$4y=5x-2$$
$$y=\tfrac{5}{4}x-\tfrac{2}{4}$$
$$y=\tfrac{5}{4}x-\tfrac{1}{2}$$
$$c=-\tfrac{1}{2}$$

3 a $y=4x+3$
$$0=4x+3-y$$
$$4x+3-y=0$$
$$4x-y+3=0$$

3 b $y=3x-2$
$$0=3x-2-y$$
$$3x-2-y=0$$
$$3x-y-2=0$$

c $y=-6x+7$
$$6x+y=7$$
$$6x+y-7=0$$

d $y=\tfrac{4}{5}x-6$

Multiply each term by 5:
$$5y=4x-30$$
$$0=4x-30-5y$$
$$4x-30-5y=0$$
$$4x-5y-30=0$$

e $y=\tfrac{5}{3}x+2$

Multiply each term by 3:
$$3y=5x+6$$
$$0=5x+6-3y$$
$$5x+6-3y=0$$
$$5x-3y+6=0$$

f $y=\tfrac{7}{3}x$

Multiply each term by 3:
$$3y=7x$$
$$0=7x-3y$$
$$7x-3y=0$$

g $y=2x-\tfrac{4}{7}$

Multiply each term by 7:
$$7y=14x-4$$
$$0=14x-4-7y$$
$$14x-4-7y=0$$
$$14x-7y-4=0$$

3 h $y = -3x + \frac{2}{9}$

Multiply each term by 9:

$$9y = -27x + 2$$
$$27x + 9y = 2$$
$$27x + 9y - 2 = 0$$

i $y = -6x - \frac{2}{3}$

Multiply each term by 3:

$$3y = -18x - 2$$
$$18x + 3y = -2$$
$$18x + 3y + 2 = 0$$

j $y = -\frac{1}{3}x + \frac{1}{2}$

Multiply each term by 6 (6 is divisible by both 3 and 2:

$$6y = -2x + 3$$
$$2x + 6y = 3$$
$$2x + 6y - 3 = 0$$

k $y = \frac{2}{3}x + \frac{5}{6}$

Multiply each term by 6 (6 is divisible by both 3 and 6:

$$6y = 4x + 5$$
$$0 = 4x + 5 - 6y$$
$$4x + 5 - 6y = 0$$
$$4x - 6y + 5 = 0$$

l $y = \frac{3}{5}x + \frac{1}{2}$

Multiply each term by 10 (10 is divisible by both 5 and 2):

$$10y = 6x + 5$$
$$0 = 6x + 5 - 10y$$
$$6x + 5 - 10y = 0$$
$$6x - 10y + 5 = 0$$

4 $y = 6x - 18$

Substitute $y = 0$:

$6x - 18 = 0$

$6x = 18$

$x = 3$

The line meets the x-axis at $P(3, 0)$.

5 $3x + 2y = 0$

$2y = -3x$

$y = -\frac{3}{2}x$

The line meets the x-axis at $y = 0$.

Substituting $y = 0$ into $y = -\frac{3}{2}x$:

$0 = -\frac{3}{2}x$

$x = 0$

The line meets the x-axis at $R(0, 0)$.

6 $5x - 4y + 20 = 0$

Substitute $x = 0$:

$5(0) - 4y + 20 = 0$

$-4y + 20 = 0$

$20 = 4y$

$4y = 20$

$y = 5$

The line meets the y-axis at $A(0, 5)$.

Substitute $y = 0$:

$5x - 4(0) + 20 = 0$

$5x + 20 = 0$

$5x = -20$

$x = -4$

The line meets the x-axis at $B(-4, 0)$.

7 a The line passes through (0, 5) and (6, 7).

$$\text{The gradient} = \frac{7-5}{6-0} = \frac{2}{6} = \frac{1}{3}$$

b So $y = \frac{1}{3}x + c$

Use the point (0, 5).

Substitute $x = 0$ and $y = 5$ into $y = \frac{1}{3}x + c$ to find c.

$5 = \frac{1}{3}(0) + c$

$c = 5$

So $y = \frac{1}{3}x + 5$

$3y = x + 15$

$x - 3y + 15 = 0$

8 a The line passes through (5, 0) and (0, 2).

The gradient $= \frac{2-0}{0-5}$

$= -\frac{2}{5}$

b $y = -\frac{2}{5}x + c$

From the coordinates (0, 2) the y-intercept is 2.

So $y = -\frac{2}{5}x + 2$

$5y = -2x + 10$

$2x + 5y - 10 = 0$

9 $ax + by + c = 0$

$by = -ax - c$

$y = -\frac{a}{b}x - \frac{c}{b}$

The gradient $= -\frac{a}{b}$

The y-intercept $= -\frac{c}{b}$

10 $ax - 2y + c = 0$

$2y = ax + c$

$y = \frac{a}{2}x + \frac{c}{2}$

The gradient $= \frac{a}{2} = 3$, so $a = 6$

The y-intercept $= \frac{c}{2} = 5$, so $c = 10$

$a = 6$, $c = 10$

11 Line l has equation $y = -2x + 6$.

For the second line:

$5x - 8y - 15 = 0$

$8y = 5x - 15$

$y = \frac{5}{8}x - \frac{15}{8}$

The lines intersect where:

$y = -2x + 6$ and $y = \frac{5}{8}x - \frac{15}{8}$ cross

So $-2x + 6 = \frac{5}{8}x - \frac{15}{8}$

Multiply through by 8.

$-16x + 48 = 5x - 15$

$21x = 63$

$x = 3$

Substituting $x = 3$ into $y = -2x + 6$:

$y = -2(3) + 6$

$y = 0$

The lines intersect at $P(3, 0)$.

12 a l_1: $y = 3x - 7$

The point of intersection is $P(-3, b)$.

Substituting $x = -3$ and $y = b$ into l_1:

$b = 3(-3) - 7$

$b = -16$

b l_2: $ax + 4y - 17 = 0$

The point $P(-3, -16)$ is on the line.

Substituting $x = -3$ and $y = -16$ into l_2:

$a(-3) + 4(-16) - 17 = 0$

$-3a - 81 = 0$

$a = -27$

Challenge

The line passes through $(0, a)$ and $(b, 0)$.

The gradient $= \frac{0-a}{b-0}$

$= -\frac{a}{b}$

The y-intercept is at $x = 0$.

This is the point $(0, a)$.

$y = -\frac{a}{b}x + a$

$by = -ax + ab$

$ax + by - ab = 0$

Straight line graphs 5C

1 a $m = 2, (x_1, y_1) = (2, 5)$
$y - y_1 = m(x - x_1)$
$y - 5 = 2(x - 2)$
$y - 5 = 2x - 4$
$y = 2x + 1$

b $m = 3, (x_1, y_1) = (-2, 1)$
$y - y_1 = m(x - x_1)$
$y - 1 = 3(x - (-2))$
$y - 1 = 3(x + 2)$
$y - 1 = 3x + 6$
$y = 3x + 7$

c $m = -1, (x_1, y_1) = (3, -6)$
$y - y_1 = m(x - x_1)$
$y - (-6) = -1(x - 3)$
$y + 6 = -x + 3$
$y = -x - 3$

d $m = -4, (x_1, y_1) = (-2, -3)$
$y - y_1 = m(x - x_1)$
$y - (-3) = -4(x - (-2))$
$y + 3 = -4(x + 2)$
$y + 3 = -4x - 8$
$y = -4x - 11$

e $m = \frac{1}{2}, (x_1, y_1) = (-4, 10)$
$y - y_1 = m(x - x_1)$
$y - 10 = \frac{1}{2}(x - (-4))$
$y - 10 = \frac{1}{2}(x + 4)$
$y - 10 = \frac{1}{2}x + 2$
$y = \frac{1}{2}x + 12$

f $m = -\frac{2}{3}, (x_1, y_1) = (-6, -1)$
$y - y_1 = m(x - x_1)$
$y - (-1) = -\frac{2}{3}(x - (-6))$
$y + 1 = -\frac{2}{3}(x + 6)$
$y + 1 = -\frac{2}{3}x - 4$
$y = -\frac{2}{3}x - 5$

g $m = 2, (x_1, y_1) = (a, 2a)$
$y - y_1 = m(x - x_1)$
$y - 2a = 2(x - a)$
$y - 2a = 2x - 2a$
$y = 2x$

h $m = -\frac{1}{2}, (x_1, y_1) = (-2b, 3b)$
$y - y_1 = m(x - x_1)$
$y - 3b = -\frac{1}{2}(x - (-2b))$
$y - 3b = -\frac{1}{2}(x + 2b)$
$y - 3b = -\frac{1}{2}x - b$
$y = -\frac{1}{2}x - b + 3b$
$y = -\frac{1}{2}x + 2b$

2 a $(x_1, y_1) = (2, 4), (x_2, y_2) = (3, 8)$

The gradient $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$$m = \frac{8-4}{3-2}$$
$$= 4$$

The equation is $y - y_1 = m(x - x_1)$
$y - 4 = 4(x - 2)$
$y - 4 = 4x - 8$
$y = 4x - 4$

b $(x_1, y_1) = (0, 2), (x_2, y_2) = (3, 5)$

The gradient $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$$m = \frac{5-2}{3-0}$$
$$= \frac{3}{3}$$
$$= 1$$

The equation is $y - y_1 = m(x - x_1)$
$y - 2 = 1(x - 0)$
$y - 2 = x$
$y = x + 2$

2 c $(x_1, y_1) = (-2, 0), (x_2, y_2) = (2, 8)$

The gradient $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$$m = \frac{8-0}{2-(-2)}$$
$$= \frac{8}{4}$$
$$= 2$$

The equation is $y - y_1 = m(x - x_1)$

$y - 0 = 2(x - (-2))$

$y = 2(x + 2)$

$y = 2x + 4$

d $(x_1, y_1) = (5, -3), (x_2, y_2) = (7, 5)$

The gradient $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$$m = \frac{5-(-3)}{7-5}$$
$$= \frac{8}{2}$$
$$= 4$$

The equation is $y - y_1 = m(x - x_1)$

$y - (-3) = 4(x - 5)$

$y + 3 = 4x - 20$

$y = 4x - 23$

e $(x_1, y_1) = (3, -1), (x_2, y_2) = (7, 3)$

The gradient $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$$m = \frac{3-(-1)}{7-3}$$
$$= \frac{4}{4}$$
$$= 1$$

The equation is $y - y_1 = m(x - x_1)$

$y - (-1) = 1(x - 3)$

$y + 1 = x - 3$

$y = x - 4$

f $(x_1, y_1) = (-4, -1), (x_2, y_2) = (6, 4)$

The gradient $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$$m = \frac{4-(-1)}{6-(-4)}$$
$$= \frac{5}{10}$$
$$= \frac{1}{2}$$

The equation is $y - y_1 = m(x - x_1)$

$y - (-1) = \frac{1}{2}(x - (-4))$

$y + 1 = \frac{1}{2}(x + 4)$

$y + 1 = \frac{1}{2}x + 2$

$y = \frac{1}{2}x + 1$

g $(x_1, y_1) = (-1, -5), (x_2, y_2) = (-3, 3)$

The gradient $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$$m = \frac{3-(-5)}{-3-(-1)}$$
$$= \frac{8}{-2}$$
$$= -4$$

The equation is $y - y_1 = m(x - x_1)$

$y - (-5) = -4(x - (-1))$

$y + 5 = -4(x + 1)$

$y + 5 = -4x - 4$

$y = -4x - 9$

h $(x_1, y_1) = (-4, -1),$

$(x_2, y_2) = (-3, -9)$

The gradient $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$$m = \frac{-9-(-1)}{-3-(-4)}$$
$$= \frac{-8}{1}$$
$$= -8$$

2 h The equation is $y - y_1 = m(x - x_1)$

$y - (-1) = -8(x - (-4))$

$y + 1 = -8(x + 4)$

$y + 1 = -8x - 32$

$y = -8x - 33$

i $(x_1, y_1) = \left(\frac{1}{3}, \frac{2}{5}\right), (x_2, y_2) = \left(\frac{2}{3}, \frac{4}{5}\right)$

The gradient $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$$m = \frac{\frac{4}{5} - \frac{2}{5}}{\frac{2}{3} - \frac{1}{3}} = \frac{\frac{2}{5}}{\frac{1}{3}} = \frac{2}{5} \times 3 = \frac{6}{5}$$

The equation is $y - y_1 = m(x - x_1)$

$y - \frac{2}{5} = \frac{6}{5}\left(x - \frac{1}{3}\right)$

$y - \frac{2}{5} = \frac{6}{5}x - \frac{2}{5}$

$y = \frac{6}{5}x$

j $(x_1, y_1) = \left(-\frac{3}{4}, \frac{1}{7}\right), (x_2, y_2) = \left(\frac{1}{4}, \frac{3}{7}\right)$

The gradient $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$$m = \frac{\frac{3}{7} - \frac{1}{7}}{\frac{1}{4} - \left(-\frac{3}{4}\right)} = \frac{\frac{2}{7}}{1} = \frac{2}{7}$$

The equation is $y - y_1 = m(x - x_1)$

$y - \frac{1}{7} = \frac{2}{7}\left(x - \left(-\frac{3}{4}\right)\right)$

$y - \frac{1}{7} = \frac{2}{7}x + \frac{3}{4}$

$y = \frac{2}{7}x + \frac{3}{4} + \frac{1}{7}$

$y = \frac{2}{7}x + \frac{5}{14}$

3 Line passes through (7, 2) and (9, −8).

$(x_1, y_1) = (7, 2), (x_2, y_2) = (9, -8)$

The gradient $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$$m = \frac{-8 - 2}{9 - 7} = \frac{-10}{2} = -5$$

The equation is $y - y_1 = m(x - x_1)$

$y - 2 = -5(x - 7)$

$y - 2 = -5x + 35$

$y + 5x - 37 = 0$

4 For the equation of AB:

The gradient $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$$m = \frac{0 - 5}{-2 - 3} = \frac{-5}{-5} = 1$$

The equation is $y - y_1 = m(x - x_1)$

$y - 5 = 1(x - 3)$

$y - 5 = x - 3$

$y = x + 2$ or $y - x - 2 = 0$

For the equation of AC:

The gradient $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$$m = \frac{-1 - 5}{4 - 3} = \frac{-6}{1} = -6$$

The equation is $y - y_1 = m(x - x_1)$

$y - 5 = -6(x - 3)$

$y - 5 = -6x + 18$

$y = -6x + 23$ or $6x + y - 23 = 0$

For the equation of BC:

$(x_1, y_1) = (-2, 0), (x_2, y_2) = (4, -1)$

4 The gradient $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$m = \dfrac{-1-0}{4-(-2)}$

$= -\dfrac{1}{6}$

The equation is $y - y_1 = m(x - x_1)$

$y - 0 = -\frac{1}{6}(x - (-2))$

$y = -\frac{1}{6}(x + 2)$

$y = -\frac{1}{6}x - \frac{1}{3} \Rightarrow 6y = -x - 2$

or $\frac{1}{6}x + y + \frac{1}{3} = 0 \Rightarrow x + 6y + 2 = 0$

5 Line through $(a, 4)$ and $(3a, 3)$.

$m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$= \dfrac{3-4}{3a-a}$

$= -\dfrac{1}{2a}$

$x + 6y + c = 0$

$6y = -x - c$

$y = -\frac{1}{6}x - \frac{1}{6}c$

$m = -\dfrac{1}{6} = -\dfrac{1}{2a}$, so $a = 3$.

As a = 3, $(a, 4)$ is the point $(3, 4)$.

Substituting $x = 3$ and $y = 4$ into $y = -\frac{1}{6}x - \frac{1}{6}c$ to find c:

$4 = -\frac{1}{6}(3) - \frac{1}{6}c$

$24 = -3 - c$

$c = -27$

$a = 3, c = -27$

6 Line through $(7a, 5)$ and $(3a, 3)$.

$m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$= \dfrac{3-5}{3a-7a}$

$= \dfrac{-2}{-4a}$

$= \dfrac{1}{2a}$

6 So $y - y_1 = m(x - x_1)$

$y - 5 = \dfrac{1}{2a}(x - 7a)$

$2ay - 10a = x - 7a$

$x - 2ay + 3a = 0$

But the equation of line l is:

$x + by - 12 = 0$

Therefore, $3a = -12$, $a = -4$.

Using the coefficients of x:

$-2a = b$

$a = -4$, $b = 8$

Challenge

a $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

b Using $y - y_1 = m(x - x_1)$:

$y - y_1 = \dfrac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

$\dfrac{y - y_1}{x - x_1} = \dfrac{y_2 - y_1}{x_2 - x_1}$

$\dfrac{y - y_1}{y_2 - y_1} = \dfrac{x - x_1}{x_2 - x_1}$

c Passes through $(-8, 4)$ and $(-1, 7)$.

$\dfrac{y - y_1}{y_2 - y_1} = \dfrac{x - x_1}{x_2 - x_1}$

$\dfrac{y-4}{7-4} = \dfrac{x-(-8)}{(-1)-(-8)}$

$\dfrac{y-4}{3} = \dfrac{x+8}{7}$

$y - 4 = \dfrac{3(x+8)}{7}$

$y = \dfrac{3}{7}x + \dfrac{24}{7} + 4$

$y = \dfrac{3}{7}x + \dfrac{52}{7}$

Straight line graphs 5D

1 $y = 4x - 8$
Substitute $y = 0$:
$4x - 8 = 0$
$4x = 8$
$x = 2$
So A has coordinates (2, 0).
For a line through A with gradient 3:
$y - y_1 = m(x - x_1)$
$y - 0 = 3(x - 2)$
$y = 3x - 6$
The equation of the line is $y = 3x - 6$.

2 $y = -2x + 8$
Substitute $x = 0$:
$y = -2(0) + 8$
$y = 8$
So B has coordinates (0, 8).

We can substitute m=2 and y-intercept 8 into y=mx + c.
Or we can calculate using the formula.
For a line through B with gradient 2:
$y - y_1 = m(x - x_1)$
$y - 8 = 2(x - 0)$
$y - 8 = 2x$
$y = 2x + 8$
The equation of the line is $y = 2x + 8$.

3 To find where the line $y = \frac{1}{2}x + 6$ crosses the x-axis, substitute $y = 0$:
$\frac{1}{2}x + 6 = 0$
$\frac{1}{2}x = -6$
$x = -12$
So C has coordinates (−12, 0).
$y - y_1 = m(x - x_1)$
$y - 0 = \frac{2}{3}(x - (-12))$
$y = \frac{2}{3}(x + 12)$
$y = \frac{2}{3}x + 8$
Multiply each term by 3:
$3y = 2x + 24$
$0 = 2x + 24 - 3y$

3 $2x - 3y + 24 = 0$
The equation of the line is
$2x - 3y + 24 = 0$.

4 To find where the line $y = \frac{1}{4}x + 2$ crosses the y-axis, substitute $x = 0$:
$y = \frac{1}{4}(0) + 2$
$y = 2$
So B has coordinates (0, 2).
C has coordinates (−5, 3).
To find the gradient of BC:
The gradient $m = \frac{y_2 - y_1}{x_2 - x_1}$

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{3-2}{-5-0} = \frac{1}{-5} = -\frac{1}{5}$$

The gradient of the line joining B and C is $-\frac{1}{5}$.

5 To find the equation of the line passing through (2, −5) and (−7, 4):
The gradient $m = \frac{y_2 - y_1}{x_2 - x_1}$

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{4-(-5)}{-7-2} = \frac{9}{-9} = -1$$

The equation is $y - y_1 = m(x - x_1)$
$y - (-5) = -1(x - 2)$
$y + 5 = -x + 2$
$y = -x - 3$
Substitute $y = 0$:
$0 = -x - 3$
$x = -3$
So the line meets the x-axis at $P(-3, 0)$.

6 To find the equation of the line passing through (−3, −5) and (4, 9):

The gradient $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{9-(-5)}{4-(-3)}$$
$$= \frac{14}{7}$$
$$= 2$$

The equation is $y - y_1 = m(x - x_1)$

$y - (-5) = 2(x - (-3))$

$y + 5 = 2(x + 3)$

$y + 5 = 2x + 6$

$y = 2x + 1$

For point G, substitute $x = 0$:

$y = 2(0) + 1 = 1$

The coordinates of G are (0, 1).

7 To find the equation of the line passing through $(3, 2\frac{1}{2})$ and $(-1\frac{1}{2}, 4)$:

The gradient $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - 2\frac{1}{2}}{-1\frac{1}{2} - 3}$$
$$= \frac{1\frac{1}{2}}{-4\frac{1}{2}}$$
$$= -\frac{1}{3}$$

The equation is $y - y_1 = m(x - x_1)$

$y - 2\frac{1}{2} = -\frac{1}{3}(x - 3)$

Multiply through by 6.

$6y - 15 = -2(x - 3)$

$6y - 15 = -2x + 6$

$6y = -2x + 21$

Substitute $x = 0$:

$6y = -2(0) + 21$

The coordinates of J are $(0, \frac{7}{2})$ or $(0, 3\frac{1}{2})$.

8 Substitute $y = x$ in the equation $y = 2x - 5$:

$x = 2x - 5$

$0 = x - 5$

$x = 5$

Substitute $x = 5$ in the equation $y = x$:

$y = 5$

The coordinates of A are (5, 5).

To find the equation of the line through A, with gradient $\frac{2}{5}$:

$$y - y_1 = m(x - x_1)$$
$$y - 5 = \tfrac{2}{5}(x - 5)$$
$$y - 5 = \tfrac{2}{5}x - 2$$
$$y = \tfrac{2}{5}x + 3$$

The equation of the line is $y = \frac{2}{5}x + 3$.

9 Substitute $y = x - 1$ in the equation $y = 4x - 10$:

$x - 1 = 4x - 10$

$-1 = 3x - 10$

$9 = 3x$

$x = 3$

Now substitute $x = 3$ into the equation $y = x - 1$:

$y = 3 - 1$

$y = 2$

The coordinates of T are (3, 2).

To find the equation of the line through T with gradient $-\frac{2}{3}$:

$$y - y_1 = m(x - x_1)$$
$$y - 2 = -\tfrac{2}{3}(x - 3)$$
$$y - 2 = -\tfrac{2}{3}x + 2$$
$$\tfrac{2}{3}x + y - 2 = 2$$
$$\tfrac{2}{3}x + y - 4 = 0$$

$2x + 3y - 12 = 0$

The equation of the line is $2x + 3y - 12 = 0$.

10 The equation of the line p is:

$y-(-12)=\frac{2}{3}(x-6)$

$y+12=\frac{2}{3}x-4$

$y=\frac{2}{3}x-16$

The equation of the line q is:

$y-5=-1(x-5)$

$y-5=-x+5$

$y=-x+10$

For the coordinates of A substitute $x=0$ into the equation $y=\frac{2}{3}x-16$.

$y=\frac{2}{3}(0)-16$

$y=-16$

The required coordinates are $A(0, -16)$.

For the coordinates of B substitute $y=0$ into the equation $y=-x+10$.

$0=-x+10$

$x=10$

The required coordinates are $B(10, 0)$.

The gradient of AB is:

$$\frac{y_2-y_1}{x_2-x_1}=\frac{0-(-16)}{10-0}$$

$$=\frac{16}{10}$$

$$=\frac{8}{5}$$

The gradient of the line joining A and B is $\frac{8}{5}$.

11 To find where the line $y=-2x+6$ crosses the x-axis, substitute $y=0$:

$0=-2x+6$

$2x=6$

$x=3$

The point P has coordinates $(3, 0)$.

$y=\frac{3}{2}x-4$

To find where the line crosses the y-axis, substitute $x=0$:

$y=\frac{3}{2}(0)-4$

$y=-4$

The point Q has coordinates $(0, -4)$.

11 The gradient of PQ is:

$$\frac{y_2-y_1}{x_2-x_1}=\frac{0-(-4)}{3-0}$$

$$=\frac{4}{3}$$

The equation of PQ is:

$y-y_1=m(x-x_1)$

Substitute $(3, 0)$:

$y-0=\frac{4}{3}(x-3)$

$y=\frac{4}{3}x-4$

The equation of the line through P and Q is $y=\frac{4}{3}x-4$.

12 To find where the line $y=3x-5$ crosses the x-axis, substitute $y=0$:

$3x-5=0$

$3x=5$

$x=\frac{5}{3}$

M has coordinates $(\frac{5}{3}, 0)$.

$y=-\frac{2}{3}x+\frac{2}{3}$

Substitute $x=0$:

$y=-\frac{2}{3}(0)+\frac{2}{3}$

$y=\frac{2}{3}$

N has coordinates $(0, \frac{2}{3})$.

The gradient of MN is:

$$\frac{y_2-y_1}{x_2-x_1}=\frac{0-\frac{2}{3}}{\frac{5}{3}-0}$$

$$=\frac{-\frac{2}{3}}{\frac{5}{3}}$$

$$=-\frac{2}{5}$$

The equation of MN is:

$y-y_1=m(x-x_1)$

Substitute $(\frac{5}{3}, 0)$:

$y-0=-\frac{2}{5}\left(x-\frac{5}{3}\right)$

$y=-\frac{2}{5}x+\frac{2}{3}$

Multiply each term by 15:

$15y=-6x+10$

$6x+15y=10$

$6x+15y-10=0$

13 To find where the line $y = 2x - 10$ crosses the x-axis, substitute $y = 0$:

$2x - 10 = 0$

$2x = 10$

$x = 5$

The coordinates of A are (5, 0).

Substitute $x = 0$ into $y = -2x + 4$:

$y = -2(0) + 4 = 4$

The coordinates of B are (0, 4).

To find the equation of AB:

The gradient $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{4-0}{0-5} = \frac{4}{-5} = -\frac{4}{5}$$

The equation is $y - y_1 = m(x - x_1)$

$y - 0 = -\frac{4}{5}(x - 5)$

Multiply through by 5.

$5y = -4(x - 5)$

$y = -\frac{4}{5}x + 4$

14 To find where the line $y = 4x + 5$ crosses the y-axis, substitute $x = 0$:

$y = 4(0) + 5 = 5$

The coordinates of C are (0, 5).

Substitute $y = 0$ in the equation

$y = -3x - 15$:

$0 = -3x - 15$

$3x = -15$

$x = -5$

The coordinates of D are (−5, 0).

To find the equation of CD:

The gradient $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{0-5}{-5-0} = \frac{-5}{-5} = 1$$

14 The equation is $y - y_1 = m(x - x_1)$

$y - 5 = 1(x - 0)$

$y = x + 5$

$x - y + 5 = 0$

15 $y = 3x - 13$

$y = x - 5$

So $3x - 13 = x - 5$

$\Rightarrow 3x = x + 8$

$\Rightarrow 2x = 8$

$\Rightarrow x = 4$

When $x = 4$, $y = 4 - 5 = -1$

The coordinates of S are (4, −1).

To find the equation of ST:

The gradient $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{2-(-1)}{-4-4} = \frac{3}{-8} = -\frac{3}{8}$$

The equation is $y - y_1 = m(x - x_1)$

$y - (-1) = -\frac{3}{8}(x - 4)$

Multiply through by 8.

$8y + 8 = -3(x - 4)$

$8y + 8 = -3x + 12$

$8y = -3x + 4$

$y = -\frac{3}{8}x + \frac{1}{2}$

16 $y = x + 7$

$y = -2x + 1$

So $x + 7 = -2x + 1$

$\Rightarrow 3x + 7 = 1$

$\Rightarrow 3x = -6$

$\Rightarrow x = -2$

When $x = -2$, $y = (-2) + 7 = 5$

The coordinates of L are (−2, 5).

To find the equation of LM:

The gradient $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

16 $\frac{y_2 - y_1}{x_2 - x_1} = \frac{1-5}{-3-(-2)}$

$= \frac{-4}{-1}$

$= 4$

The equation is $y - y_1 = m(x - x_1)$

$M = (-3,1)$

$y - 1 = 4(x - - 3)$

$y - 1 = 4(x + 3)$

$y - 1 = 4x + 12$

$y = 4x + 13$

Straight line graphs 5E

1 a $y = 5x - 2$, $m = 5$
$15x - 3y + 9 = 0$
Parallel lines have the same gradient.
Rearrange the second equation to give:
$3y = 15x + 9$
$y = 5x + 3$, $m = 5$
The lines have the same gradients so are parallel.

b $7x + 14y - 1 = 0$
$y = \frac{1}{2}x + 9$, $m = \frac{1}{2}$
Parallel lines have the same gradient.
Rearrange the first equation to give:
$14y = -7x + 1$
$y = -\frac{1}{2}x + \frac{1}{14}$, $m = -\frac{1}{2}$
The lines have different gradients so are not parallel.

c $4x - 3y - 8 = 0$
$3x - 4y - 8 = 0$
Parallel lines have the same gradient.
Rearrange the first equation to give:
$3y = 4x - 8$
$y = \frac{4}{3}x - \frac{8}{3}$, $m = \frac{4}{3}$
Rearrange the second equation to give:
$4y = 3x - 8$
$y = \frac{3}{4}x - 2$, $m = \frac{3}{4}$
The lines have different gradients so are not parallel.

2 The gradient of r is:

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{8-4}{6-1} = \frac{4}{5}$$

The gradient of s is:

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{9-(-3)}{20-5} = \frac{12}{15} = \frac{4}{5}$$

2 The gradients are equal, so the lines are parallel.

3 If $A(-6, 2)$, $B(4, 8)$, $C(6, 1)$ and $D(-9, -8)$ are coordinates of a trapezium, line AB is parallel to CD or BC is parallel to DA.
Parallel lines have the same gradient.

$$\text{The gradient of } AB = \frac{y_2 - y_1}{x_2 - x_1} = \frac{8-2}{4-(-6)} = \frac{6}{10} = \frac{3}{5}$$

$$\text{The gradient of } CD = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-8-1}{-9-6} = \frac{-9}{-15} = \frac{3}{5}$$

Gradient of AB = gradient of CD.
AB is parallel to CD, therefore $ABCD$ is a trapezium.

4 The line is parallel to $y = 5x + 8$, so $m = 5$.
The line intercepts the y-axis at $(0, 3)$, so $c = 3$.
Using $y = mx + c$, the equation of the line is $y = 5x + 3$.

5 The line is parallel to $y = -\frac{2}{5}x + 1$, so $m = -\frac{2}{5}$.
The line intercepts the y-axis at $(0, -4)$, so $c = -4$.

5 Using $y = mx + c$, the equation of the line is
$y = -\frac{2}{5}x - 4$.
Multiply each term by 5:
$$5y = -2x - 20$$
$$2x + 5y = -20$$
$$2x + 5y + 20 = 0$$

6
$$3x + 6y + 11 = 0$$
$$6y + 11 = -3x$$
$$6y = -3x - 11$$
$$y = -\frac{3}{6}x - \frac{11}{6}$$
$$y = -\frac{1}{2}x - \frac{11}{6}$$
The line is parallel to $y = -\frac{1}{2}x - \frac{11}{6}$,
so $m =$.
The line intercepts the y-axis at (0, 7),
so $c = 7$.
Using $y = mx + c$, the equation of the line is $y = -\frac{1}{2}x + 7$.

7
$$2x - 3y - 1 = 0$$
$$2x - 1 = 3y$$
$$3y = 2x - 1$$
$$y = \frac{2}{3}x - \frac{1}{3}$$
The line is parallel to $y = \frac{2}{3}x - \frac{1}{3}$, so
$m = \frac{2}{3}$.
The line intercepts the y-axis at (0, 0),
so $c = 0$.
Using $y = mx + c$:
$$y = \frac{2}{3}x + 0$$
$$y = \frac{2}{3}x$$

8 The gradient of a line parallel to $y = 4x + 1$ is 4.
$$y - y_1 = m(x - x_1)$$
$$y - 7 = 4(x - (-2))$$
$$y - 7 = 4(x + 2)$$
$$y - 7 = 4x + 8$$
$$y = 4x + 15$$
$$0 = 4x + 15 - y$$
$$4x - y + 15 = 0$$
The equation of the line is
$4x - y + 15 = 0$.

Straight line graphs 5F

1 a The gradients of the lines are 4 and $-\frac{1}{4}$.
The product of the gradients is $4\times-\frac{1}{4}=-1$.
The lines are perpendicular.

b The gradients of the lines are $\frac{2}{3}$ and $\frac{2}{3}$, i.e. they have the same gradient.
The lines are parallel.

c The gradients of the lines are $\frac{1}{5}$ and 5.
The product of the gradients is $\frac{1}{5}\times5=1$.
The lines are neither perpendicular nor parallel.

d The gradients of the lines are -3 and $\frac{1}{3}$.
The product of the gradients is $-3\times\frac{1}{3}=-1$
The lines are perpendicular.

e The gradients of the lines are $\frac{3}{5}$ and $-\frac{5}{3}$.
The product of the gradients is $\frac{3}{5}\times-\frac{5}{3}=-1$.
The lines are perpendicular.

f The gradients of the lines are $\frac{5}{7}$ and $\frac{5}{7}$, i.e. they have the same gradient.
The lines are parallel.

g The gradient of $y = 5x - 3$ is 5.

$$5x-y+4=0$$
$$5x+4=y$$
$$y=5x+4$$

The gradient of $5x - y + 4 = 0$ is 5.
The lines have the same gradient.
The lines are parallel.

h

$$5x-y-1=0$$
$$5x-1=y$$
$$y=5x-1$$

The gradient of $5x - y - 1 = 0$ is 5.
The gradient of $y = -\frac{1}{5}x$ is $-\frac{1}{5}x$.
The product of the gradients is $5\times-\frac{1}{5}x=-1$.
So the lines are perpendicular.

i The gradient of $y = -\frac{3}{2}x + 8$ is $-\frac{3}{2}$.

$$2x-3y-9=0$$
$$2x-9=3y$$
$$3y=2x-9$$
$$y=\frac{2}{3}x-3$$

The gradient of $2x - 3y - 9 = 0$ is $\frac{2}{3}$.
The product of the gradients is $\frac{2}{3}\times-\frac{3}{2}=-1$.
So the lines are perpendicular.

j

$$4x-5y+1=0$$
$$4x+1=5y$$
$$5y=4x+1$$
$$y=\frac{4}{5}x+\frac{1}{5}$$

The gradient of $4x - 5y + 1 = 0$ is $\frac{4}{5}$.

$$8x-10y-2=0$$
$$8x-2=10y$$
$$10y=8x-2$$
$$y=\frac{8}{10}x-\frac{2}{10}$$
$$y=\frac{4}{5}x-\frac{1}{5}$$

The gradient of $8x - 10y - 2 = 0$ is $\frac{4}{5}$.
The lines have the same gradient, they are parallel.

k

$$3x+2y-12=0$$
$$3x+2y=12$$
$$2y=-3x+12$$
$$y=-\frac{3}{2}x+6$$

1 k The gradient of $3x + 2y - 12 = 0$ is $-\frac{3}{2}$.

$$2x+3y-6=0$$
$$2x+3y=6$$
$$3y=-2x+6$$
$$y=-\tfrac{2}{3}x+2$$

The gradient of $2x + 3y - 6 = 0$ is $-\frac{2}{3}$.

The product of the gradients is $-\frac{3}{2}\times-\frac{2}{3}=1$.

So the lines are neither parallel nor perpendicular.

l

$$5x-y+2=0$$
$$5x+2=y$$
$$y=5x+2$$

The gradient of $5x - y + 2 = 0$ is 5.

$$2x+10y-4=0$$
$$2x+10y=4$$
$$10y=-2x+4$$
$$y=-\tfrac{2}{10}x+\tfrac{4}{10}$$
$$y=-\tfrac{1}{5}x+\tfrac{2}{5}$$

The gradient of $2x + 10y - 4 = 0$ is $-\frac{1}{5}$.

The product of the gradients is $5\times-\frac{1}{5}=-1$.

So the lines are perpendicular.

2 The gradient of $y = 6x - 9$ is 6.
So the gradient of the perpendicular line is $-\frac{1}{6}$.

The line goes through the point (0, 1).

$$y-y_1=m(x-x_1)$$
$$y-1=-\tfrac{1}{6}(x-0)$$
$$y=-\tfrac{1}{6}x+1$$

3 Rearrange $3x + 8y - 11 = 0$
$8y = -3x + 11$
$y=-\frac{3}{8}x+\frac{11}{8}$, and the gradient is $-\frac{3}{8}$.

So the gradient of the perpendicular line is $\frac{8}{3}$.

3 The line goes through the point (0, −8).

$$y-y_1=m(x-x_1)$$
$$y-(-8)=\tfrac{8}{3}(x-0)$$
$$y=\tfrac{8}{3}x-8$$

4 The gradient of $y = 3x + 5$ is 3.
The gradient of a line perpendicular to $y = 3x + 5$ is $-\frac{1}{3}$.

The line goes through the point (6, −2).

$$y-y_1=m\left(x-x_1\right)$$
$$y-\left(-2\right)=-\tfrac{1}{3}\left(x-6\right)$$
$$y+2=-\tfrac{1}{3}x+2$$
$$y=-\tfrac{1}{3}x$$

The equation of the line is $y=-\frac{1}{3}x$.

5 The gradient of a line perpendicular to $y = 3x + 6$ is $-\frac{1}{3}$.

The line goes through the point (−2, 5).

$$y-y_1=m\left(x-x_1\right)$$
$$y-5=-\tfrac{1}{3}\left(x-\left(-2\right)\right)$$
$$y-5=-\tfrac{1}{3}\left(x+2\right)$$
$$y-5=-\tfrac{1}{3}x-\tfrac{2}{3}$$
$$y=-\tfrac{1}{3}x+\tfrac{13}{3}$$

6 The gradient of the line $4x - 6y + 7 = 0$ is $\frac{2}{3}$.

The gradient of a line perpendicular to $4x - 6y + 7 = 0$ is $-\dfrac{1}{\frac{2}{3}}=-\dfrac{3}{2}$.

The line goes through the point (3, 4).

$$y-y_1=m\left(x-x_1\right)$$
$$y-4=-\tfrac{3}{2}\left(x-3\right)$$
$$y-4=-\tfrac{3}{2}x+\tfrac{9}{2}$$
$$y=-\tfrac{3}{2}x+\tfrac{17}{2}$$

7 The gradient of a line perpendicular to $y = \frac{2}{3}x + 5$ is $-\frac{1}{\frac{2}{3}} = -\frac{3}{2}$.

The line goes through the point (5, −5).

$$y - y_1 = m(x - x_1)$$
$$y - (-5) = -\tfrac{3}{2}(x - 5)$$
$$y + 5 = -\tfrac{3}{2}(x - 5)$$

Multiply each term by 2:

$$2y + 10 = -3(x - 5)$$
$$2y + 10 = -3x + 15$$
$$3x + 2y + 10 = 15$$
$$3x + 2y - 5 = 0$$

The equation of the line is $3x + 2y - 5 = 0$.

8 The gradient of a line perpendicular to $y = -\frac{4}{7}x + 5$ is $-\frac{1}{-\frac{4}{7}} = \frac{7}{4}$.

The line goes through the point (−2, −3).

$$y - y_1 = m(x - x_1)$$
$$y - (-3) = \tfrac{7}{4}(x - (-2))$$
$$y + 3 = \tfrac{7}{4}(x + 2)$$

Multiply each term by 4:

$$4y + 12 = 7(x + 2)$$
$$4y + 12 = 7x + 14$$
$$4y = 7x + 2$$
$$0 = 7x + 2 - 4y$$
$$7x - 4y + 2 = 0$$

The equation of the line is $7x - 4y + 2 = 0$.

9 $(x_1, y_1) = (-3, 0)$, $(x_2, y_2) = (3, -2)$

The gradient of l is:

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{-2 - 0}{3 - (-3)}$$
$$= -\frac{2}{6}$$
$$\frac{y_2 - y_1}{x_2 - x_1} = -\frac{1}{3}$$

$(x_1, y_1) = (1, 8)$, $(x_2, y_2) = (-1, 2)$

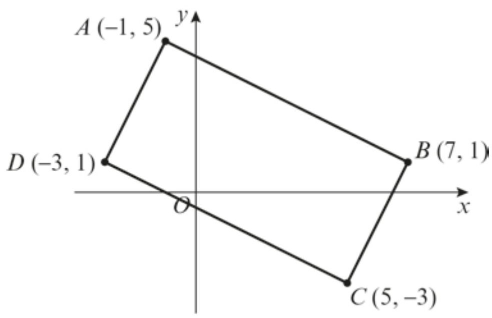
The gradient of n is:

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{2 - 8}{-1 - 1}$$
$$= \frac{-6}{-2}$$
$$= 3$$

The product of the gradients is $-\frac{1}{3} \times 3 = -1$

So the lines are perpendicular.

10



The gradient of AB is:

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - 1}{-1 - 7}$$
$$= \frac{4}{-8}$$
$$= -\frac{1}{2}$$

The gradient of DC is:

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{-3 - 1}{5 - (-3)}$$
$$= \frac{-4}{8}$$
$$= -\frac{1}{2}$$

The gradient of AB is the same as the gradient of DC, so the lines are parallel.

10 The gradient of *AD* is:

$$\frac{y_2-y_1}{x_2-x_1}=\frac{5-1}{-1-(-3)}$$
$$=-\frac{4}{-1+3}$$
$$=\frac{4}{2}$$
$$=2$$

The gradient of *BC* is:

$$\frac{y_2-y_1}{x_2-x_1}=\frac{-3-1}{5-7}$$
$$=\frac{-4}{-2}$$
$$=2$$

The gradient of *AD* is the same as the gradient of BC, so the lines are parallel. The line *AD* is perpendicular to the line *AB*, since $2\times-\frac{1}{2}=-1$.

So *ABCD* is a rectangle.

11 a The line l_1, $5x + 11y - 7 = 0$, crosses the x-axis when $y = 0$, so:

$5x + 11(0) - 7 = 0$

$x = \frac{7}{5}$

The point *A* is $(\frac{7}{5}, 0)$

b Rearranging $5x + 11y - 7 = 0$ to find the gradient gives:

$11y = -5x + 7$

$y=-\frac{5}{11}x+\frac{7}{11}$

The gradient is $-\frac{5}{11}$.

So the gradient of the perpendicular line is $\frac{11}{5}$.

$$y-y_1=m(x-x_1)$$
$$y-0=\tfrac{11}{5}(x-\tfrac{7}{5})$$
$$y=\tfrac{11}{5}x-\tfrac{77}{25}$$

l_2 : $55x-25y-77=0$

12 The gradient of line *AB* is:

$$\frac{y_2-y_1}{x_2-x_1}=\frac{0-4}{-3-0}$$
$$=\frac{-4}{-3}$$
$$=\frac{4}{3}$$

The gradient of the perpendicular *BC* is $-\frac{3}{4}$.

The gradient of the line *BC* is:

$$\frac{y_2-y_1}{x_2-x_1}=\frac{c-0}{0-(-3)}$$
$$=\frac{c}{3}$$
$$\frac{c}{3}=-\frac{3}{4}$$
$$c=-\frac{9}{4}$$

Straight line graphs 5G

1 a $(x_1, y_1) = (0, 1), (x_2, y_2) = (6, 9)$

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(6-0)^2 + (9-1)^2}$$
$$= \sqrt{6^2 + 8^2}$$
$$= \sqrt{36 + 64}$$
$$= \sqrt{100}$$
$$= 10$$

b $(x_1, y_1) = (4, -6), (x_2, y_2) = (9, 6)$

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(9-4)^2 + (6-(-6))^2}$$
$$= \sqrt{5^2 + 12^2}$$
$$= \sqrt{25 + 144}$$
$$= \sqrt{169}$$
$$= 13$$

c $(x_1, y_1) = (3, 1), (x_2, y_2) = (-1, 4)$

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(-1-3)^2 + (4-1)^2}$$
$$= \sqrt{(-4)^2 + 3^2}$$
$$= \sqrt{16 + 9}$$
$$= \sqrt{25}$$
$$= 5$$

d $(x_1, y_1) = (3, 5), (x_2, y_2) = (4, 7)$

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(4-3)^2 + (7-5)^2}$$
$$= \sqrt{1^2 + 2^2}$$
$$= \sqrt{1 + 4}$$
$$= \sqrt{5}$$

e $(x_1, y_1) = (0, -4), (x_2, y_2) = (5, 5)$

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(5-0)^2 + (5-(-4))^2}$$
$$= \sqrt{5^2 + 9^2}$$
$$= \sqrt{25 + 81}$$
$$= \sqrt{106}$$

f $(x_1, y_1) = (-2, -7), (x_2, y_2) = (5, 1)$

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(5-(-2))^2 + (1-(-7))^2}$$
$$= \sqrt{(5+2)^2 + (1+7)^2}$$
$$= \sqrt{7^2 + 8^2}$$
$$= \sqrt{49 + 64}$$
$$= \sqrt{113}$$

2 $A(-3, 5)$, $B(-2, -2)$ and $C(3, -7)$.
Lines are congruent if they are the same length.

Using the distance formula, and taking the unknown length as d:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

For the line AB:

$(x_1, y_1) = (-3, 5), (x_2, y_2) = (-2, -2)$

$$AB = \sqrt{((-2)-(-3))^2 + ((-2)-5)^2}$$
$$= \sqrt{1^2 + (-7)^2}$$
$$= \sqrt{50}$$

For the line BC:

$(x_1, y_1) = (-2, -2), (x_2, y_2) = (3, -7)$

$$BC = \sqrt{(3-(-2))^2 + ((-7)-(-2))^2}$$
$$= \sqrt{5^2 + (-5)^2}$$

2 $BC = \sqrt{50}$
$AB = BC$, $\therefore$ they are congruent.

3 $P(11, -8)$, $Q(4, -3)$ and $R(7, 5)$.

Using the distance formula, and taking the unknown length as d:
$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

For the line PQ:
$(x_1, y_1) = (11, -8), (x_2, y_2) = (4, -3)$
$PQ = \sqrt{(4-11)^2 + ((-3)-(-8))^2}$
$= \sqrt{(-7)^2 + 5^2}$
$= \sqrt{74}$

For the line QR:
$(x_1, y_1) = (4, -3), (x_2, y_2) = (7, 5)$
$QR = \sqrt{(7-4)^2 + (5-(-3))^2}$
$= \sqrt{3^2 + 8^2}$
$= \sqrt{73}$

$PQ \neq QR$, therefore the two lines are not congruent.

4 The distance between the points $(-1, 13)$ and $(x, 9)$ is $\sqrt{65}$.

Using the distance formula, and taking the length as d:
$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$
$65 = (x - (-1))^2 + (9 - 13)^2$
$65 = (x + 1)^2 + (-4)^2$
$65 = x^2 + 2x + 1 + 16$
$x^2 + 2x - 48 = 0$
$(x + 8)(x - 6) = 0$
$x = -8$ or $x = 6$

5 The distance between the points $(2, y)$ and $(5, 7)$ is $3\sqrt{10}$.

Using the distance formula, and taking the length as d:
$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$
$90 = (5-2)^2 + (7-y)^2$
$90 = 3^2 + 49 - 14y + y^2$
$y^2 - 14y - 32 = 0$
$(y - 16)(y + 2) = 0$
$y = 16$ or $y = -2$

6 a l_1: $y = 2x + 4$, gradient $= 2$
l_2: $6x - 3y - 9 = 0$

Rearrange line l_2 to give:
$3y = 6x - 9$
$y = 2x - 3$, gradient $= 2$

Lines l_1 and l_2 both have gradient 2 so they are parallel.

b Line l_3 is perpendicular to line l_1 so has gradient $-\frac{1}{2}$.

It also passes through the point $(3, 10)$.

$y - y_1 = m(x - x_1)$
$y - 10 = -\frac{1}{2}(x - 3)$
$2y - 20 = -x + 3$
$x + 2y - 23 = 0$ is the equation of l_3

c l_2: $y = 2x - 3$
l_3: $2y = -x + 23$

Dividing through by 2:
$y = -\frac{1}{2}x + \frac{23}{2}$

At the point of intersection the two expressions for y are equal, so:
$2x - 3 = -\frac{1}{2}x + \frac{23}{2}$

6 c Then multiplying through by 2:

$4x - 6 = -x + 23$

$5x = 29$

$x = \frac{29}{5}$

Substituting $x = \frac{29}{5}$ into $y = 2x - 3$:

$y = 2\left(\frac{29}{5}\right) - 3$

$= \frac{43}{5}$

The point of intersection of the lines l_1 and l_2 is $\left(\frac{29}{5}, \frac{43}{5}\right)$.

d l_1 and l_2 are parallel so the shortest distance between them is the perpendicular distance.

l_3 is perpendicular to l_1 and therefore is perpendicular to l_2.

l_2 and l_3 intersect at (, $\frac{43}{5}$).

Now work out the point of intersection for lines l_1 and l_3.

l_1: $y = 2x + 4$

l_3: $y = -\frac{1}{2}x + \frac{23}{2}$

$2x + 4 = -\frac{1}{2}x + \frac{23}{2}$

$4x + 8 = -x + 23$

$5x = 15, x = 3$

When $x = 3, y = 10$

The point of intersection of the lines l_1 and l_3 is (3, 10).

Now find the distance, d, between (3, 10) and $\left(\frac{29}{5}, \frac{43}{5}\right)$.

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$= \sqrt{(\frac{29}{5} - 3)^2 + (\frac{43}{5} - 10)^2}$

$= \sqrt{(\frac{14}{5})^2 + (-\frac{7}{5})^2}$

$= \sqrt{\frac{245}{25}}$

$= \frac{1}{5}\sqrt{245}$

$= \frac{1}{5}\sqrt{49 \times 5}$

6 d $d = \dfrac{7\sqrt{5}}{5}$

The perpendicular distance between l_1 and l_2 is $\frac{7}{5}\sqrt{5}$.

7 Point P is on the line $y = -3x + 4$.

Its distance, d, from (0, 0) is $\sqrt{34}$.

$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$

$34 = (x - 0)^2 + (y - 0)^2$

$34 = x^2 + y^2$

Solve $34 = x^2 + y^2$ and $y = -3x + 4$ simultaneously.

$34 = x^2 + (-3x + 4)^2$

$34 = x^2 + 9x^2 - 24x + 16$

$10x^2 - 24x - 18 = 0$

$5x^2 - 12x - 9 = 0$

$(5x + 3)(x - 3) = 0$

$x = -\frac{3}{5}$ or $x = 3$

When $x = \frac{3}{5}$, $y = -3\left(-\frac{3}{5}\right) + 4$

$y = \frac{29}{5}$

When $x = 3$, $y = -3(3) + 4$

$y = -5$

So P is the point $\left(-\frac{3}{5}, \frac{29}{5}\right)$ or $(3, -5)$.

8 a In a scalene triangle, all three sides have different lengths: $AB \neq BC \neq AC$.

$A(2, \frac{29}{5} 7)$, $B(5, -6)$ and $C(8, -6)$.

$AB = \sqrt{(5-2)^2 + ((-6)-7)^2}$

$= \sqrt{3^2 + (-13)^2}$

$= \sqrt{178}$

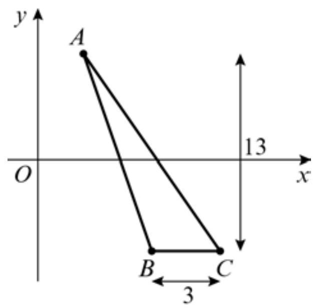
$BC = \sqrt{(8-5)^2 + ((-6)-(-6))^2}$

$= \sqrt{3^2 + 0^2}$

$= 3$

8a $AC = \sqrt{(8-2)^2 + ((-6)-7)^2}$
$= \sqrt{6^2 + (-13)^2}$
$= \sqrt{205}$
$AB \neq BC \neq AC$ therefore ABC is a scalene triangle.

b Draw a sketch and labels the points A, B and C.



Area of ΔABC $= \frac{1}{2}bh$
$= \frac{1}{2} \times 3 \times 13$
$= 19.5$

9 a l_1: $y = 7x - 3$
l_2: $4x + 3y - 41 = 0$

Substituting l_1 into l_2 gives:
$4x + 3(7x - 3) - 41 = 0$
$4x + 21x - 9 - 41 = 0$
$25x = 50$
$x = 2$

Substituting $x = 2$ into $y = 7x - 3$ gives $y = 11$.
A is the point $(2, 11)$.

b When l_2 crosses the x-axis, $y = 0$.
So $4x + 3(0) - 41 = 0$
$4x = 41$
$x = \frac{41}{4}$
B is the point $(\frac{41}{4}, 0)$.

9 c The base of ΔAOB is $\frac{41}{4}$
The height of ΔAOB is 11

Area $\Delta AOB = \frac{1}{2} \times \frac{41}{4} \times 11$
$= \frac{451}{8}$

10 a l_1: $4x - 5y - 10 = 0$ intersects the x-axis at A, so $y = 0$.

$4x - 5(0) - 10 = 0$
$4x = 10$
$x = \frac{5}{2}$
A is the point $\left(\frac{5}{2}, 0\right)$.

b l_2: $4x - 2y + 20 = 0$ intersects the x-axis at B, so $y = 0$.

$4x - 2(0) + 20 = 0$
$4x = -20$
$x = -5$

B is the point $(-5, 0)$.

c l_1: $4x = 5y + 10$, l_2: $4x = 2y - 20$

Where the lines intersect:
$5y + 10 = 2y - 20$
$3y = -30$
$y = -10$

Substituting $y = -10$ into $4x = 5y + 10$:
$4x = -40$
$x = -10$

l_1 and l_2 intersect at the point $(-10, -10)$.

d The base of ΔABC is $7\frac{1}{2}$.
The height of the triangle is 10.
Area ΔABC $= \frac{1}{2} \times 7\frac{1}{2} \times 10$
$= \frac{75}{2}$

11 a $R(5, -2)$ and $S(9, 0)$ lie on a straight line.

The gradient, m, of the line is:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0-(-2)}{9-5} = \frac{2}{4} = \frac{1}{2}$$

The equation of the line is:

$$y - y_1 = m(x - x_1)$$
$$y-(-2) = \tfrac{1}{2}(x-5)$$
$$y+2 = \tfrac{1}{2}x - \tfrac{5}{2}$$
$$y = \tfrac{1}{2}x - \tfrac{9}{2}$$

b l_2 is perpendicular to l_1 so has gradient -2.

The equation of the line is:

$$y - y_1 = m(x - x_1)$$
$$y-(-2) = -2(x-5)$$
$$y+2 = -2x+10$$
$$y = -2x+8$$

c The y-intercept for line l_2 is 8.
T is the point $(0, 8)$.

d
$$RS = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} = \sqrt{(9-5)^2+(0-(-2))^2} = \sqrt{4^2+2^2} = \sqrt{20} = \sqrt{4\times 5} = 2\sqrt{5}$$

11 d
$$TR = \sqrt{(5-0)^2+((-2)-8)^2} = \sqrt{5^2+(-10)^2} = \sqrt{125} = 5\sqrt{5}$$

e The base of ΔRST is RS, $2\sqrt{5}$.
The height of ΔRST is RT, $5\sqrt{5}$.

Area $\Delta RST = \tfrac{1}{2}\times 2\sqrt{5}\times 5\sqrt{5}$
$= 25$

12 a l_1 has gradient $m = -\tfrac{1}{4}$ and passes through the point $(-4, 14)$.

$$y - y_1 = m(x - x_1)$$
$$y-14 = -\tfrac{1}{4}(x-(-4))$$
$$4y-56 = -x-4$$
$$x+4y-52 = 0$$

b When l_1 crosses the y-axis, $x = 0$.
$$0+4y-52=0$$
$$y = 13$$
A is the point $(0, 13)$.

c l_2 has gradient, $m = 3$ and passes through the point $(0, 0)$.
$y = 3x$

To find point B, substitute l_2 into l_1:
$$x+4(3x)-52=0$$
$$13x-52=0$$
$$x = 4$$

Substitute $x = 4$ into $y = 3x$.
$y = 12$
B is the point $(4, 12)$.

d The base of ΔOAB is $OA = 13$.
The height of ΔOAB is the distance of B from the y-axis $= 4$
Area $\Delta OAB = \tfrac{1}{2}\times 13\times 4 = 26$

Straight line graphs 5H

1 a i Gradient $= \dfrac{y_2 - y_1}{x_2 - x_1}$

$$= \frac{600 - 200}{12 - 4}$$
$$= \frac{400}{8}$$
$$= 50$$
$$k = 50$$

Direct proportion equations go through the origin so $c = 0$.

ii $d = kt + c \quad k = 50, c = 0$

$d = 50t$

b i Gradient $= \dfrac{y_2 - y_1}{x_2 - x_1}$

$$= \frac{9 - 3}{30 - 10}$$
$$= \frac{6}{20}$$
$$= \frac{3}{10}$$

$k = \frac{3}{10}$

ii $C = kt + c \quad k = \frac{3}{10}, c = 0$

$C = \frac{3}{10}t$

c i Gradient $= \dfrac{y_2 - y_1}{x_2 - x_1}$

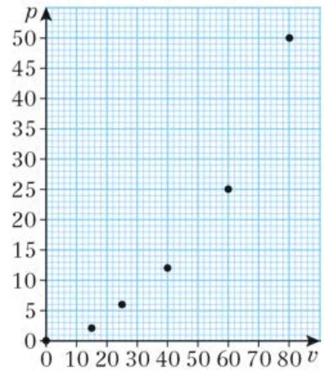
$$= \frac{18 - 6}{30 - 10}$$
$$= \frac{12}{20}$$
$$= \frac{3}{5}$$

$k = \frac{3}{5}$

ii $p = kt + c \quad k = \frac{3}{5}, c = 0$

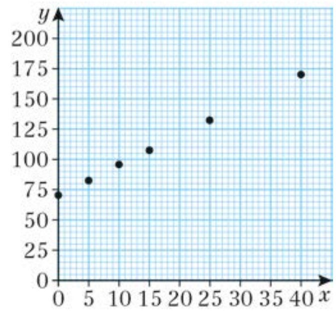
$p = \frac{3}{5}t$

2 a



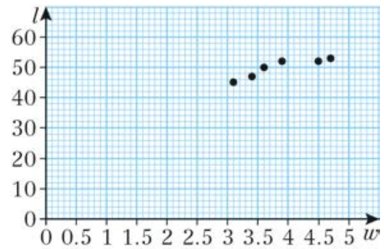
The points do not lie on a straight line, so a linear model is not appropriate.

b



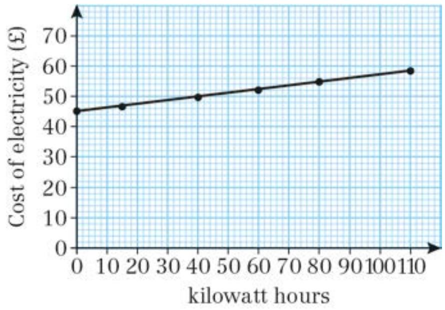
The points lie on a straight line so a linear model is appropriate.

c



The points do not lie on a straight line, so a linear model is not appropriate.

3 a



b The points lie on a straight line so a linear model is appropriate.

c Gradient $= \dfrac{y_2 - y_1}{x_2 - x_1}$

$= \dfrac{58.2 - 45}{110 - 0}$

$= \dfrac{13.2}{110}$

$= 0.12$

$E = ah + b$

a is the gradient = 0.12

b is the y-intercept = 45

$E = 0.12h + 45$

d a = £0.12, this is the cost of 1 kilowatt hour of electricity.

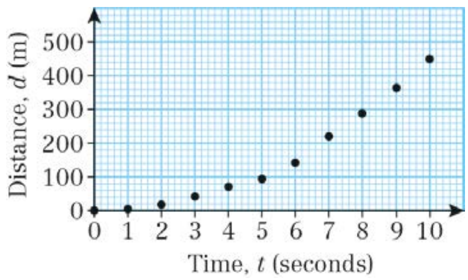
b = £45, this is the fixed charge for the electricity supply (per month or per quarter).

e When $h = 65$:

$E = 0.12(65) + 45$

= £52.80

4 a



b The points do not lie on a straight line, so a linear model is not appropriate.

5 a $(x_1, y_1) = (6, 7100)$

$(x_2, y_2) = (13, 9550)$

Gradient $= \dfrac{y_2 - y_1}{x_2 - x_1}$

$= \dfrac{9550 - 7100}{13 - 6}$

$= \dfrac{2450}{7}$

$= 350$

$C = ad + b$

$C = 350d + b$

Substituting $d = 6$ and $C = 7100$ into $C = 350d + b$ gives:

$7100 = 350(6) + b$

$b = 5000$

$C = 350d + 5000$

b a = £350, this is the daily fee charged by the web designer.

b = £5000, this is the flat rate fee charged by the web designer.

c Substitute $C = 13\,400$ into $C = 350d + 5000$ to give:

$13\,400 = 350d + 5000$

$d = 24$

The designer spent 24 days working on the website.

6 a $(x_1, y_1) = (9, 48.2)$, $(x_2, y_2) = (20, 68)$

Gradient $= \dfrac{y_2 - y_1}{x_2 - x_1}$

$= \dfrac{68 - 48.2}{20 - 9}$

$= \dfrac{19.8}{11}$

$= 1.8$

$F = 1.8C + b$

Substituting $C = 9$ and $F = 48.2$ into $F = 1.8C + b$ gives:

$48.2 = 1.8(9) + b$

$b = 32$

$F = 1.8C + 32$

6 b $a = 1.8$ which is the increase in the Fahrenheit temperature for every 1 degree increase in the Celsius temperature.
$b = 32$ which is the Fahrenheit temperature when the Celsius temperature is zero.

c Substitute $F = 101.3$ into $F = 1.8C + 32$ to give:
$101.3 = 1.8C + 32$
$C = 38.5$
The temperature 101.3 °F is 38.5 °C.

d $F = 1.8C + 32$
When $F = C$:
$F = 1.8F + 32$
$-0.8F = 32$
$F = -40$
−40 °F is the same as −40 °C.

7 a Gradient = 750
Intercept on the vertical axis = 17 500
$n = 750t + 17\,500$

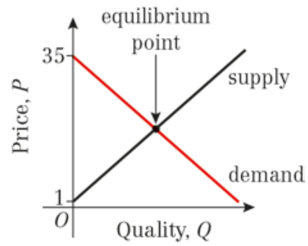
b The assumption is that the number of homes receiving internet connection will increase by the same amount each year.

8 a The data can be approximated to a linear model as all of the points lie close to the line of best fit shown.

b Gradient $= \dfrac{y_2 - y_1}{x_2 - x_1}$
$= \dfrac{177 - 165}{27 - 24}$
$= 4$
$h = 4f + b$
Substituting $f = 24$ and $h = 165$ into $h = 4f + b$ gives:
$165 = 4(24) + b$
$b = 69$
$h = 4f + 69$

8 c Substituting $f = 26.5$ into $h = 4f + 69$ gives:
$h = 4(26.5) + 69$
$= 175$ cm

9 a



b Solve $P = -\frac{3}{4}Q + 35$ and $P = \frac{2}{3}Q + 1$ simultaneously:
$-\frac{3}{4}Q + 35 = \frac{2}{3}Q + 1$
$34 = \frac{17}{12}Q$
$Q = 24$
Substituting $Q = 24$ into P gives:
$P = \frac{2}{3}Q + 1$: P
$= \frac{2}{3}(24) + 1$
$= 17$
$P = 17$, $Q = 24$

Straight line graphs, Mixed Exercise 5

1 a Gradient $m = -\frac{5}{12}$, $(x_1, y_1) = (2, 1)$

The equation of the line is:

$y - y_1 = m(x - x_1)$

$y - 1 = -\frac{5}{12}(x - 2)$

$y - 1 = -\frac{5}{12}x + \frac{5}{6}$

$y = -\frac{5}{12}x + \frac{11}{6}$

b Substitute $(k, 11)$ into $y = -\frac{5}{12}x + \frac{11}{6}$

$11 = -\frac{5}{12}k + \frac{11}{6}$

$11 - \frac{11}{6} = -\frac{5}{12}k$

$\frac{55}{6} = -\frac{5}{12}k$

Multiply each side by 12:

$110 = 5k$

$k = -22$

2 a The gradient of AB is:

$\frac{y_2 - y_1}{x_2 - x_1} = \frac{1}{3}$

So:

$\frac{(2k-1)-1}{8-k} = \frac{1}{3}$

$\frac{2k-1-1}{8-k} = \frac{1}{3}$

$\frac{2k-2}{8-k} = \frac{1}{3}$

Multiply each side by $(8 - k)$:

$2k - 2 = \frac{1}{3}(8 - k)$

Multiply each term by 3:

$6k - 6 = 8 - k$

$7k - 6 = 8$

$7k = 14$

$k = 2$

b $k = 2$.

So A and B have coordinates $(2, 1)$ and $(8, 3)$.

2 b The equation of the line is:

$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$

$\frac{y-1}{3-1} = \frac{x-2}{8-2}$

$\frac{y-1}{2} = \frac{x-2}{6}$

Multiply each side by 2:

$y - 1 = \frac{1}{3}(x - 2)$

$y - 1 = \frac{1}{3}x - \frac{2}{3}$

$y = \frac{1}{3}x + \frac{1}{3}$

3 a The equation of L_1 is:

$y - y_1 = m(x - x_1)$

$y - 2 = \frac{1}{7}(x - 2)$

$y - 2 = \frac{1}{7}x - \frac{2}{7}$

$y = \frac{1}{7}x + \frac{12}{7}$

The equation of L_2 is:

$y - y_1 = (x - x_1)$

$y - 8 = -1(x - 4)$

$y - 8 = -x + 4$

$y = -x + 12$

b Solve $y = \frac{1}{7}x + \frac{12}{7}$ and $y = -x + 12$ simultaneously.

$-x + 12 = \frac{1}{7}x + \frac{12}{7}$

$12 = \frac{8}{7}x + \frac{12}{7}$

$10\frac{2}{7} = \frac{8}{7}x$

$x = \frac{10\frac{2}{7}}{\frac{8}{7}}$

$= 9$

Substitute $x = 9$ into $y = -x + 12$:

$y = -9 + 12$

$= 3$

The lines intersect at $C(9, 3)$.

4 a The equation of l is:

$$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}$$

$$\frac{y-0}{6-0}=\frac{x-1}{5-1}$$

$$\frac{y}{6}=\frac{x-1}{4}$$

Multiply each side by 6:

$$y=6\frac{(x-1)}{4}$$

$$y=\frac{3}{2}(x-1)$$

$$y=\frac{3}{2}x-\frac{3}{2}$$

b Solve $2x + 3y = 15$ and $y=\frac{3}{2}x-\frac{3}{2}$ simultaneously.

Substitute:

$$2x+3\left(\tfrac{3}{2}x-\tfrac{3}{2}\right)=15$$

$$2x+\tfrac{9}{2}x-\tfrac{9}{2}=15$$

$$\tfrac{13}{2}x-\tfrac{9}{2}=15$$

$$\tfrac{13}{2}x=\tfrac{39}{2}$$

$$13x=39$$

$$x=3$$

Substitute $x = 3$ into $y=\frac{3}{2}x-\frac{3}{2}$:

$$y=\tfrac{3}{2}(3)-\tfrac{3}{2}$$

$$=\tfrac{9}{2}-\tfrac{3}{2}$$

$$=\tfrac{6}{2}$$

$$=3$$

The coordinates of C are (3, 3).

5 $(x_1, y_1) = (1, 3)$, $(x_2, y_2) = (-19, -19)$

The equation of L is:

$$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}$$

$$\frac{y-3}{-19-3}=\frac{x-1}{-19-1}$$

$$\frac{y-3}{-22}=\frac{x-1}{-20}$$

Multiply each side by −22:

$$y-3=\tfrac{-22}{-20}(x-1)$$

$$y-3=\tfrac{11}{10}(x-1)$$

Multiply each term by 10:

$$10y-30=11(x-1)$$

$$10y-30=11x-11$$

$$10y=11x+19$$

$$0=11x-10y+19$$

The equation of L is $11x - 10y + 19 = 0$.

6 a $(x_1, y_1) = (2, 2)$, $(x_2, y_2) = (6, 0)$

The equation of l_1 is:

$$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}$$

$$\frac{y-2}{0-2}=\frac{x-2}{6-2}$$

$$\frac{y-2}{-2}=\frac{x-2}{4}$$

Multiply each side by −2:

$$y-2=-\tfrac{1}{2}(x-2)\ \left(\text{Note:}-\tfrac{2}{4}=-\tfrac{1}{2}\right)$$

$$y-2=-\tfrac{1}{2}x+1$$

$$y=-\tfrac{1}{2}x+3$$

b The equation of l_2 is:

$$y-y_1=m(x-x_1)$$

$$y-0=\tfrac{1}{4}(x-(-9))$$

$$y=\tfrac{1}{4}(x+9)$$

$$y=\tfrac{1}{4}x+\tfrac{9}{4}$$

7 $A\left(1,3\sqrt{3}\right), B\left(2+\sqrt{3},3+4\sqrt{3}\right)$

The gradient of the line through A and B is:

$$\frac{y_2-y_1}{x_2-x_1}=\frac{3+4\sqrt{3}-3\sqrt{3}}{2+\sqrt{3}-1}$$

$$=\frac{3+\sqrt{3}}{1+\sqrt{3}}$$

7 Rationalising the denominator:

$$\frac{(3+\sqrt{3})\times(1-\sqrt{3})}{(1+\sqrt{3})\times(1-\sqrt{3})}=\frac{3-3\sqrt{3}+\sqrt{3}-3}{1-3}$$
$$=\frac{-2\sqrt{3}}{-2}$$
$$=\sqrt{3}$$

The equation of the line is:

$y=\sqrt{3}x+c$

Substituting $x = 1$ and $y = 3\sqrt{3}$ into $y=\sqrt{3}x+c$:

$3\sqrt{3}=\sqrt{3}+c$

$c=2\sqrt{3}$

The equation of line l is:

$y=\sqrt{3}x+2\sqrt{3}$

Line l meets the x-axis when $y = 0$.
When $y = 0$, $x = -2$.
C is the point $(-2, 0)$.

8 a $A(-4, 6)$, $B(2, 8)$
The gradient of AB is:

$$\frac{y_2-y_1}{x_2-x_1}=\frac{8-6}{2-(-4)}$$
$$=\frac{2}{6}$$
$$=\frac{1}{3}$$

The gradient of a line perpendicular to AB is:

$$-\frac{1}{\frac{1}{3}}=-3$$

The equation of p is:

$y-y_1=m(x-x_1)$

$y-8=-3(x-2)$

$y-8=-3x+6$

$y=-3x+14$

b Substitute $x = 0$ in the equation for AB:
$y = -3(0) + 14 = 14$
The coordinates of C are $(0, 14)$.

9 a The line passes through $A(0, 4)$ and is perpendicular to l: $2x - y - 1 = 0$.

$2x-y-1=0$

$2x-1=y$

$y=2x-1$

The gradient of $2x - y - 1 = 0$ is 2.
The gradient of a line perpendicular to $2x - y - 1 = 0$ is $-\frac{1}{2}$.
The equation of the line m is:

$y-y_1=m(x-x_1)$

$y-4=-\frac{1}{2}(x-0)$

$y-4=-\frac{1}{2}x$

$y=-\frac{1}{2}x+4$

Or, since A is a y-intercept, the equation can be written once the gradient is known i.e $y=-\left(\frac{1}{2}\right)x+4$.

b To find P, solve $y=-\frac{1}{2}x+4$ and $2x - y - 1 = 0$ simultaneously.
Substitute:

$2x-\left(-\frac{1}{2}x+4\right)-1=0$

$2x+\frac{1}{2}x-4-1=0$

$\frac{5}{2}x-5=0$

$5x=10$

$x=2$

Substitute $x = 2$ into $y=-\frac{1}{2}x+4$:

$y=-\frac{1}{2}(2)+4$

$=-1+4$

$=3$

The lines intersect at $P(2, 3)$, as required.

c A line parallel to the line m has gradient $-\frac{1}{2}$.
The equation of the line n is:

$y-y_1=m(x-x_1)$

$y-0=-\frac{1}{2}(x-3)$

$y=-\frac{1}{2}x+\frac{3}{2}$

9 c To find Q solve $2x - y - 1 = 0$ and $y = -\frac{1}{2}x + \frac{3}{2}$ simultaneously.

Substitute:

$$2x - \left(-\tfrac{1}{2}x + \tfrac{3}{2}\right) - 1 = 0$$
$$2x + \tfrac{1}{2}x - \tfrac{3}{2} - 1 = 0$$
$$\tfrac{5}{2}x - \tfrac{5}{2} = 0$$
$$\tfrac{5}{2}x = \tfrac{5}{2}$$
$$x = 1$$

Substitute $x = 1$ into $y = -\frac{1}{2}x + \frac{3}{2}$:

$$y = -\tfrac{1}{2}(1) + \tfrac{3}{2}$$
$$= -\tfrac{1}{2} + \tfrac{3}{2}$$
$$= 1$$

The lines intersect at $Q(1, 1)$.

10 $A(0, -2)$ and $B(6, 7)$

The gradient of the line through A and B is:

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{7-(-2)}{6-0}$$
$$= \frac{9}{6}$$
$$= \frac{3}{2}$$

The equation of the line through A and B is:

$y = \frac{3}{2}x + c$

Substituting $x = 0$ and $y = -2$ into $y = \frac{3}{2}x + c$:

$-2 = \frac{3}{2}(0) + c$ so $c = -2$

As in Q9, the point A is the y-intercept so the equation can be written once the gradient has been calculated.

l_1: $y = \frac{3}{2}x - 2$

l_2: $x + y = 8$

To find point D, solve simultaneously by substituting l_1 into l_2.

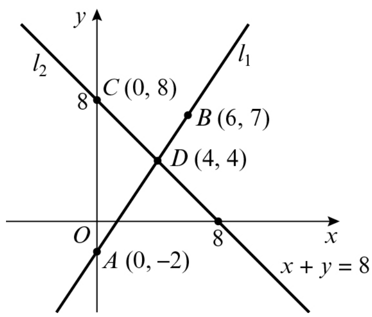
$$x + \tfrac{3}{2}x - 2 = 8$$
$$\tfrac{5}{2}x = 10$$
$$x = 4$$

10 when $x = 4$,

$4 + y = 8$,

$y = 4$

$\therefore$ D is the point $(4, 4)$.



The base of the triangle AC is 10 units.

The height of the triangle is 4 units.

Area ΔACD is $\frac{1}{2} \times 10 \times 4 = 20$ units^2

11 a $A(2, 16)$ and $B(12, -4)$

The equation of l_1 through A and B is:

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$
$$\frac{y-16}{-4-16} = \frac{x-2}{12-2}$$
$$\frac{y-16}{-20} = \frac{x-2}{10}$$

Multiply each side by -20:

$$y - 16 = -2(x-2) \quad \left(\text{Note: } -\frac{20}{10} = -2\right)$$
$$y - 16 = -2x + 4$$
$$y = -2x + 20$$
$$2x + y = 20$$

b The equation of l_2 through $C(-1, 1)$ with gradient $\frac{1}{3}$ is:

$$y - y_1 = m(x - x_1)$$
$$y - 1 = \tfrac{1}{3}(x - (-1))$$
$$y - 1 = \tfrac{1}{3}(x + 1)$$
$$y - 1 = \tfrac{1}{3}x + \tfrac{1}{3}$$
$$y = \tfrac{1}{3}x + \tfrac{4}{3}$$

12 a $A(-1, -2)$, $B(7, 2)$ and $C(k, 4)$

The gradient of AB is:

$$\frac{y_2-y_1}{x_2-x_1}=\frac{2-(-2)}{7-(-1)}$$
$$=\frac{4}{8}$$
$$=\frac{1}{2}$$

b Since ABC is a right angle the gradient of BC is:

$$\frac{-1}{\frac{1}{2}}=-2$$

$$\text{So}\ \frac{y_2-y_1}{x_2-x_1}=-2$$
$$\Rightarrow \frac{4-2}{k-7}=-2$$
$$\Rightarrow \frac{2}{k-7}=-2$$

Multiply each side by $(k-7)$:

$$2=-2(k-7)$$
$$2=-2k+14$$
$$2k=12$$
$$k=6$$

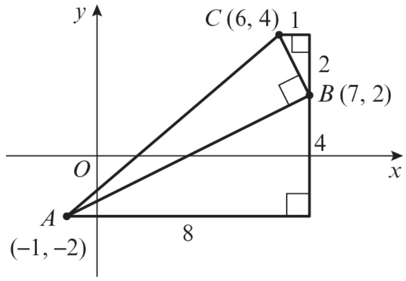
c The equation of the line passing through B and C is:

$$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}$$
$$\frac{y-2}{4-2}=\frac{x-7}{6-7}$$
$$\frac{y-2}{2}=\frac{x-7}{-1}$$

Multiply each side by 2:

$$y-2=-2(x-7)$$
$$y-2=-2x+14$$
$$y=-2x+16$$
$$2x+y=16$$
$$2x+y-16=0$$

d Remember angle B is a right angle.



Use the diagram or the distance formula to find the lengths AB and BC.

$$AB=\sqrt{8^2+4^2}$$
$$=\sqrt{80}$$
$$BC=\sqrt{1^2+2^2}$$
$$=\sqrt{5}$$

$$\text{Area of } \Delta ABC=\tfrac{1}{2}\times\sqrt{80}\times\sqrt{5}$$
$$=\tfrac{1}{2}\times\sqrt{400}$$
$$=\tfrac{1}{2}\times 20$$
$$=10\ \text{units}^2$$

13 a The equation of the line through $(-1, 5)$ and $(4, -2)$ is:

$$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}$$
$$\frac{y-5}{-2-5}=\frac{x-(-1)}{4-(-1)}$$
$$\frac{y-5}{-7}=\frac{x+1}{5}$$

Multiply each side by -35:

$$5(y-5)=-7(x+1)$$
$$5y-25=-7x-7$$
$$\left(\text{Note: } \tfrac{-35}{-7}=5 \text{ and } \tfrac{-35}{5}=-7\right)$$
$$7x+5y-25=-7$$
$$7x+5y-18=0$$

13 b For the coordinates of A, substitute $y = 0$:

$$7x+5(0)-18=0$$
$$7x-18=0$$
$$7x=18$$
$$x=\tfrac{18}{7}$$

The coordinates of A are $\left(\tfrac{18}{7},0\right)$.

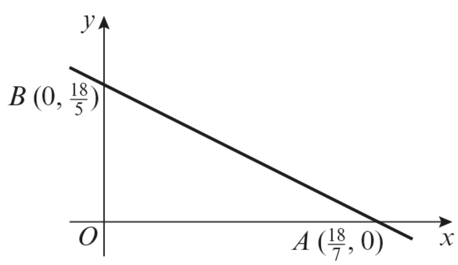
For the coordinates of B, substitute $x = 0$:

$$7(0)+5y-18=0$$
$$5y-18=0$$
$$5y=18$$
$$y=\tfrac{18}{5}$$

The coordinates of B are $\left(0,\tfrac{18}{5}\right)$.

The area of ΔOAB is:

$$\tfrac{1}{2}\times\tfrac{18}{7}\times\tfrac{18}{5}=\tfrac{162}{35}$$



14 a Rearrange l_1: $4y + x = 0$ into the form $y = mx + c$:

$$4y=-x$$
$$y=-\tfrac{1}{4}x$$

l_1 has gradient $-\tfrac{1}{4}$ and it meets the coordinate axes at (0, 0).

l_2 has gradient 2 and it meets the y-axis at (0, −3).

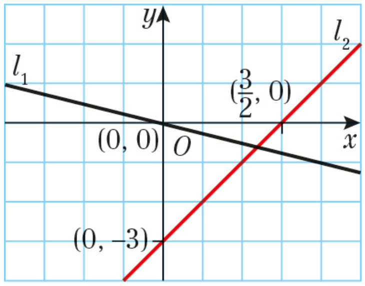
l_2 meets the x-axis when $y = 0$.

Substitute $y = 0$ into the equation:

$$0=2x-3$$
$$2x=3$$
$$x=\tfrac{3}{2}$$

l_2 meets the x-axis at $\left(\tfrac{3}{2},0\right)$.

14 a



b Solve $4y + x = 0$ and $y = 2x - 3$ simultaneously.

Substitute:

$$4(2x-3)+x=0$$
$$8x-12+x=0$$
$$9x-12=0$$
$$9x=12$$
$$x=\tfrac{12}{9}$$
$$x=\tfrac{4}{3}$$

Now substitute $x=\tfrac{4}{3}$ into $y = 2x - 3$:

$$y=2\left(\tfrac{4}{3}\right)-3$$
$$=\tfrac{8}{3}-3$$
$$=-\tfrac{1}{3}$$

The coordinates of A are $\left(\tfrac{4}{3},-\tfrac{1}{3}\right)$.

c The gradient of l_1 is $-\tfrac{1}{4}$.

The gradient of a line perpendicular to l_1 is $-\dfrac{1}{-\frac{1}{4}}=4$.

The equation of this line is:

$$y-y_1=m(x-x_1)$$
$$y-\left(-\tfrac{1}{3}\right)=4\left(x-\tfrac{4}{3}\right)$$
$$y+\tfrac{1}{3}=4x-\tfrac{16}{3}$$
$$y=4x-\tfrac{17}{3}$$

Multiply each term by 3:

$$3y=12x-17$$
$$0=12x-3y-17$$

The equation of the line is $12x - 3y - 17 = 0$.

15 a $A(4, 6)$ and $B(12, 2)$
The gradient of the line l_1 through A and B is:

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{2-6}{12-4} = \frac{-4}{8} = -\frac{1}{2}$$

The equation of l_1 is:
$y = -\frac{1}{2}x + c$
Substituting $x = 4$ and $y = 6$ into $y = -\frac{1}{2}x + c$:
$6 = -\frac{1}{2}(4) + c$
$c = 8$
$y = -\frac{1}{2}x + c$
$x + 2y - 16 = 0$

b The gradient of the line l_2 is $-\frac{2}{3}$, the y-intercept is 0.
$y = -\frac{2}{3}x$

c Solve $x + 2y - 16 = 0$ and $y = -\frac{2}{3}x$ simultaneously.

$$x + 2(-\tfrac{2}{3}x) - 16 = 0$$
$$x - \tfrac{4}{3}x - 16 = 0$$
$$-\tfrac{1}{3}x = 16$$
$$x = -48$$

When $x = -48$:
$y = -\frac{2}{3}(-48)$
$y = 32$
C is the point $(-48, 32)$.

d The gradient of OA is:

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{6-0}{4-0} = \frac{3}{2}$$

d The gradient of OC is:

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{32-0}{(-48)-0} = -\frac{2}{3}$$

$\frac{3}{2} \times -\frac{2}{3} = -1$.

Therefore the lines OA and OC are perpendicular.

e
$$OA = \sqrt{(4-0)^2 + (6-0)^2} = \sqrt{52} = \sqrt{4 \times 13} = 2\sqrt{13}$$
$$OC = \sqrt{((-48)-0)^2 + (32-0)^2} = \sqrt{3328} = \sqrt{256 \times 13} = 16\sqrt{13}$$

f Area of $\Delta OAB = \frac{1}{2} \times 16\sqrt{13} \times 2\sqrt{13}$
$= 208$ units^2

16 a $(4a, a)$ and $(-3a, 2a)$
The distance d between the points is:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{((-3a) - 4a)^2 + (2a - a)^2}$$
$$= \sqrt{49a^2 + a^2}$$
$$= \sqrt{50a^2}$$
$$= \sqrt{25 \times 2a^2}$$
$$= 5a\sqrt{2}$$

b For points $(4, 1)$ and $(-3, 2)$, $a = 1$.
Substitute $a = 1$ into $5a\sqrt{2}$.
Distance $= 5\sqrt{2}$

c For points $(12, 3)$ and $(-9, 6)$, $a = 3$.
Substitute $a = 3$ into $5a\sqrt{2}$.
Distance $= 15\sqrt{2}$

16 d For points (−20, −5) and (15, −10), $a = -5$.

Substitute $a = -5$ into $5a\sqrt{2}$.

Distance = $25\sqrt{2}$

17 a (x, y) is a point on $y = 3x$, so its coordinates are $(x, 3x)$.

The distance between $A(-1, 5)$ and $(x, 3x)$ is:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(x-(-1))^2 + (3x-5)^2}$$
$$= \sqrt{x^2 + 2x + 1 + 9x^2 - 30x + 25}$$
$$= \sqrt{10x^2 - 28x + 26}$$

b $\sqrt{10x^2 - 28x + 26} = \sqrt{74}$

$$10x^2 - 28x + 26 = 74$$
$$10x^2 - 28x - 48 = 0$$
$$5x^2 - 14x - 24 = 0$$
$$(5x+6)(x-4) = 0$$

$x = -\frac{6}{5}$ or $x = 4$

When $x = -\frac{6}{5}$, $y = 3(-\frac{6}{5}) = -\frac{18}{5}$

When $x = 4$, $y = 3(4) = 12$

The points are $B(-\frac{6}{5}, -\frac{18}{5})$ and $C(4, 12)$.

c The gradient of the line $y = 3x$ is 3, so the perpendicular line has gradient $-\frac{1}{3}$.

Its equation is:

$y = -\frac{1}{3}x + c$

When $x = -1$ and $y = 5$:

$5 = -\frac{1}{3}(-1) + c$

$c = \frac{14}{3}$

$y = -\frac{1}{3}x + \frac{14}{3}$

d Solving $y = -\frac{1}{3}x + \frac{14}{3}$ and $y = 3x$ simultaneously:

$3x = -\frac{1}{3}x + \frac{14}{3}$

17 d $9x = -x + 14$

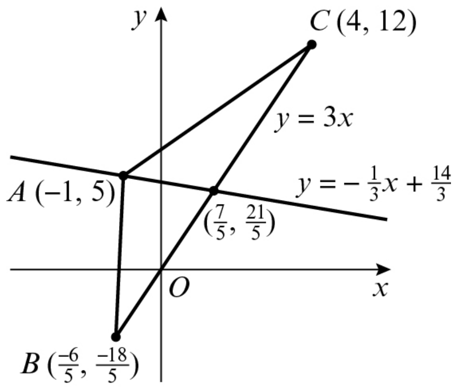
$10x = 14$

$x = \frac{7}{5}$

When $x = \frac{7}{5}$: $y = 3(\frac{7}{5}) = \frac{21}{5}$

The point is $(\frac{7}{5}, \frac{21}{5})$

e



$$BC = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(4-(-\tfrac{6}{5}))^2 + (12-(-\tfrac{18}{5}))^2}$$
$$= \sqrt{(\tfrac{26}{5})^2 + (\tfrac{78}{5})^2}$$
$$= \sqrt{\tfrac{6760}{25}}$$

Distance from $A(-1, 5)$ to $(\frac{7}{5}, \frac{21}{5})$ is:

$$\sqrt{(\tfrac{7}{5}-(-1))^2 + (\tfrac{21}{5}-5)^2}$$
$$= \sqrt{(\tfrac{12}{5})^2 + (-\tfrac{4}{5})^2}$$
$$= \sqrt{\tfrac{160}{25}}$$

Area of triangle is:

$$\tfrac{1}{2} \times \sqrt{\tfrac{6760}{25}} \times \sqrt{\tfrac{160}{25}} = \tfrac{520}{25}$$

$= 20.8$ units^2

18 a Gradient of the line of best fit is:

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{30\,000 - 23\,000}{3900 - 3200}$$
$$= \frac{7000}{700}$$
$$= 10$$

18 b $C = aP + b$
$a = 10$
Using the point (3200, 23 000):
$23\,000 = 10(3200) + b$
$b = -9000$
$C = 10P - 9000$

c a is the gradient, which is the increase in carbon dioxide emissions in millions of tonnes for every 1 million tonnes of oil pollution.

d The model is not valid for small values of P as a negative amount of carbon dioxide emissions is not possible.

Challenge

1 $(-2, -2)$, $B(13, 8)$ and $C(-4, 14)$

The equation of AB is:

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

$$\frac{y-(-2)}{8-(-2)} = \frac{x-(-2)}{13-(-2)}$$

$$\frac{y+2}{10} = \frac{x+2}{15}$$

$$3y + 6 = 2x + 4$$

$$3y = 2x - 2$$

$$y = \tfrac{2}{3}x - \tfrac{2}{3}$$

The gradient of AB = $\frac{2}{3}$.

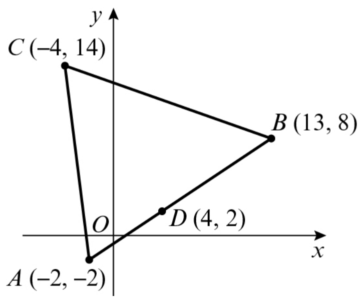
The gradient of a line perpendicular to AB = $-\frac{3}{2}$.

The equation of the perpendicular to AB through $C(-4, 14)$ is:

$$y - 14 = -\tfrac{3}{2}(x - (-4))$$

$$y - 14 = -\tfrac{3}{2}x - 6$$

$$y = -\tfrac{3}{2}x + 8$$



Point D is where the line and the perpendicular intersect.
Solve the equations $y = \frac{2}{3}x - \frac{2}{3}$ and $y = -\frac{3}{2}x + 8$ simultaneously.

$$\tfrac{2}{3}x - \tfrac{2}{3} = -\tfrac{3}{2}x + 8$$

Multiply each term by 6.

$$4x - 4 = -9x + 48$$

$$13x = 52$$

$$x = 4$$

Now substitute $x = 4$ into $y = -\frac{3}{2}x + 8$:

$$y = -\tfrac{3}{2}(4) + 8$$

$$y = 2$$

D is the point (4, 2).

Challenge

1 $AB=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

$=\sqrt{(13-(-2))^2+(8-(-2))^2}$

$=\sqrt{15^2+10^2}$

$=\sqrt{325}$

$CD=\sqrt{(4-(-4))^2+(2-14)^2}$

$=\sqrt{8^2+(-12)^2}$

$=\sqrt{208}$

Area of $\Delta ABC=\frac{1}{2}\times\sqrt{325}\times\sqrt{208}$

$= 130$ units^2

2 $A(3, 8)$, $B(9, 9)$ and $C(5, 2)$

The gradient of AB is:

$\frac{y_2-y_1}{x_2-x_1}=\frac{9-8}{9-3}$

$=\frac{1}{6}$

l_1 is perpendicular to AB, so its gradient is -6. It passes through C, so its equation is:

$y=-6x+c$

$2=-6(5)+c$

$c=32$

The equation of l_1 is $y=-6x+32$.

The gradient of BC is:

$\frac{y_2-y_1}{x_2-x_1}=\frac{2-9}{5-9}$

$=\frac{7}{4}$

l_2 is perpendicular to BC, so its gradient is $-\frac{4}{7}$. It passes through A, so its equation is:

$y=-\frac{4}{7}x+c$

$8=-\frac{4}{7}(3)+c$

$c=\frac{68}{7}$

The equation of l_2 is $y=-\frac{4}{7}x+\frac{68}{7}$.

2 The gradient of AC is:

$\frac{y_2-y_1}{x_2-x_1}=\frac{2-8}{5-3}$

$=-\frac{6}{2}$

$=-3$

l_3 is perpendicular to BC, so its gradient is $\frac{1}{3}$. It passes through B, so its equation is:

$y=\frac{1}{3}x+c$

$9=\frac{1}{3}(9)+c$

$c=6$

The equation of l_3 is $y=\frac{1}{3}x+6$.

Solve l_1 and l_2 simultaneously.

$-6x+32=-\frac{4}{7}x+\frac{68}{7}$

$-42x+224=-4x+68$

$38x=156$

$x=\frac{78}{19}$

$y=-6(\frac{78}{19})+32=\frac{140}{19}$

Their point of intersection is $(\frac{78}{19}, \frac{140}{19})$.

Now solve l_2 and l_3 simultaneously.

$-\frac{4}{7}x+\frac{68}{7}=\frac{1}{3}x+6$

$-12x+204=7x+126$

$19x=78$

$x=\frac{78}{19}$

$y=\frac{1}{3}(\frac{78}{19})+6=\frac{140}{19}$

Their point of intersection is $(\frac{78}{19},\frac{140}{19})$.

Therefore, l_1, l_2 and l_3 all intersect at $(\frac{78}{19},\frac{140}{19})$.

3 $A(0, 0)$, $B(a, b)$ and $C(c, 0)$

The gradient of AB is:

$\frac{y_2-y_1}{x_2-x_1}=\frac{b-0}{a-0}$

$=\frac{b}{a}$

l_1 is perpendicular to AB so its gradient is $-\frac{a}{b}$.

3 It passes through C so its equation is:

$$y = -\frac{a}{b}x + k \text{ where } k \text{ is the } y\text{-intercept.}$$

At C, $x = c$ and $y = 0$.

$$0 = -\frac{ac}{b} + k$$

$$k = \frac{ac}{b}$$

The equation of line l_1 is:

$$y = -\frac{a}{b}x + \frac{ac}{b}$$

The gradient of BC is:

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{0 - b}{c - a}$$

$$= \frac{-b}{c - a}$$

l_2 is perpendicular to BC so its gradient is $\frac{c - a}{b}$.

It passes through A, so its equation is:

$$y = \frac{c - a}{b}x + K \text{ where } K \text{ is the } y\text{-intercept.}$$

At A, $x = 0$, $y = 0$.

$$0 = \frac{c - a}{b}(0) + K$$

$$K = 0$$

The equation of line l_2 is $y = \frac{c - a}{b}x$.

l_3 is the vertical line through (a, b), so its equation is $x = a$.

Solve l_1 and l_3 simultaneously.

$$y = -\frac{a^2}{b} + \frac{ac}{b}$$

$$= \frac{a(c - a)}{b}$$

The intersection of l_1 and l_3 is the point $\left(a, \frac{a(c - a)}{b}\right)$.

Now solve l_2 and l_3 simultaneously.

$$y = \frac{a(c - a)}{b}$$

The intersection of l_2 and l_3 is the point $\left(a, \frac{a(c - a)}{b}\right)$.

Therefore, l_1, l_2 and l_3 all intersect at $\left(a, \frac{a(c - a)}{b}\right)$.

Circles 6A

1 a $(x_1, y_1) = (4, 2), (x_2, y_2) = (6, 8)$

So $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{4+6}{2}, \frac{2+8}{2}\right) = \left(\frac{10}{2}, \frac{10}{2}\right) = (5, 5)$

b $(x_1, y_1) = (0, 6), (x_2, y_2) = (12, 2)$

So $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{0+12}{2}, \frac{6+2}{2}\right) = \left(\frac{12}{2}, \frac{8}{2}\right) = (6, 4)$

c $(x_1, y_1) = (2, 2), (x_2, y_2) = (-4, 6)$

So $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{2+(-4)}{2}, \frac{2+6}{2}\right) = \left(\frac{-2}{2}, \frac{8}{2}\right) = (-1, 4)$

d $(x_1, y_1) = (-6, 4), (x_2, y_2) = (6, -4)$

So $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{-6+6}{2}, \frac{4+(-4)}{2}\right) = \left(\frac{0}{2}, \frac{0}{2}\right) = (0, 0)$

e $(x_1, y_1) = (7, -4), (x_2, y_2) = (-3, 6)$

So $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{7+(-3)}{2}, \frac{-4+6}{2}\right) = \left(\frac{4}{2}, \frac{2}{2}\right) = (2, 1)$

f $(x_1, y_1) = (-5, -5), (x_2, y_2) = (-11, 8)$

So $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{-5+(-11)}{2}, \frac{-5+8}{2}\right) = \left(\frac{-16}{2}, \frac{3}{2}\right) = \left(-8, \frac{3}{2}\right)$

g $(x_1, y_1) = (6a, 4b), (x_2, y_2) = (2a, -4b)$

So $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{6a+2a}{2}, \frac{4b+(-4b)}{2}\right) = \left(\frac{8a}{2}, \frac{0}{2}\right) = (4a, 0)$

h $(x_1, y_1) = (-4u, 0), (x_2, y_2) = (3u, -2v)$

So $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{-4u+3u}{2}, \frac{0+(-2v)}{2}\right) = \left(\frac{-u}{2}, -v\right)$

1 i $(x_1, y_1) = (a+b, 2a-b), (x_2, y_2) = (3a-b, -b)$

So $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = \left(\frac{a+b+3a-b}{2}, \frac{2a-b+(-b)}{2}\right) = \left(\frac{4a}{2}, \frac{2a-2b}{2}\right) = (2a, a-b)$

j $(x_1, y_1) = (4\sqrt{2}, 1), (x_2, y_2) = (2\sqrt{2}, 7)$

So $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = \left(\frac{4\sqrt{2}+2\sqrt{2}}{2}, \frac{1+7}{2}\right) = \left(\frac{6\sqrt{2}}{2}, \frac{8}{2}\right) = (3\sqrt{2}, 4)$

k $(x_1, y_1) = (\sqrt{2}-\sqrt{3}, 3\sqrt{2}+4\sqrt{3}), (x_2, y_2) = (3\sqrt{2}+\sqrt{3}, -\sqrt{2}+2\sqrt{3})$

So $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = \left(\frac{\sqrt{2}-\sqrt{3}+3\sqrt{2}+\sqrt{3}}{2}, \frac{3\sqrt{2}+4\sqrt{3}+(-\sqrt{2}+2\sqrt{3})}{2}\right)$

$= \left(\frac{4\sqrt{2}}{2}, \frac{2\sqrt{2}+6\sqrt{3}}{2}\right)$

$= (2\sqrt{2}, \sqrt{2}+3\sqrt{3})$

2 A(−2, 5) and B(a, b), midpoint M(4, 3)

Midpoint = $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

$(4, 3) = \left(\frac{-2+a}{2}, \frac{5+b}{2}\right)$

$4 = \frac{-2+a}{2}$ and $3 = \frac{5+b}{2}$

$a = 10$ and $b = 1$

3 $(x_1, y_1) = (-4, 6), (x_2, y_2) = (7, 8)$

So $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = \left(\frac{-4+7}{2}, \frac{6+8}{2}\right) = \left(\frac{3}{2}, \frac{14}{2}\right) = \left(\frac{3}{2}, 7\right)$

The centre is $\left(\frac{3}{2}, 7\right)$.

4 $(x_1, y_1) = \left(\frac{4a}{5}, \frac{-3b}{4}\right), (x_2, y_2) = \left(\frac{2a}{5}, \frac{5b}{4}\right)$

So $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = \left(\frac{\frac{4a}{5}+\frac{2a}{5}}{2}, \frac{\frac{-3b}{4}+\frac{5b}{4}}{2}\right) = \left(\frac{\frac{6a}{5}}{2}, \frac{\frac{2b}{4}}{2}\right) = \left(\frac{3a}{5}, \frac{b}{4}\right)$

4 The centre is $\left(\frac{3a}{5}, \frac{b}{4}\right)$.

5 a A(−3, −4) and B(6, 10)

centre of circle = midpoint = $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = \left(\frac{-3+6}{2}, \frac{-4+10}{2}\right) = \left(\frac{3}{2}, 3\right)$

b $y = 2x$

When $x = \frac{3}{2}$, $y = 2(\frac{3}{2}) = 3$, therefore the centre of the circle $(\frac{3}{2}, 3)$ lies on the line $y = 2x$.

6 $J\left(\frac{3}{4}, \frac{4}{3}\right)$ and $K\left(-\frac{1}{2}, 2\right)$

centre of circle = midpoint = $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = \left(\frac{\frac{3}{4}+\left(-\frac{1}{2}\right)}{2}, \frac{\frac{4}{3}+2}{2}\right) = \left(\frac{1}{8}, \frac{5}{3}\right)$

$y = 8x + b$

At $\left(\frac{1}{8}, \frac{5}{3}\right)$, $\frac{5}{3} = 8\left(\frac{1}{8}\right) + b$

$b = \frac{2}{3}$

7 $(x_1, y_1) = (0, -2), (x_2, y_2) = (6, -5)$

So $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = \left(\frac{0+6}{2}, \frac{-2+(-5)}{2}\right) = \left(\frac{6}{2}, \frac{-7}{2}\right) = \left(3, -\frac{7}{2}\right)$

Substitute $x = 3$ and $y = -\frac{7}{2}$ into $x - 2y - 10 = 0$:

$(3) - 2\left(-\frac{7}{2}\right) - 10 = 3 + 7 - 10 = 0$

So the centre is on the line $x - 2y - 10 = 0$.

8 $(x_1, y_1) = (a, b), (x_2, y_2) = (2, -3)$

The centre is (6, 1) so

$$\left(\frac{a+2}{2}, \frac{b+(-3)}{2}\right) = (6, 1)$$

$$\frac{a+2}{2} = 6$$

8 $a+2=12$

$a=10$

$$\frac{b+(-3)}{2}=1$$

$$\frac{b-3}{2}=1$$

$$b-3=2$$

$$b=5$$

The coordinates of G are (10, 5).

9 $(x_1, y_1)=(p,q),(x_2,y_2)=(3a,-7a)$

The centre is ($-2a$, $5a$) so

$$\left(\frac{p+3a}{2},\frac{q+(-7a)}{2}\right)=(-2a,5a)$$

$$\frac{p+3a}{2}=-2a$$

$$p+3a=-4a$$

$$p=-7a$$

$$\frac{q+(-7a)}{2}=5a$$

$$\frac{q-7a}{2}=5a$$

$$q-7a=10a$$

$$q=17a$$

The coordinates of C are ($-7a$, $17a$).

10 $(x_1, y_1)=(3,p),(x_2,y_2)=(q,4)$ so

$$\left(\frac{3+q}{2},\frac{p+4}{2}\right)=(5,6)$$

$$\frac{3+q}{2}=5$$

$$3+q=10$$

$$q=7$$

$$\frac{p+4}{2}=6$$

$$p+4=12$$

$$p=8$$

so $p=8, q=7$

11 $(x_1, y_1) = (-4, 2a), (x_2, y_2) = (3b, -4)$ so

$$\left(\frac{-4+3b}{2}, \frac{2a-4}{2}\right) = (b, 2a)$$

$$\frac{-4+3b}{2} = b$$

$$-4+3b = 2b$$

$$-4 = -b$$

$$b = 4$$

$$\frac{2a-4}{2} = 2a$$

$$2a-4 = 4a$$

$$-4 = 2a$$

$$a = -2$$

so $a = -2, b = 4$

Challenge

a $B(7, 11)$ and $C(p, q)$. Midpoint of BC is $M(8, 5)$

Midpoint = $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

$(8, 5) = \left(\frac{7+p}{2}, \frac{11+q}{2}\right)$

$8 = \frac{7+p}{2}$ and $5 = \frac{11+q}{2}$

$p = 9$ and $q = -1$

b $A(3, 5)$ and $B(7, 11)$

Midpoint AB = = $\left(\frac{3+7}{2}, \frac{5+11}{2}\right)$ = (5, 8)

Equation of line joining (5, 8) and $M(8, 5)$:

$$\frac{y-8}{5-8} = \frac{x-5}{8-5}$$

$$y-8 = -(x-5)$$

$$y = -x+13$$

c Gradient of line AC = $\frac{y_2-y_1}{x_2-x_1} = \frac{(-1)-5}{9-3} = -1$

The gradient of the line $y = -x + 13$ is -1. The two gradients are equal so the two lines are parallel.

Circles 6B

1 a $A(-5, 8)$ and $B(7, 2)$

Midpoint = $\left(\frac{-5+7}{2}, \frac{8+2}{2}\right)$

$= (1, 5)$

The gradient of the line segment $AB = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2-8}{7-(-5)} = -\frac{1}{2}$

So the gradient of the line perpendicular to AB is 2.

The equation of the perpendicular line is

$y - y_1 = m(x - x_1)$

$m = 2$ and $(x_1, y_1) = (1, 5)$

So $y - 5 = 2(x - 1)$

$y = 2x + 3$

b $C(-4, 7)$ and $D(2, 25)$

Midpoint = $\left(\frac{-4+2}{2}, \frac{7+25}{2}\right) = (-1, 16)$

The gradient of the line segment $CD = \frac{y_2 - y_1}{x_2 - x_1} = \frac{25-7}{2-(-4)} = 3$

So the gradient of the line perpendicular to CD is $-\frac{1}{3}$.

The equation of the perpendicular line is

$y - y_1 = m(x - x_1)$

$m = -\frac{1}{3}$ and $(x_1, y_1) = (-1, 16)$

So $y - 16 = -\frac{1}{3}(x - (-1))$

$y = -\frac{1}{3}x + \frac{47}{3}$

c $E(3, -3)$ and $F(13, -7)$

Midpoint = $\left(\frac{3+13}{2}, \frac{(-3)+(-7)}{2}\right) = (8, -5)$

The gradient of the line segment $EF = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-7-(-3)}{13-3} = -\frac{2}{5}$

So the gradient of the line perpendicular to EF is $\frac{5}{2}$.

The equation of the perpendicular line is

$y - y_1 = m(x - x_1)$

1 c $m = \frac{5}{2}$ and $(x_1, y_1) = (8, -5)$

So $y - (-5) = \frac{5}{2}(x - 8)$

$y + 5 = \frac{5}{2}x - 20$

$y = \frac{5}{2}x - 25$

d $P(-4, 7)$ and $Q(-4, -1)$

Midpoint = $\left(\frac{-4+(-4)}{2}, \frac{7+(-1)}{2}\right) = (-4, 3)$

P and Q both have x-coordinates of -4, so this is the line $x = -4$. So the perpendicular to PQ is a horizontal line with gradient 0.

The equation of the perpendicular line is

$y - y_1 = m(x - x_1)$

$m = 0$ and $(x_1, y_1) = (-4, 3)$

So $y - 3 = 0(x - (-4))$

$y = 3$

e $S(4, 11)$ and $T(-5, -1)$

Midpoint = $\left(\frac{4+(-5)}{2}, \frac{11+(-1)}{2}\right) = \left(-\frac{1}{2}, 5\right)$

The gradient of the line segment $ST = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1-11}{-5-4} = \frac{4}{3}$

So the gradient of the line perpendicular to ST is $-\frac{3}{4}$.

The equation of the perpendicular line is

$y - y_1 = m(x - x_1)$

$m = -\frac{3}{4}$ and $(x_1, y_1) = \left(-\frac{1}{2}, 5\right)$

So $y - 5 = -\frac{3}{4}\left(x - \left(-\frac{1}{2}\right)\right)$

$y - 5 = -\frac{3}{4}x - \frac{3}{8}$

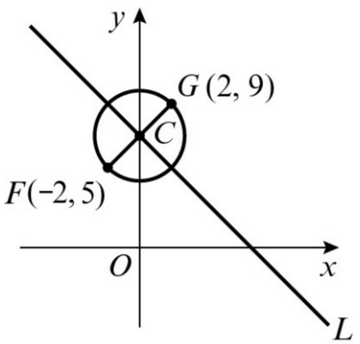
$y = -\frac{3}{4}x + \frac{37}{8}$

1 f $X(13, 11)$ and $Y(5, 11)$

Midpoint = $\left(\frac{13+5}{2}, \frac{11+11}{2}\right) = (9, 11)$

The y-coordinates of points X and Y are both 11, so this is the line $y = 11$.
So the equation of the perpendicular line is $x = \text{a}$.
The line passes through the point (9, 11) so a = 9.
$x = 9$

2



The gradient of FG is

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{9-5}{2-(-2)} = \frac{4}{4} = 1$$

The gradient of a line perpendicular to FG is

$$\frac{-1}{(1)} = -1.$$

C is the mid-point of FG, so the coordinates of C are

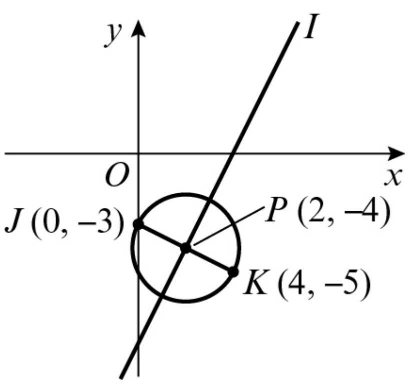
$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = \left(\frac{-2+2}{2}, \frac{5+9}{2}\right) = \left(\frac{0}{2}, \frac{14}{2}\right) = (0, 7)$$

The equation of l is

$$y - y_1 = m(x - x_1)$$
$$y - 7 = -1(x - 0)$$
$$y - 7 = -x$$
$$y = -x + 7$$

Or we could have recognised immediately that (0, 7) is the y-intercept.

3



The gradient of JK is

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{-5-(-3)}{4-0} = \frac{-5+3}{4} = -\frac{2}{4} = -\frac{1}{2}$$

The gradient of a line perpendicular to JK is

$$\frac{-1}{\left(-\frac{1}{2}\right)} = 2$$

3 P is the mid-point of JK, so the coordinates of P are

$$\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)=\left(\frac{0+4}{2},\frac{-3+(-5)}{2}\right)=\left(\frac{4}{2},-\frac{8}{2}\right)=(2,-4)$$

The equation of l is

$$y-y_1=m(x-x_1)$$
$$y-(-4)=2(x-2)$$
$$y+4=2x-4$$
$$0=2x-y-4-4$$
$$2x-y-8=0$$

4 a $A(-4, -9)$ and $B(6, -3)$

Midpoint = $\left(\frac{-4+6}{2},\frac{-9+(-3)}{2}\right)$ = (1, −6)

The gradient of the line segment AB = $\frac{y_2-y_1}{x_2-x_1}=\frac{-3-(-9)}{6-(-4)}=\frac{3}{5}$

So the gradient of the line perpendicular to AB is $-\frac{5}{3}$.

The equation of the perpendicular line is

$y-y_1=m(x-x_1)$

$m=-\frac{5}{3}$ and $(x_1,y_1)=(1,-6)$

So $y-(-6)=-\frac{5}{3}(x-1)$

$$y+6=-\frac{5}{3}x+\frac{5}{3}$$
$$y=-\frac{5}{3}x-\frac{13}{3}$$

b $C(11, 5)$ and $D(-1, 9)$

Midpoint = $\left(\frac{11+(-1)}{2},\frac{5+9}{2}\right)$ = (5, 7)

The gradient of the line segment CD = $\frac{y_2-y_1}{x_2-x_1}=\frac{9-5}{-1-11}=-\frac{1}{3}$

So the gradient of the line perpendicular to CD is 3.

The equation of the perpendicular line is

$y-y_1=m(x-x_1)$

$m=3$ and $(x_1,y_1)=(5,7)$

So $y-7=3(x-5)$

$$y=3x-8$$

4 c Solve the two perpendicular bisectors simultaneously

$$-\frac{5}{3}x-\frac{13}{3}=3x-8$$

$$-5x-13=9x-24$$

$$14x=11$$

$x=\frac{11}{14}$, so $y=3\left(\frac{11}{14}\right)-8=-\frac{79}{14}$

$$\left(\frac{11}{14},-\frac{79}{14}\right)$$

5 $X(7, -2)$ and $Y(4, q)$

The gradient of the line segment $XY=\frac{y_2-y_1}{x_2-x_1}=\frac{q-(-2)}{4-7}=\frac{q+2}{-3}$

From the equation of the perpendicular bisector of PQ, $y = 4x + b$, the gradient is 4

Therefore, the gradient of $XY=-\frac{1}{4}$, so $-\frac{1}{4}=\frac{q+2}{-3}$

$q=-\frac{5}{4}$

Midpoint of $XY=\left(\frac{7+4}{2},\frac{-2+\left(-\frac{5}{4}\right)}{2}\right)=\left(\frac{11}{2},-\frac{13}{8}\right)$

Substituting $x=\frac{11}{2}$ and $y=-\frac{13}{8}$ into $y = 4x + b$ gives

$$-\frac{13}{8}=4\left(\frac{11}{2}\right)+b$$

$$b=-\frac{189}{8}$$

So $b=-\frac{189}{8}$, $q=-\frac{5}{4}$

Challenge

a $P(6, 9)$ and $Q(3, -3)$

Midpoint of $PQ = \left(\frac{6+3}{2}, \frac{9+(-3)}{2}\right) = \left(\frac{9}{2}, 3\right)$

The gradient of the line segment $PQ = \frac{-3-9}{3-6} = 4$

So the gradient of the line perpendicular to PQ is $-\frac{1}{4}$.

The equation of the perpendicular line is

$y - y_1 = m(x - x_1)$

$m = -\frac{1}{4}$ and $(x_1, y_1) = \left(\frac{9}{2}, 3\right)$

So $y - 3 = -\frac{1}{4}\left(x - \frac{9}{2}\right)$

$y = -\frac{1}{4}x + \frac{33}{8}$

$R(-9, 3)$ and $Q(3, -3)$

Midpoint of $RQ = \left(\frac{(-3)+3}{2}, \frac{3+(-9)}{2}\right) = (-3, 0)$

The gradient of the line segment $RQ = \frac{3-(-3)}{-9-3} = -\frac{1}{2}$

So the gradient of the line perpendicular to RQ is 2.

The equation of the perpendicular line is

$y - y_1 = m(x - x_1)$

$m = 2$ and $(x_1, y_1) = (-3, 0)$

So $y - 0 = 2(x - (-3))$

$y = 2x + 6$

$P(6, 9)$ and $R(-9, 3)$

Midpoint of $PR = \left(\frac{6+(-9)}{2}, \frac{9+3}{2}\right) = \left(-\frac{3}{2}, 6\right)$

The gradient of the line segment $PR = \frac{3-9}{-9-6} = \frac{2}{5}$

So the gradient of the line perpendicular to PR is $-\frac{5}{2}$.

The equation of the perpendicular line is

$y - y_1 = m(x - x_1)$

$m = -\frac{5}{2}$ and $(x_1, y_1) = \left(-\frac{3}{2}, 6\right)$

a So $y-6=-\frac{5}{2}\left(x-\left(-\frac{3}{2}\right)\right)$

$$y-6=-\frac{5}{2}x-\frac{15}{4}$$

$$y=-\frac{5}{2}x+\frac{9}{4}$$

b Solving each pair of pair of perpendicular bisectors simultaneously

PQ: $y = -\frac{1}{4}x + \frac{33}{8}$ and RQ: $y = 2x + 6$

$$-\frac{1}{4}x+\frac{33}{8}=2x+6$$

$$-2x+33=16x+48$$

$$18x=-15$$

$$x = -\frac{5}{6}, y = 2\left(-\frac{5}{6}\right)+6 = \frac{13}{3}$$

Lines PQ and RQ intersect at the point $\left(-\frac{5}{6},\frac{13}{3}\right)$

RQ: $y = 2x + 6$ and PR: $y = -\frac{5}{2}x + \frac{9}{4}$

$$2x+6=-\frac{5}{2}x+\frac{9}{4}$$

$$8x+24=-10x+9$$

$$18x=-15$$

$$x = -\frac{5}{6}, y = 2\left(-\frac{5}{6}\right) + 6 = \frac{13}{3}$$

Therefore, all three perpendicular bisectors meet at the point $\left(-\frac{5}{6},\frac{13}{3}\right)$

Circles 6C

1 a $(x_1, y_1) = (3, 2), r = 4$

So $(x-3)^2 + (y-2)^2 = 4^2$

$(x-3)^2 + (y-2)^2 = 16$

b $(x_1, y_1) = (-4, 5), r = 6$

So $(x-(-4))^2 + (y-5)^2 = 6^2$

$(x+4)^2 + (y-5)^2 = 36$

c $(x_1, y_1) = (5, -6), r = 2\sqrt{3}$

So $(x-5)^2 + (y-(-6))^2 = (2\sqrt{3})^2$

$(x-5)^2 + (y+6)^2 = 2^2(\sqrt{3})^2$

$(x-5)^2 + (y+6)^2 = 4 \times 3$

$(x-5)^2 + (y+6)^2 = 12$

d $(x_1, y_1) = (2a, 7a), r = 5a$

So $(x-2a)^2 + (y-7a)^2 = (5a)^2$

$(x-2a)^2 + (y-7a)^2 = 25a^2$

e $(x_1, y_1) = (-2\sqrt{2}, -3\sqrt{2}), r = 1$

So $(x-(-2\sqrt{2}))^2 + (y-(-3\sqrt{2}))^2 = 1^2$

$(x+2\sqrt{2})^2 + (y+3\sqrt{2})^2 = 1$

2 a $(x+5)^2 + (y-4)^2 = 9^2$

$(x-(-5))^2 + (y-4)^2 = 9^2$

The centre of the circle is (–5, 4) and the radius is 9.

b $(x-7)^2 + (y-1)^2 = 16$

$(x-7)^2 + (y-1)^2 = 4^2$

The centre of the circle is (7, 1) and the radius is 4.

2 c $\left(x+4\right)^2+y^2=25$

$\left(x-(-4)\right)^2+\left(y-0\right)^2=5^2$

The centre of the circle is (–4, 0) and the radius is 5.

d $\left(x+4a\right)^2+\left(y+a\right)^2=144a^2$

$\left(x-(-4a)\right)^2+\left(y-(-a)\right)^2=\left(12a\right)^2$

The centre of the circle is (–4a, –a) and the radius is 12a.

e $\left(x-3\sqrt{5}\right)^2+\left(y+\sqrt{5}\right)^2=27$

$\left(x-3\sqrt{5}\right)^2+\left(y-\left(-\sqrt{5}\right)\right)^2=\left(\sqrt{27}\right)^2$

Now $\sqrt{27}=\sqrt{9\times 3}=\sqrt{9}\times\sqrt{3}=3\sqrt{3}$

The centre of the circle is $\left(3\sqrt{5},-\sqrt{5}\right)$ and the radius is $3\sqrt{3}$.

3 a Substitute $x = 4$, $y = 8$ into $(x – 2)^2 + (y – 5)^2 = 13$

$(x – 2)^2 + (y – 5)^2 = (4 – 2)^2 + (8 – 5)^2 = 2^2 + 3^2 = 4 + 9 = 13$ ✓

So the circle passes through (4, 8).

b Substitute $x = 0$, $y = –2$ into $(x + 7)^2 + (y – 2)^2 = 65$

$(x +7)^2 + (y – 2)^2 = (0 + 7)^2 + (–2 – 2)^2 = 7^2 + (–4)^2 = 49 + 16 = 65$ ✓

So the circle passes through (0, –2).

c Substitute $x = 7$, $y = –24$ into $x^2 + y^2 = 25^2$

$x^2 + y^2 = 7^2 + (–24)^2 = 49 + 576 = 625 = 25^2$ ✓

So the circle passes through (7, –24).

d Substitute $x = 6a$, $y = –3a$ into $(x – 2a)^2 + (y + 5a)^2 = 20a^2$

$(x – 2a)^2 + (y + 5a)^2 = (6a – 2a)^2 + (–3a + 5a)^2 = (4a)^2 + (2a)^2 = 16a^2 + 4a^2 = 20a^2$ ✓

So the circle passes through (6a, –3a).

e Substitute $x=\sqrt{5}, y=-\sqrt{5}$ into $\left(x-3\sqrt{5}\right)^2+\left(y-\sqrt{5}\right)^2=\left(2\sqrt{10}\right)^2$

$$\left(x-3\sqrt{5}\right)^2+\left(y-\sqrt{5}\right)^2=\left(\sqrt{5}-3\sqrt{5}\right)^2+\left(-\sqrt{5}-\sqrt{5}\right)^2=\left(-2\sqrt{5}\right)^2+\left(-2\sqrt{5}\right)^2$$

$$=4\times 5+4\times 5=20+20=40=\left(\sqrt{40}\right)^2$$

Now $\sqrt{40}=\sqrt{4\times 10}=\sqrt{4}\times\sqrt{10}=2\sqrt{10}$ ✓

So the circle passes through $\left(\sqrt{5},-\sqrt{5}\right)$.

4 The radius of the circle is

$$\begin{aligned}\sqrt{(x_2-x_1)^2+(y_2-y_1)^2} &= \sqrt{(4-8)^2+((-2)-1)^2}\\ &= \sqrt{4^2+3^2}\\ &= \sqrt{16+9}\\ &= \sqrt{25}\\ &= 5\end{aligned}$$

The centre of the circle is (8, 1) and the radius is 5.
So $(x-8)^2+(y-1)^2=5^2$
or $(x-8)^2+(y-1)^2=25$

5 P (5, 6) and Q (−2, 2)
The centre of the circle is

$$\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)=\left(\frac{5+(-2)}{2},\frac{6+2}{2}\right)=\left(\frac{3}{2},\frac{8}{2}\right)=\left(\frac{3}{2},4\right)$$

The radius of the circle is

$$\begin{aligned}\sqrt{(x_2-x_1)^2+(y_2-y_1)^2} &= \sqrt{\left(5-\frac{3}{2}\right)^2+(6-4)^2}\\ &= \sqrt{\left(\frac{7}{2}\right)^2+(2)^2}\\ &= \sqrt{\frac{49}{4}+4}\\ &= \sqrt{\frac{49}{4}+\frac{16}{4}}\\ &= \sqrt{\frac{65}{4}}\end{aligned}$$

So the equation of the circle is

$$\left(x-\frac{3}{2}\right)^2+(y-4)^2=\left(\sqrt{\frac{65}{4}}\right)^2 \text{ or } \left(x-\frac{3}{2}\right)^2+(y-4)^2=\frac{65}{4}$$

6 Substitute $x=1,\ y=-3$ into $(x-3)^2+(y+4)^2=r^2$

$$\begin{aligned}(1-3)^2+(-3+4)^2 &= r^2\\ (-2)^2+(1)^2 &= r^2\\ 5 &= r^2\end{aligned}$$

So $r=\sqrt{5}$

7 a Substitute $(2,2)$ into $(x-2)^2+(y-4)^2=r^2$

$$(2-2)^2+(2-4)^2=r^2$$
$$0^2+(-2)^2=r^2$$
$$r^2=4$$
$$r=2$$

b The distance between (2, 2) and $\left(2+\sqrt{3},5\right)$ is

$$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}=\sqrt{\left(2+\sqrt{3}-2\right)^2+(5-2)^2}$$
$$=\sqrt{\left(\sqrt{3}\right)^2+3^2}$$
$$=\sqrt{3+9}$$
$$=\sqrt{12}$$

The distance between (2, 2) and $\left(2-\sqrt{3},5\right)$ is

$$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}=\sqrt{\left(2-\sqrt{3}-2\right)^2+(5-2)^2}$$
$$=\sqrt{\left(-\sqrt{3}\right)^2+(3)^2}$$
$$=\sqrt{3+9}$$
$$=\sqrt{12}$$

The distance between $\left(2+\sqrt{3},5\right)$ and $\left(2-\sqrt{3},5\right)$ is

$$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}=\sqrt{\left(\left(2-\sqrt{3}\right)-\left(2+\sqrt{3}\right)\right)^2+(5-5)^2}$$
$$=\sqrt{\left(2-\sqrt{3}-2-\sqrt{3}\right)^2+0^2}$$
$$=\sqrt{\left(-2\sqrt{3}\right)^2}$$
$$=\sqrt{4\times 3}$$
$$=\sqrt{12}$$

PQ, *QR* and *PR* all equal $\sqrt{12}$.
So ΔPQR is equilateral.

8 a Rearrange $x^2+y^2-4x-11=0$ into the form $(x-a)^2+y^2=r^2$

$x^2-4x-11+y^2=0$

Completing the square gives

$(x-2)^2-4-11+y^2=0$

$(x-2)^2+y^2=15$

b Centre of the circle = (2, 0), radius = $\sqrt{15}$

9 a $x^2+y^2-10x+4y-20=0$

$x^2-10x+y^2+4y-20=0$

Completing the square gives

$(x-5)^2-25+(y+2)^2-4-20=0$

$(x-5)^2+(y+2)^2=49$

b Centre of the circle = (5, −2), radius = 7

10 a $x^2+y^2-2x+8y-8=0$

$x^2-2x+y^2+8y-8=0$

Completing the square gives

$(x-1)^2-1+(y+4)^2-16-8=0$

$(x-1)^2+(y+4)^2=25$

Centre of the circle = (1, −4), radius = 5

b $x^2+y^2+12x-4y=9$

$x^2+12x+y^2-4y=9$

Completing the square gives

$(x+6)^2-36+(y-2)^2-4=9$

$(x+6)^2+(y-2)^2=49$

Centre of the circle = (−6, 2), radius = 7

c $x^2+y^2-6y=22x-40$

$x^2-22x+y^2-6y=-40$

Completing the square gives

$(x-11)^2-121+(y-3)^2-9=-40$

$(x-11)^2+(y-3)^2=90$

Centre of the circle = (11, 3), radius = $\sqrt{90}=3\sqrt{10}$

d $x^2+y^2+5x-y+4=2y+8$

$x^2+5x+y^2-3y=4$

Completing the square gives

$$\left(x+\frac{5}{2}\right)^2-\frac{25}{4}+\left(y-\frac{3}{2}\right)^2-\frac{9}{4}=4$$

10 d $\left(x+\frac{5}{2}\right)^2+\left(y-\frac{3}{2}\right)^2=\frac{25}{2}$

Centre of the circle = $\left(-\frac{5}{2},\frac{3}{2}\right)$, radius = $\frac{5}{\sqrt{2}}=\frac{5\sqrt{2}}{2}$

e $2x^2 + 2y^2 - 6x + 5y = 2x - 3y - 3$
$2x^2 - 8x + 2y^2 + 8y = -3$
Completing the square gives
$2((x-2)^2 - 4) + 2((y+2)^2 - 4) = -3$
$2(x-2)^2 - 8 + 2(y+2)^2 - 8 = -3$
$2(x-2)^2 + 2(y+2)^2 = 13$
$(x-2)^2 + (y+2)^2 = \frac{13}{2}$
Centre of the circle = (2, −2), radius = $\sqrt{\frac{13}{2}}=\frac{\sqrt{26}}{2}$

11 a $x^2 + y^2 + 12x + 2y = k$
$x^2 + 12x + y^2 + 2y = k$
$(x+6)^2 - 36 + (y+1)^2 - 1 = k$
$(x+6)^2 + (y+1)^2 = k + 37$
Centre of the circle = (−6, −1)

b A circle must have a positive radius, so $k + 37 > 0$
So $k > -37$

12 $x^2 + y^2 + 6x - 14y = 483$
$x^2 + 6x + y^2 - 14y = 483$
Completing the square gives
$(x+3)^2 - 9 + (y-7)^2 - 49 = 483$
$(x+3)^2 + (y-7)^2 = 541$
Centre of the circle = (−3, 7), radius = $\sqrt{541}$
$P(7, -14)$, midpoint of the line = (−3, 7)

Midpoint = $\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$

$(-3, 7) = \left(\frac{7+x_2}{2},\frac{-14+y_2}{2}\right)$

$-3=\frac{7+x_2}{2}$ and $7=\frac{-14+y_2}{2}$

$x_1=-13, x_2=28$

$Q(-13, 28)$

13 $(x - k)^2 + y^2 = 41$, $(3, 4)$
Substitute $x = 3$ and $y = 4$ into the equation $(x - k)^2 + y^2 = 41$
$(3 - k)^2 + 4^2 = 41$
$k^2 - 6k + 9 + 16 = 41$
$k^2 - 6k - 16 = 0$
$(k + 2)(k - 8) = 0$
$k = -2$ or $k = 8$

Challenge

1 $(x - k)^2 + (y - 2)^2 = 50$, $(4, -5)$
Substitute $x = 4$ and $y = -5$ into the equation $(x - k)^2 + (y - 2)^2 = 50$
$(4 - k)^2 + (-5 - 2)^2 = 50$
$k^2 - 8k + 16 + 49 = 50$
$k^2 - 8k + 15 = 0$
$(k - 3)(k - 5) = 0$
$k = 3$ or $k = 5$
$(x - 3)^2 + (y - 2)^2 = 50$ or $(x - 5)^2 + (y - 2)^2 = 50$

2 $x^2 + y^2 + 2fx + 2gy + c = 0$
Rearranging the equation:
$x^2 + 2fx + y^2 + 2gy + c = 0$
Completing the square gives
$(x + f)^2 - f^2 + (y + g)^2 - g^2 + c = 0$
$(x + f)^2 + (y + g)^2 = f^2 + g^2 - c$
The centre of the circle is $(-f, -g)$ and the radius is $\sqrt{f^2 + g^2 - c}$.

Circles 6D

1 Substitute $y = 0$ into $(x-1)^2+(y-3)^2=45$

$$\begin{aligned}(x-1)^2+(-3)^2&=45\\(x-1)^2+9&=45\\(x-1)^2&=36\\x-1&=\pm\sqrt{36}\\x-1&=\pm 6\end{aligned}$$

So $x-1=6 \Rightarrow x=7$
and $x-1=-6 \Rightarrow x=-5$
The circle meets the x-axis at (7, 0) and (–5, 0).

2 Substitute $x = 0$ into $(x-2)^2+(y+3)^2=29$

$$\begin{aligned}(-2)^2+(y+3)^2&=29\\4+(y+3)^2&=29\\(y+3)^2&=25\\y+3&=\pm\sqrt{25}\\y+3&=\pm 5\end{aligned}$$

So $y+3=5 \Rightarrow y=2$
and $y+3=-5 \Rightarrow y=-8$
The circle meets the y-axis at (0, 2) and (0, –8).

3 Substitute $y = x + 4$ into $(x-3)^2+(y-5)^2=34$

$$\begin{aligned}(x-3)^2+((x+4)-5)^2&=34\\(x-3)^2+(x+4-5)^2&=34\\(x-3)^2+(x-1)^2&=34\\x^2-6x+9+x^2-2x+1&=34\\2x^2-8x+10&=34\\2x^2-8x-24&=0\\x^2-4x-12&=0\\(x-6)(x+2)&=0\end{aligned}$$

3 So $x = 6$ and $x = –2$
Substitute $x = 6$ into $y = x + 4$
$y = 6 + 4$
$y = 10$
Substitute $x = –2$ into $y = x + 4$
$y = –2 + 4$
$y = 2$
The coordinates of A and B are (6, 10) and (–2, 2).

4 Rearranging $x + y + 5 = 0$
$y + 5 = –x$
$y = –x –5$
and $x^2 + 6x + y^2 + 10y - 31 = 0$
$$(x+3)^2 + (y+5)^2 = 65$$
Substitute $y = –x –5$ into $(x+3)^2 + (y+5)^2 = 65$
$$(x+3)^2 + ((-x-5)+5)^2 = 65$$
$$(x+3)^2 + (-x-5+5)^2 = 65$$
$$(x+3)^2 + (-x)^2 = 65$$
$$x^2 + 6x + 9 + x^2 = 65$$
$$2x^2 + 6x + 9 = 65$$
$$2x^2 + 6x - 56 = 0$$
$$x^2 + 3x - 28 = 0$$
$$(x+7)(x-4) = 0$$
So $x = –7$ and $x = 4$
Substitute $x = –7$ into $y = –x –5$
$y = – (–7) –5$
$y = 7–5$
$y = 2$
Substitute $x = 4$ into $y = x – 5$
$y = –(4) –5$
$y = –4 –5$
$y = –9$
So the line meets the circle at (–7, 2) and (4, –9).

5 $x^2 - 4x + y^2 = 21$

Completing the square gives $(x-2)^2 + y^2 = 25$
Substitute $y = x – 10$ into $(x – 2)^2 + y^2 = 25$

5 $(x-2)^2+(x-10)^2=25$

$x^2-4x+4+x^2-20x+100=25$

$2x^2-24x+104=25$

$2x^2-24x+79=0$

Now $b^2-4ac=(-24)^2-4(2)(79)=576-632=-56$

As $b^2-4ac<0$ then $2x^2-24x+79=0$ has no real roots.

So the line does not meet the circle.

6 a Rearranging $x + y = 11$

$y = 11 - x$

Substitute $y = 11 - x$ into $x^2 + (y - 3)^3 = 32$

$x^2+((11-x)-3)^2=32$

$x^2+(11-x-3)^2=32$

$x^2+(8-x)^2=32$

$x^2+64-16x+x^2=32$

$2x^2-16x+64=32$

$2x^2-16x+32=0$

$x^2-8x+16=0$

$(x-4)(x-4)=0$

The line meets the circle at $x = 4$ (only).

So the line is a tangent.

b Substitute $x = 4$ into $y = 11 - x$

$y = 11 - (4)$

$y = 11 - 4$

$y = 7$

The point of intersection is (4, 7).

7 a Substitute $y = 2x - 2$ into $(x-2)^2+(y-2)^2=20$

$(x-2)^2+((2x-2)-2)^2=20$

$(x-2)^2+(2x-4)^2=20$

$x^2-4x+4+4x^2-16x+16=20$

$5x^2-20x+20=20$

$5x^2-20x=0$

$5x(x-4)=0$

So $x = 0$ and $x = 4$

7 **a** Substitute $x = 0$ into $y = 2x - 2$
$y = 2(0) - 2$
$y = 0 - 2$
$y = -2$
Substitute $x = 4$ into $y = 2x - 2$
$y = 2(4) - 2$
$y = 8 - 2$
$y = 6$
So the coordinates of A and B are $(0, -2)$ and $(4, 6)$.

b The length of AB is

$$\begin{aligned}\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} &= \sqrt{(4-0)^2 + (6-(-2))^2} \\ &= \sqrt{4^2 + (6+2)^2} \\ &= \sqrt{4^2 + 8^2} \\ &= \sqrt{16+64} \\ &= \sqrt{80} \\ &= \sqrt{4 \times 20} \\ &= \sqrt{4} \times \sqrt{20} \\ &= 2\sqrt{20}\end{aligned}$$

The radius of the circle $(x-2)^2 + (y-2)^2 = 20$ is $\sqrt{20}$.
So the length of the chord AB is twice the length of the radius.
AB is a diameter of the circle.

Alternative method: substitute $x = 2$, $y = 2$ into $y = 2x - 2$
$2 = 2(2) - 2 = 4 - 2 = 2$
So the line $y = 2x - 2$ joining A and B passes through the centre $(2, 2)$ of the circle.
So AB is a diameter of the circle.

8 **a** Substitute $x = 3$, $y = 10$ into $x + y = a$
$(3) + (10) = a$
So $a = 13$

b Substitute $x = 3$, $y = 10$ into $(x - p)^2 + (y - 6)^2 = 20$

$$\begin{aligned}(3-p)^2 + (10-6)^2 &= 20 \\ (3-p)^2 + 4^2 &= 20 \\ (3-p)^2 + 16 &= 20 \\ (3-p)^2 &= 4\end{aligned}$$

8 b $(3-p)=\sqrt{4}$

$3-p=\pm 2$

So $3-p=2 \Rightarrow p=1$

and $3-p=-2 \Rightarrow p=5$

9 a Substitute $y = x - 5$ into $(x - 4)^2 + (y + 7)^2 = 50$

$(x - 4)^2 + (x - 5 + 7)^2 = 50$

$x^2 - 8x + 16 + x^2 + 4x + 4 = 50$

$2x^2 - 4x - 30 = 0$

$x^2 - 2x - 15 = 0$

$(x + 3)(x - 5) = 0$

$x = -3$ or $x = 5$

when $x = -3$, $y = -3 - 5 = -8$

when $x = 5$, $y = 5 - 5 = 0$

$A(-3, -8)$ and $B(5, 0)$ or vice versa

b Midpoint = $\left(\frac{-3+5}{2}, \frac{-8+0}{2}\right)$ = (1, −4)

The gradient of the line segment AB = $\frac{y_2 - y_1}{x_2 - x_1}$

$=\frac{0-(-8)}{5-(-3)}$

$=1$

So the gradient of the line perpendicular to AB is −1.

Using $y - y_1 = m(x - x_1)$, $m = -1$ and (x_1, y_1) = (1, −4)

So the equation of the perpendicular line is $y - (-4) = -(x - 1)$

$y = -x - 3$

c Centre of the circle = (4, −7)

Substitute $x = 4$ into $y = -x - 3$

$y = -4 - 3 = -7$

Therefore, the perpendicular bisector of AB passes through the centre of the circle (4, −7)

d Base OB = 5 units, height of the triangle = 8 units

Area OAB = $\frac{1}{2} \times 5 \times 8 = 20$ units^2

10 a Substitute $y = kx$ into $x^2 - 10x + y^2 - 12y + 57 = 0$

$x^2 - 10x + (kx)^2 - 12kx + 57 = 0$

$(1 + k^2)x^2 - (10 + 12k)x + 57 = 0$

10 a For two distinct points of intersection, $b^2 - 4ac > 0$

$$(-(10 + 12k))^2 - 4(1 + k^2)(57) > 0$$
$$144k^2 + 240k + 100 - 228k^2 - 228 > 0$$
$$-84k^2 + 240k - 128 > 0$$
$$21k^2 - 60k + 32 < 0$$

b Using the formula, $k = \dfrac{60 \pm \sqrt{(-60)^2 - 4(21)(32)}}{2(21)}$

$$k = \frac{60 \pm \sqrt{912}}{42}$$

$k = 0.71$ or $k = 2.15$,
$0.71 < k < 2.15$

11 $x^2 + 2x + y^2 = k$
Completing the square gives
$(x + 1)^2 - 1 + y^2 = k$
$$y^2 = k + 1 - (x + 1)^2$$
Using the equation of the line $y = 4x - 1$
$$y^2 = (4x - 1)^2$$
Solving the equations simultaneously gives
$$k+1-(x+1)^2 = (4x-1)^2$$
$$k+1-x^2-2x-1 = 16x^2-8x+1$$
$$17x^2-6x-k+1=0$$

The line and the circle do not intersect so there are no solutions.
Using the discriminant: $b^2 - 4ac < 0$
$$36-4(17)(-k+1)<0$$
$$36-68+68k<0$$
$$68k<32$$
$$k<\frac{8}{17}$$

12 Substitute $y = 2x + 5$ into $x^2+kx+y^2=4$
$$x^2+kx+(2x+5)^2=4$$
$$x^2+kx+4x^2+20x+25=4$$
$$5x^2+(20+k)x+21=0$$
For one point of intersection, $b^2 - 4ac = 0$
$$(20+k)^2-4(5)(21)=0$$
$$k^2+40k+400-420=0$$
$$k^2+40k-20=0$$

12 Using the formula, $k = \dfrac{-40 \pm \sqrt{40^2 - 4(1)(-20)}}{2(1)}$

$$= \frac{-40 \pm \sqrt{1680}}{2}$$

$$= -20 \pm \sqrt{420}$$

$$= -20 \pm 2\sqrt{105}$$

$k = -20 + 2\sqrt{105}$ or $k = -20 - 2\sqrt{105}$

Circles 6E

1 a $(x + 1)^2 + (y + 6)^2 = r^2$, (2, 3)

Substitute $x = 2$ and $y = 3$ into the equation $(x + 1)^2 + (y + 6)^2 = r^2$

$$\begin{aligned}(2+1)^2+(3+6)^2&=r^2\\ 9+81&=r^2\\ r&=\sqrt{90}\\ &=3\sqrt{10}\end{aligned}$$

b The line has equation $x + 3y - 11 = 0$

$3y = -x + 11$

$$y = -\frac{1}{3}x+\frac{11}{3}$$

The gradient of the line is $-\frac{1}{3}$

The gradient of the radius from the centre of the circle (−1, −6) to (2, 3) is:

$$\frac{y_2-y_1}{x_2-x_1}=\frac{3-(-6)}{2-(-1)}=\frac{9}{3}=3$$

As $-\frac{1}{3}\times 3 = -1$, the line and the radius to the point (2, 3) are perpendicular.

2 a The radius of the circle is

$$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}=\sqrt{(4-1)^2+(6-(-2))^2}=\sqrt{3^2+8^2}=\sqrt{9+64}=\sqrt{73}$$

The equation of the circle is

$$(x-4)^2+(y-6)^2=\left(\sqrt{73}\right)^2$$

or $(x-4)^2+(y-6)^2=73$

b The gradient of the line joining (1, –2) and (4, 6) is

$$\frac{y_2-y_1}{x_2-x_1}=\frac{6-(-2)}{4-1}=\frac{6+2}{3}=\frac{8}{3}$$

The gradient of the tangent is $\frac{-1}{\left(\frac{8}{3}\right)}=-\frac{3}{8}$

The equation of the tangent to the circle at (1, –2) is

$$\begin{aligned}y-y_1&=m(x-x_1)\\ y-(-2)&=-\frac{3}{8}(x-1)\end{aligned}$$

2 b

$$y+2=-\frac{3}{8}(x-1)$$
$$8y+16=-3(x-1)$$
$$8y+16=-3x+3$$
$$3x+8y+16=3$$
$$3x+8y+13=0$$

3 a $A(-1, -9)$ and $B(7, -5)$

Midpoint = $\left(\frac{-1+7}{2}, \frac{-9+(-5)}{2}\right) = (3, -7)$

The gradient of the line segment $AB = \frac{y_2-y_1}{x_2-x_1} = \frac{-5-(-9)}{7-(-1)} = \frac{1}{2}$

So the gradient of a line perpendicular to AB is -2.
The equation of the perpendicular line is

$y-y_1=m(x-x_1)$

$m = -2$ and $(x_1, y_1) = (3, -7)$

So $y-(-7) = -2(x-3)$

$y = -2x - 1$

b Centre of the circle is $(1, -3)$
Substitute $x = 1$ into the equation $y = -2x - 1$
$y = -2(1) - 1 = -3$
Therefore, the perpendicular bisector of AB, $y = -2x - 1$, passes through the centre of the circle $(1, -3)$

4 a $P(3, 1)$ and $Q(5, -3)$

Midpoint = $\left(\frac{3+5}{2}, \frac{1+(-3)}{2}\right) = (4, -1)$

The gradient of the line segment $PQ = \frac{y_2-y_1}{x_2-x_1} = \frac{-3-1}{5-3} = -2$

So the gradient of the line perpendicular to PQ is $\frac{1}{2}$.

The equation of the perpendicular line is

$y-y_1=m(x-x_1)$

m = $\frac{1}{2}$ and $(x_1, y_1) = (4, -1)$

So $y-(-1) = \frac{1}{2}(x-4)$

$y = \frac{1}{2}x - 3$

4 b Complete the square for $x^2 - 4x + y^2 + 4y = 2$

$(x-2)^2 - 4 + (y+2)^2 - 4 = 2$

$(x-2)^2 + (y+2)^2 = 10$

Centre of the circle is $(2, -2)$

Substitute $x = 2$ into the equation $y = \frac{1}{2}x - 3$

$y = \frac{1}{2}(2) - 3 = -2$

Therefore, the perpendicular bisector of PQ, $y = \frac{1}{2}x - 3$, passes through the centre of the circle $(2, -2)$

5 a Substitute $x = -7$ and $y = -6$ into $x^2 + 18x + y^2 - 2y + 29$

$$\begin{aligned} x^2 + 18x + y^2 - 2y + 29 &= (-7)^2 + 18(-7) + (-6)^2 - 2(-6) + 29 \\ &= 49 - 126 + 36 + 12 + 29 \\ &= 0 \end{aligned}$$

The point P satisfies the equation, so P lies on C.

b Completing the square gives

$(x+9)^2 + (y-1)^2 = 53$

The centre of the circle, A, is $(-9, 1)$.

The gradient of $CP = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-6-1}{-7-(-9)} = \frac{-7}{2}$

Gradient of the tangent is $\frac{2}{7}$

$$y - y_1 = m(x - x_1)$$

$$y - (-6) = \frac{2}{7}(x - (-7))$$

$$y = \frac{2}{7}x - 4$$

c The tangent intersects the y-axis at $x = 0$

$y = \frac{2}{7}(0) - 4 = -4$

$R(0, -4)$

d Height of triangle = radius of circle = $\sqrt{53}$

Base of triangle = distance PR

$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(-7-0)^2 + (-6-(-4))^2} = \sqrt{53}$

Area $APR = \frac{1}{2} \times \sqrt{53} \times \sqrt{53} = 26.5$ units^2

6 a The centre of the circle $(x+4)^2+(y-1)^2=242$ is $(-4,1)$.

The gradient of the line joining (–4, 1) and (7, –10) is

$$\frac{y_2-y_1}{x_2-x_1}=\frac{-10-1}{7-(-4)}=\frac{-11}{7+4}=-\frac{11}{11}=-1$$

The gradient of the tangent is $\frac{-1}{(-1)}=1$.

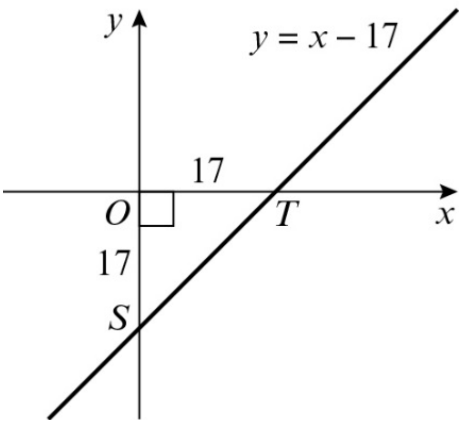
The equation of the tangent is

$$\begin{aligned} y-y_1&=m(x-x_1)\\ y-(-10)&=1(x-7)\\ y+10&=x-7\\ y&=x-17 \end{aligned}$$

Substitute $x = 0$ into $y = x - 17$
$y = 0 - 17$
$y = -17$
So the coordinates of S are (0, –17)

Substitute $y = 0$ into $y = x - 17$
$0 = x - 17$
$x = 17$
So the coordinates of T are (17, 0).

b



The area of ΔOST is $\frac{1}{2}\times17\times17=144.5$

7 $(x+5)^2+(y+3)^2=80$
Gradient of tangent = 2, so $y = 2x + \text{c}$

Diameter of the circle that touches l_1 and l_2 has gradient $-\frac{1}{2}$ and passes through the centre of the circle $(-5, -3)$

$y = -\frac{1}{2}x + d$

$-3 = -\frac{1}{2}(-5) + d$

$d = -\frac{11}{2}$

$y = -\frac{1}{2}x - \frac{11}{2}$ is the equation of the diameter that touches l_1 and l_2.

Solve the equation of the diameter and circle simultaneously:

$$(x+5)^2 + \left(-\frac{1}{2}x - \frac{5}{2}\right)^2 = 80$$

$$x^2 + 10x + 25 + \frac{1}{4}x^2 + \frac{5}{2}x + \frac{25}{4} - 80 = 0$$

$$4x^2 + 40x + 100 + x^2 + 10x + 25 - 320 = 0$$

$$5x^2 + 50x - 195 = 0$$

$$x^2 + 10x - 39 = 0$$

$$(x+13)(x-3) = 0$$

$x = -13$ or $x = 3$

When $x = -13$, $y = -\frac{1}{2}(-13) - \frac{11}{2} = 1$

When $x = 3$, $y = -\frac{1}{2}(3) - \frac{11}{2} = -7$

$(-13, 1)$ and $(3, -7)$ are the coordinates where the diameter touches lines l_1 and l_2.

Substitute these coordinates into the equation $y = 2x + c$

When $x = -13$, $y = 1$, $1 = 2(-13) + c$, $c = 27$, $y = 2x + 27$
When $x = 3$, $y = -7$, $-7 = 2(3) + c$, $c = -13$, $y = 2x - 13$
l_1: $y = 2x + 27$
l_2: $y = 2x - 13$

8 a $(x-3)^3 + (y-p)^2 = 5$ and $2x + y - 5 = 0$
So $y = -2x + 5$

Solve the equations simultaneously:

$$(x-3)^3 + (-2x + 5 - p)^2 = 5$$

$$x^2 - 6x + 9 + 4x^2 - 20x + 4px + 25 - 10p + p^2 - 5 = 0$$

$$5x^2 - 26x + 4px + 29 - 10p + p^2 = 0$$

8 **a** Using the discriminant for one solution:

$$b^2 - 4ac = 0$$
$$(-26 + 4p)^2 - 4(5)(29 - 10p + p^2) = 0$$
$$16p^2 - 208p + 676 - 20p^2 + 200p - 580 = 0$$
$$-4p^2 - 8p + 96 = 0$$
$$p^2 + 2p - 24 = 0$$
$$(p - 4)(p + 6) = 0$$

$p = 4$ or $p = -6$

b When $p = 4$, $(x - 3)^3 + (y - 4)^2 = 5$
When $p = -6$, $(x - 3)^3 + (y + 6)^2 = 5$
(3, 4) and (3, −6)

9 **a** The centre of the circle, Q, is (11, −5)
To find the radius of the circle:

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(5-11)^2 + (3-(-5))^2}$$
$$= \sqrt{(-6)^2 + 8^2}$$
$$= 10$$

$(x - 11)^2 + (y + 5)^2 = 100$

b The gradient of $PQ = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{3-(-5)}{5-11} = \dfrac{4}{-3}$

Gradient of the tangent is $\dfrac{3}{4}$

$$y - y_1 = m(x - x_1)$$
$$y - 3 = \frac{3}{4}(x - 5)$$
$$y = \frac{3}{4}x - \frac{3}{4}$$

c Midpoint of $PQ = \left(\dfrac{11+5}{2}, \dfrac{-5+3}{2}\right) = (8, -1)$

Gradient of l_2 is $\dfrac{3}{4}$ as the line is parallel to l_1.

$$y - y_1 = m(x - x_1)$$
$$y - (-1) = \frac{3}{4}(x - 8)$$
$$y = \frac{3}{4}x - 7$$

9 c Solve $y = \frac{3}{4}x - 7$ and $(x - 11)^2 + (y + 5)^2 = 100$ simultaneously

$$(x-11)^2 + \left(\frac{3}{4}x - 2\right)^2 = 100$$

$$x^2 - 22x + 121 + \frac{9}{16}x^2 - 3x + 4 - 100 = 0$$
$$25x^2 - 400x + 400 = 0$$
$$x^2 - 16x + 16 = 0$$

Using the formula, $x = \frac{16 \pm \sqrt{192}}{2} = 8 \pm 4\sqrt{3}$

When $x = 8 + 4\sqrt{3}$, $y = \frac{3}{4}(8 + 4\sqrt{3}) - 7 = -1 + 3\sqrt{3}$

When $x = 8 - 4\sqrt{3}$, $y = \frac{3}{4}(8 - 4\sqrt{3}) - 7 = -1 - 3\sqrt{3}$

$A(8 - 4\sqrt{3},\ -1 - 3\sqrt{3})$ and $B(8 + 4\sqrt{3},\ -1 + 3\sqrt{3})$

d $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$$= \sqrt{(8 + 4\sqrt{3} - (8 - 4\sqrt{3}))^2 + (-1 + 3\sqrt{3} - (-1 - 3\sqrt{3}))^2}$$
$$= \sqrt{(8\sqrt{3})^2 + (6\sqrt{3})^2}$$
$$= \sqrt{192 + 108}$$
$$= \sqrt{300}$$
$$= 10\sqrt{3}$$

10 a $M = \left(\frac{2+10}{2}, \frac{3+1}{2}\right) = (6, 2)$

Gradient $RS = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1-3}{10-2} = -\frac{1}{4}$

Gradient of l is 4

$y - y_1 = m(x - x_1)$

$y - 2 = 4(x - 6)$

$y = 4x - 22$ is the equation of l

b Using $y = 4x - 22$ when $x = a$, $y = -2$:
$-2 = 4a - 22$, $a = 5$

c radius = distance $CR = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(2-5)^2 + (3-(-2))^2} = \sqrt{34}$

Centre of circle is $(5, -2)$

Equation of circle is $(x - 5)^2 + (y + 2)^2 = 34$

10 d Solve $y = 4x - 22$ and $(x - 5)^2 + (y + 2)^2 = 34$ simultaneously

$$(x - 5)^2 + (4x - 20)^2 = 34$$
$$x^2 - 10x + 25 + 16x^2 - 160x + 400 - 34 = 0$$
$$17x^2 - 170x + 391 = 0$$
$$x^2 - 10x + 23 = 0$$

Using the formula, $x = \dfrac{10 \pm \sqrt{8}}{2} = 5 \pm \sqrt{2}$

When $x = 5+\sqrt{2}$, $y = 4(5+\sqrt{2}) - 22 = -2+4\sqrt{2}$

When $x = 5-\sqrt{2}$, $y = 4(5-\sqrt{2}) - 22 = -2-4\sqrt{2}$

$A(5+\sqrt{2}, -2+4\sqrt{2})$ and $B(5-\sqrt{2}, -2-4\sqrt{2})$

11 a $x^2 - 4x + y^2 - 6y = 7$

$(x - 2)^2 - 4 + (y - 3)^2 - 9 = 7$

$$(x - 2)^2 + (y - 3)^2 = 20$$

Substitute $x = 3y - 17$ into $(x - 2)^2 + (y - 3)^2 = 20$

$$(3y - 19)^2 + (y - 3)^2 = 20$$
$$9y^2 - 114y + 361 + y^2 - 6y + 9 - 20 = 0$$
$$10y^2 - 120y + 350 = 0$$
$$y^2 - 12y + 35 = 0$$
$$(y - 7)(y - 5) = 0$$

$y = 7$ or 5

when $y = 7$, $x = 3(7) - 17 = 4$

when $y = 5$, $x = 3(5) - 17 = -2$

$P(-2, 5)$ and $Q(4, 7)$

b Centre of circle $T = (2, 3)$ and $P(-2, 5)$

Gradient of $PT = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{5-3}{-2-2} = -\dfrac{1}{2}$

Gradient of the tangent is 2

$y - y_1 = m(x - x_1)$

$y - 5 = 2(x + 2)$

$y = 2x + 9$

Gradient of $QT = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{7-3}{4-2} = 2$

Gradient of the tangent is $-\dfrac{1}{2}$

$y - y_1 = m(x - x_1)$

$y - 7 = -\dfrac{1}{2}(x - 4)$

$y = -\dfrac{1}{2}x + 9$

11 c Gradient $PQ = \dfrac{7-5}{4-(-2)} = \dfrac{1}{3}$

Midpoint of $PQ = \left(\dfrac{-2+4}{2}, \dfrac{5+7}{2}\right) = (1, 6)$

Gradient of the perpendicular bisector is −3

$y - y_1 = m(x - x_1)$

$y - 6 = -3(x - 1)$

$y = -3x + 9$

d $y = 2x + 9$, $y = -\dfrac{1}{2}x + 9$ and $y = -3x + 9$

Solve $y = 2x + 9$ and $y = -\dfrac{1}{2}x + 9$ simultaneously

$2x + 9 = -\dfrac{1}{2}x + 9$

$4x + 18 = -x + 18$

$x = 0, y = 9$

(0, 9)

Solve $y = 2x + 9$ and $y = -3x + 9$ simultaneously

$2x + 9 = -3x + 9$

$x = 0, y = 9$

(0, 9)

Therefore all three lines intersect at (0, 9)

Challenge

1 y-intercept = −2 so $y = mx - 2$

Substitute $y = mx - 2$ into $(x-7)^2 + (y+1)^2 = 5$

$$(x-7)^2 + (mx-1)^2 = 5$$

$$x^2 - 14x + 49 + m^2x^2 - 2mx + 1 - 5 = 0$$

$$(1+m^2)x^2 - (14+2m)x + 45 = 0$$

Using the discriminant when there are only one root

$$b^2 - 4ac = 0$$

$$(-(14+2m))^2 - 4(1+m^2)(45) = 0$$

$$4m^2 + 56m + 196 - 180 - 180m^2 = 0$$

$$176m^2 - 56m - 16 = 0$$

$$22m^2 - 7m - 2 = 0$$

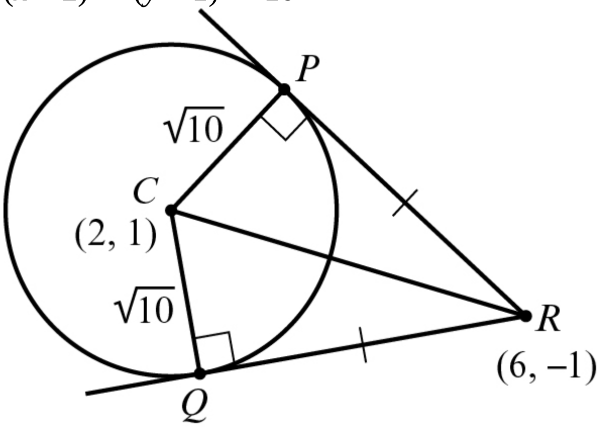
$$(11m+2)(2m-1) = 0$$

$m = -\frac{2}{11}$ or $m = \frac{1}{2}$

As m is positive, $m = \frac{1}{2}$

Therefore the equation of the line is $y = \frac{1}{2}x - 2$

2 a $(x-2)^2 + (y-1)^2 = 10$



Radius = $\sqrt{10}$

$CR = \sqrt{(6-2)^2 + (-1-1)^2} = \sqrt{4^2 + (-2)^2} = \sqrt{20}$

Using Pythagoras' theorem, $PR = \sqrt{(\sqrt{20})^2 - (\sqrt{10})^2} = \sqrt{20-10} = \sqrt{10}$

$QR = \sqrt{(\sqrt{20})^2 - (\sqrt{10})^2} = \sqrt{20-10} = \sqrt{10}$

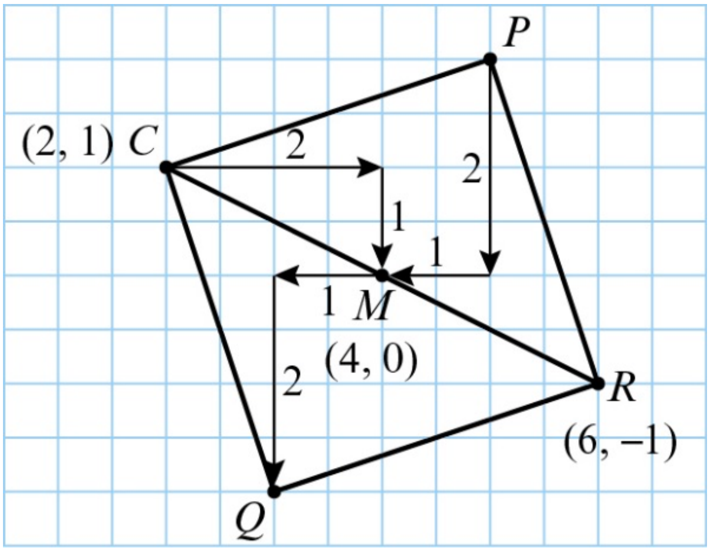
$CP = PR = QR = CQ = \sqrt{10}$, so all four sides of the quadrilateral are the same length.

Using circle theorems, angle CPR = angle CQR = 90° (A radius meets a tangent at 90°.)

2 b Since $QR = CQ$ the angles QCR and QRC are equal and are each 45 degrees. The same is true for angles CRP and RCP. Therefore all the angles at Q, P, C and R are 90°. Therefore, $CPQR$ is a square.

$CPQR$ is a square so it's diagonals bisect at right angles and are equal.

Midpoint of square = midpoint of $CR = \left(\frac{2+6}{2}, \frac{1-1}{2}\right) = (4, 0)$



$P(5, 2)$ and $Q(3, -2)$

Gradient of $PR = \frac{-1-2}{6-5} = -3$

$R(6, -1)$, $m = -3$

$y - y_1 = m(x - x_1)$

$y + 1 = -3(x - 6)$

$y = -3x + 17$

QR is perpendicular to PR, so gradient of $QR = \frac{1}{3}$

$R(6, -1)$, $m = \frac{1}{3}$

$y - y_1 = m(x - x_1)$

$y + 1 = \frac{1}{3}(x - 6)$

$y = \frac{1}{3}x - 3$

The equations are $y = -3x + 17$ and $y = \frac{1}{3}x - 3$

Circles 6F

1 a $U(-2, 8)$, $V(7, 7)$ and $W(-3, -1)$

$UV^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$

$= (7+2)^2 + (7-8)^2$

$= 82$

$VW^2 = (-3-7)^2 + (-1-7)^2$

$= 164$

$UW^2 = (-3+2)^2 + (-1-8)^2$

$= 82$

Use Pythagoras' theorem to show $UV^2 + UW^2 = VW^2$

$82 + 82 = 164 = VW^2$

Therefore, UVW is a right-angled triangle.

b UVW is a right-angled triangle, therefore VW is the diameter of the circle.

Centre of circle = Midpoint of VW

Midpoint $= \left(\frac{7+(-3)}{2}, \frac{7+(-1)}{2}\right) = (2, 3)$

c Radius of the circle is $\frac{1}{2}$ of $VW = \frac{\sqrt{164}}{2} = \sqrt{\frac{164}{4}} = \sqrt{41}$

$(x-2)^2 + (y-3)^2 = 41$

2 a $A(2, 6)$, $B(5, 7)$ and $C(8, -2)$

Use Pythagoras' theorem to show $AB^2 + BC^2 = AC^2$

$AB^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 = (5-2)^2 + (7-6)^2 = 10$

$BC^2 = (8-5)^2 + (-2-7)^2 = 90$

$AC^2 = (8-2)^2 + (-2-6)^2 = 100$

Therefore, ABC is a right-angled triangle and AC is the diameter of the circle.

b Centre of circle = Midpoint of AC

Midpoint $= \left(\frac{2+8}{2}, \frac{6+(-2)}{2}\right) = (5, 2)$

Radius of the circle is $\frac{1}{2}$ of $AC = \frac{\sqrt{100}}{2} = 5$

$(x-5)^2 + (y-2)^2 = 25$

c Base of triangle = $AB = \sqrt{10}$ units

Height of triangle = $BC = \sqrt{90}$ units

Area of triangle $ABC = \frac{1}{2} \times \sqrt{10} \times \sqrt{90} = 15$ units^2

3 a i $A(-3, 19)$ and $B(9, 11)$

Midpoint = $\left(\frac{-3+9}{2}, \frac{19+11}{2}\right) = (3, 15)$

The gradient of the line segment $AB = \frac{y_2 - y_1}{x_2 - x_1} = \frac{11-19}{9-(-3)} = -\frac{2}{3}$

So the gradient of the line perpendicular to AB is $\frac{3}{2}$.

The equation of the perpendicular line is

$y - y_1 = m(x - x_1)$

$m = \frac{3}{2}$ and $(x_1, y_1) = (3, 15)$

So $y - 15 = \frac{3}{2}(x - 3)$

$y = \frac{3}{2}x + \frac{21}{2}$

ii $A(-3, 19)$ and $C(-15, 1)$

Midpoint = $\left(\frac{-3-15}{2}, \frac{19+1}{2}\right) = (-9, 10)$

The gradient of the line segment $AC = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1-19}{-15+3} = \frac{3}{2}$

So the gradient of the line perpendicular to AC is $-\frac{2}{3}$.

The equation of the perpendicular line is

$y - y_1 = m(x - x_1)$

$m = -\frac{2}{3}$ and $(x_1, y_1) = (-9, 10)$

So $y - 10 = -\frac{2}{3}(x + 9)$

$y = -\frac{2}{3}x + 4$

b Solve $y = \frac{3}{2}x + \frac{21}{2}$ and $y = -\frac{2}{3}x + 4$ simultaneously

$\frac{3}{2}x + \frac{21}{2} = -\frac{2}{3}x + 4$

$9x + 63 = -4x + 24$

$13x = -39$

$x = -3$, $y = -\frac{2}{3}(-3) + 4 = 6$

So, the coordinates of the centre of the circle are $(-3, 6)$

3 c Radius = distance from (−3, 6) to (9, 11)

$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2} = \sqrt{(9+3)^2+(11-6)^2} = \sqrt{12^2+5^2} = 13$

$(x+3)^2 + (y-6)^2 = 169$

4 a i $P(-11, 8)$ and $Q(-6, -7)$

Midpoint = $\left(\frac{-11-6}{2}, \frac{8-7}{2}\right) = \left(-\frac{17}{2}, \frac{1}{2}\right)$

The gradient of the line segment $PQ = \frac{y_2-y_1}{x_2-x_1} = \frac{-7-8}{-6+11} = -3$

So the gradient of the line perpendicular to PQ is $\frac{1}{3}$.

The equation of the perpendicular line is

$y - y_1 = m(x - x_1)$

$m = \frac{1}{3}$ and $(x_1, y_1) = \left(-\frac{17}{2}, \frac{1}{2}\right)$

So $y - \frac{1}{2} = \frac{1}{3}\left(x + \frac{17}{2}\right)$

$y = \frac{1}{3}x + \frac{10}{3}$

ii $Q(-6, -7)$ and $R(4, -7)$

QR is the line $y = -7$.

Midpoint = $\left(\frac{-6+4}{2}, \frac{-7-7}{2}\right) = (-1, -7)$

The equation of the perpendicular line is $x = -1$.

b Solve $y = \frac{1}{3}x + \frac{10}{3}$ and $x = -1$ simultaneously to find the centre of the circle:

$\frac{1}{3}(-1) + \frac{10}{3} = y$

$y = 3$

The centre of the circle is (−1, 3)

Radius = distance from (−1, 3) to (4, −7)

$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2} = \sqrt{(4+1)^2+(-7-3)^2} = \sqrt{125}$

$(x+1)^2 + (y-3)^2 = 125$

5 $R(-2, 1)$ and $S(4, 3)$

Midpoint = $\left(\frac{-2+4}{2}, \frac{1+3}{2}\right)$ = (1, 2)

The gradient of the line segment $RS = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-1}{4+2} = \frac{1}{3}$

So the gradient of the line perpendicular to RS is −3.
The equation of the perpendicular line is

$y - y_1 = m(x - x_1)$

$m = -3$ and $(x_1, y_1) = (1, 2)$

So $y - 2 = -3(x - 1)$

$y = -3x + 5$

$S(4, 3)$ and $T(10, -5)$

Midpoint = $\left(\frac{4+10}{2}, \frac{3-5}{2}\right)$ = (7, −1)

The gradient of the line segment $ST = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-5-3}{10-4} = -\frac{4}{3}$

So the gradient of the line perpendicular to ST is $\frac{3}{4}$.

The equation of the perpendicular line is

$y - y_1 = m(x - x_1)$

$m = \frac{3}{4}$ and $(x_1, y_1) = (7, -1)$

So $y + 1 = \frac{3}{4}(x - 7)$

$y = \frac{3}{4}x - \frac{25}{4}$

Solve $y = -3x + 5$ and $y = \frac{3}{4}x - \frac{25}{4}$ simultaneously

$-3x + 5 = \frac{3}{4}x - \frac{25}{4}$

$-12x + 20 = 3x - 25$

$15x = 45$

$x = 3$, $y = -3(3) + 5 = -4$

So the centre of the circle is (3, −4)

Radius = distance from centre (3, −4) to (−2, 1)

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(-2-3)^2 + (1+4)^2} = \sqrt{50}$$

The equation of the circle is $(x - 3)^2 + (y + 4)^2 = 50$

6 a $A(3, 15)$, $B(-13, 3)$ and $C(-7, -5)$

Using Pythagoras' theorem $AB^2 + BC^2 = AC^2$

$AB^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 = (-13 - 3)^2 + (3 - 15)^2 = 256 + 144 = 400$

$BC^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 = (-7 + 13)^2 + (-5 - 3)^2 = 36 + 64 = 100$

$AC^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 = (-7 - 3)^2 + (-5 - 15)^2 = 100 + 400 = 500$

Therefore ABC is a right-angled triangle.

b Centre of circle = midpoint of $AC = \left(\frac{3-7}{2}, \frac{15-5}{2}\right) = (-2, 5)$

Radius $= \frac{1}{2}$ of $AC = \frac{1}{2}$ of $\sqrt{500} = \frac{10\sqrt{5}}{2} = 5\sqrt{5}$

Equation of circle: $(x + 2)^2 + (y - 5)^2 = (5\sqrt{5})^2$ or $(x + 2)^2 + (y - 5)^2 = 125$

c We know that A, B and C all lie on the circumference of the circle.

$D(8, 0)$, substitute $x = 8$ and $y = 0$ into the equation of the circle:

$(8 + 2)^2 + (0 - 5)^2 = 100 + 25 = 125$

Therefore, $D(8, 0)$ lies on the circumference of the circle $(x + 2)^2 + (y - 5)^2 = 125$

7 a $A(-1, 9)$, $B(6, 10)$, $C(7, 3)$, $D(0, 2)$

The length of AB is

$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(6-(-1))^2 + (10-9)^2} = \sqrt{7^2 + 1^2} = \sqrt{49+1} = \sqrt{50}$

The length of BC is

$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(7-6)^2 + (3-10)^2} = \sqrt{1^2 + (-7)^2} = \sqrt{1+49} = \sqrt{50}$

The length of CD is

$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(0-7)^2 + (2-3)^2} = \sqrt{(-7)^2 + (-1)^2} = \sqrt{49+1} = \sqrt{50}$

The length of DA is

$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(-1-0)^2 + (9-2)^2} = \sqrt{(-1)^2 + 7^2} = \sqrt{1+49} = \sqrt{50}$

The sides of the quadrilateral are equal.

The gradient of AB is

$\frac{y_2 - y_1}{x_2 - x_1} = \frac{10-9}{6-(-1)} = \frac{1}{7}$

7 a The gradient of BC is

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{3-10}{7-6} = \frac{-7}{1} = -7$$

The product of the gradients $= \left(\frac{1}{7} \times -7\right) = -1.$

So the line AB is perpendicular to BC.
So the quadrilateral $ABCD$ is a square.

b The area $= \sqrt{50} \times \sqrt{50} = 50$

c The mid-point of AC is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{-1+7}{2}, \frac{9+3}{2}\right) = \left(\frac{6}{2}, \frac{12}{2}\right) = (3, 6)$$

So the centre of the circle is (3, 6).

8 a $D(-12, -3)$, $E(-10, \text{b})$, $F(2, -5)$

Using Pythagoras' theorem $DE^2 + EF^2 = DF^2$

$$\begin{aligned} DE^2 &= (x_2 - x_1)^2 + (y_2 - y_1)^2 \\ &= (-10 + 12)^2 + (b + 3)^2 \\ &= 4 + b^2 + 6b + 9 \\ &= b^2 + 6b + 13 \end{aligned}$$

$$\begin{aligned} EF^2 &= (x_2 - x_1)^2 + (y_2 - y_1)^2 \\ &= (2 + 10)^2 + (-5 - b)^2 \\ &= 144 + b^2 + 10b + 25 \\ &= b^2 + 10b + 169 \end{aligned}$$

$$\begin{aligned} DF^2 &= (x_2 - x_1)^2 + (y_2 - y_1)^2 \\ &= (2 + 12)^2 + (-5 + 3)^2 \\ &= 196 + 4 \\ &= 200 \end{aligned}$$

$$\begin{aligned} b^2 + 6b + 13 + b^2 + 10b + 169 &= 200 \\ 2b^2 + 16b - 18 &= 0 \\ b^2 + 8b - 9 &= 0 \\ (b + 9)(b - 1) &= 0 \end{aligned}$$

$b = -9$ or $b = 1$

As $b > 0$, $b = 1$.

b Centre of circle = midpoint of $DF = \left(\frac{-12+2}{2}, \frac{-3-5}{2}\right) = (-5, -4)$

Distance of radius $= \frac{1}{2}$ of $DF = \frac{1}{2}$ of $\sqrt{200} = \frac{10\sqrt{2}}{2} = 5\sqrt{2}$

Equation of circle: $(x + 5)^2 + (y + 4)^2 = (5\sqrt{2})^2 = 50$

9 a $x^2 + 2x + y^2 - 24y - 24 = 0$

Completing the square gives:

$(x + 1)^2 - 1 + (y - 12)^2 - 144 - 24 = 0$

$(x + 1)^2 + (y - 12)^2 = 169$

Centre of the circle is $(-1, 12)$ and the radius of the circle is 13.

b If AB is the diameter of the circle then the midpoint of AB is the centre of the circle.

Midpoint of $AB = \left(\frac{-13+11}{2}, \frac{17+7}{2}\right) = (-1, 12)$

Therefore, AB is the diameter of the circle.

c The point C lies on the x-axis, so $y = 0$.

Substitute $y = 0$ into the equation of the circle.

$(x + 1)^2 + (0 - 12)^2 = 169$

$x^2 + 2x + 1 + 144 = 169$

$x^2 + 2x - 24 = 0$

$(x + 6)(x - 4) = 0$

$x = -6$, $x = 4$

As x is negative, $x = -6$

The coordinates of C are $(-6, 0)$

Circles, Mixed Exercise 6

1 a QR is the diameter of the circle so the centre, C, is the midpoint of QR

Midpoint $= \left(\dfrac{11+(-5)}{2}, \dfrac{12+0}{2}\right) = (3, 6)$

$C(3, 6)$

b Radius $= \dfrac{1}{2}$ of diameter $= \dfrac{1}{2}$ of $QR = \dfrac{1}{2}$ of $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

$= \dfrac{1}{2}$ of $\sqrt{(-5-11)^2+(0-12)^2}$

$= \dfrac{1}{2}$ of $\sqrt{400}$

$= \dfrac{1}{2}$ of $20 = 10$ units

c Circle with centre (3, 6) and radius 10:
$(x-3)^2 + (y-6)^2 = 100$

d $P(13, 6)$ lies on the circle if P satisfies the equation, so substitute $x = 13$ and $y = 6$ into the equation of the circle:
$(13-3)^2 + (6-6)^2 = 100 + 0 = 100$
Therefore, P lies on the circle.

2 The distance between (0, 0) and (5, –2) is

$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2} = \sqrt{(5-0)^2+(-2-0)^2} = \sqrt{5^2+(-2)^2} = \sqrt{25+4} = \sqrt{29}$

The radius of the circle is $\sqrt{30}$.

As $\sqrt{29} < \sqrt{30}$ $(0,0)$ lies inside the circle.

3 a $x^2 + 3x + y^2 + 6y = 3x - 2y - 7$
$x^2 + y^2 + 8y = -7$
Completing the square gives:
$(x-0)^2 + (y+4)^2 - 16 = -7$
$(x-0)^2 + (y+4)^2 = 9$
Centre of the circle is $(0, -4)$ and the radius is 3.

b The circle intersects the y-axis at $x = 0$
$(0-0)^2 + (y+4)^2 = 9$
$y^2 + 8y + 16 = 9$
$y^2 + 8y + 7 = 0$
$(y+1)(y+7) = 0$
$y = -1$ or $y = -7$
$(0, -1)$ and $(0, -7)$

3 c At the x-axis, $y = 0$

$x^2 + 0^2 + 8(0) = -7$

$x^2 = -7$

There are no real solutions, so the circle does not intersect the x-axis.

4 a The centre of $(x-8)+(y-8)^2 = 117$ is $(8,8)$.

Substitute (8, 8) into $(x+1)^2 + (y-3)^2 = 106$

$(8+1)^2 + (8-3)^2 = 9^2 + 5^2 = 81 + 25 = 106$ ✓

So (8, 8) lies on the circle $(x+1)^2 + (y-3)^2 = 106$.

b As Q is the centre of the circle $(x+1)^2 + (y-3)^2 = 106$ and P lies on this circle, the length PQ must equal the radius.

So $PQ = \sqrt{106}$

Alternative method: Work out the distance between $P(8, 8)$ and $Q(-1, 3)$ using the distance formula.

5 a Substitute $(-1,0)$ into $x^2 + y^2 = 1$

$(-1)^2 + (0)^2 = 1 + 0 = 1$ ✓

So $(-1,0)$ is on the circle.

Substitute $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ into $x^2 + y^2 = 1$

$\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4} + \frac{3}{4} = 1$ ✓

So $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ is on the circle.

Substitute $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ into $x^2 + y^2 = 1$

$\left(\frac{1}{2}\right)^2 + \left(-\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4} + \frac{3}{4} = 1$ ✓

So $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ is on the circle.

5 b The distance between $(-1,0)$ and $\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$ is

$$\begin{aligned}\sqrt{(x_2-x_1)^2+(y_2-y_1)^2} &= \sqrt{\left(\frac{1}{2}-(-1)\right)^2+\left(\frac{\sqrt{3}}{2}-0\right)^2}\\ &= \sqrt{\left(\frac{1}{2}+1\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\\ &= \sqrt{\left(\frac{3}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\\ &= \sqrt{\frac{9}{4}+\frac{3}{4}}\\ &= \sqrt{\frac{12}{4}}\\ &= \sqrt{3}\end{aligned}$$

The distance between $(-1,0)$ and $\left(\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$ is

$$\begin{aligned}\sqrt{(x_2-x_1)^2+(y_2-y_1)^2} &= \sqrt{\left(\frac{1}{2}-(-1)\right)^2+\left(-\frac{\sqrt{3}}{2}-0\right)^2}\\ &= \sqrt{\left(\frac{1}{2}+1\right)^2+\left(-\frac{\sqrt{3}}{2}\right)^2}\\ &= \sqrt{\left(\frac{3}{2}\right)^2+\left(-\frac{\sqrt{3}}{2}\right)^2}\\ &= \sqrt{\frac{9}{4}+\frac{3}{4}}\\ &= \sqrt{\frac{12}{4}}\\ &= \sqrt{3}\end{aligned}$$

5 b The distance between $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{\left(\frac{1}{2} - \frac{1}{2}\right)^2 + \left(-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right)^2}$$
$$= \sqrt{0^2 + \left(-\sqrt{3}\right)^2}$$
$$= \sqrt{0 + 3}$$
$$= \sqrt{3}$$

So AB, BC and AC all equal $\sqrt{3}$.
ΔABC is equilateral.

6 a $(x - k)^2 + (y - 3k)^2 = 13$, $(3, 0)$
Substitute $x = 3$ and $y = 0$ into the equation of the circle.
$(3 - k)^2 + (0 - 3k)^2 = 13$
$9 - 6k + k^2 + 9k^2 - 13 = 0$
$10k^2 - 6k - 4 = 0$
$5k^2 - 3k - 2 = 0$
$(5k + 2)(k - 1) = 0$
$k = -\frac{2}{5}$ or $k = 1$

b As $k > 0$, $k = 1$
Equation of the circle is $(x - 1)^2 + (y - 3)^2 = 13$

7 $x^2 + px + y^2 + 4y = 20$, $y = 3x - 9$
Substitute $y = 3x - 9$ into the equation $x^2 + px + y^2 + 4y = 20$
$x^2 + px + (3x - 9)^2 + 4(3x - 9) = 20$
$x^2 + px + 9x^2 - 54x + 81 + 12x - 36 - 20 = 0$
$10x^2 + (p - 42)x + 25 = 0$
There are no solutions, so using the discriminant $b^2 - 4ac < 0$:
$(p - 42)^2 - 4(10)(25) < 0$
$(p - 42)^2 < 1000$
$p - 42 < \pm\sqrt{1000}$
$p < 42 \pm \sqrt{1000}$
$p < 42 \pm 10\sqrt{10}$
$42 - 10\sqrt{10} < p < 42 + 10\sqrt{10}$

8 Substitute $x = 0$ into $y = 2x - 8$
$y = 2(0) - 8$
$y = -8$

Substitute $y = 0$ into $y = 2x - 8$
$0 = 2x - 8$
$2x = 8$
$x = 4$

The line meets the coordinate axes at $(0,-8)$ and $(4,0)$.
The coordinates of the centre of the circle are at the midpoint:

$$\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)=\left(\frac{0+4}{2},\frac{-8+0}{2}\right)=\left(\frac{4}{2},\frac{-8}{2}\right)=(2,-4)$$

The length of the diameter is

$$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}=\sqrt{(4-0)^2+(0-(-8))^2}=\sqrt{4^2+8^2}=\sqrt{80}=\sqrt{16\times 5}=4\sqrt{5}$$

So the length of the radius is $\frac{4\sqrt{5}}{2}=2\sqrt{5}.$

The centre of the circle is $(2,-4)$ and the radius is $2\sqrt{5}.$

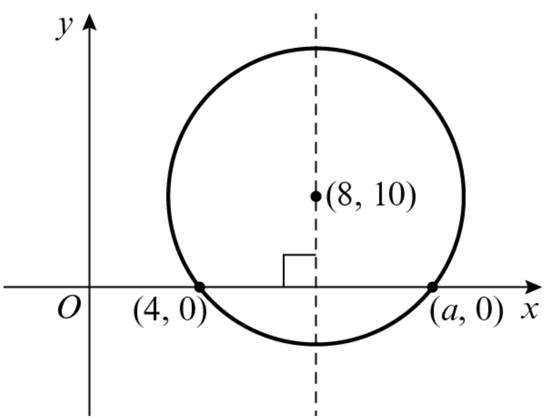
The equation of the circle is

$$(x-x_1)^2+(y-y_1)^2=r^2$$
$$(x-2)^2+(y-(-4))^2=(2\sqrt{5})^2$$
$$(x-2)^2+(y+4)^2=20$$

9 **a** The radius is

$$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}=\sqrt{(8-4)^2+(10-0)^2}=\sqrt{4^2+10^2}=\sqrt{16+100}=\sqrt{116}=2\sqrt{29}$$

b



9 b The centre is on the perpendicular bisector of $(4,0)$ and $(a,0.)$ So

$$\frac{4+a}{2}=8$$
$$4+a=16$$
$$a=12$$

10 Substitute y = 0 into $(x-5)^2+y^2=36$

$$(x-5)^2=36$$
$$x-5=\sqrt{36}$$
$$x-5=\pm 6$$
So $x-5=6 \Rightarrow x=11$

and $x-5=-6 \Rightarrow x=-1$

The coordinates of P and Q are $(-1,0)$ and $(11,0)$.

11 Substitute x = 0 into $(x+4)^2+(y-7)^2=121$

$$4^2+(y-7)^2=121$$
$$16+(y-7)^2=121$$
$$(y-7)^2=105$$
$$y-7=\pm\sqrt{105}$$
So $y=7\pm\sqrt{105}$

The values of m and n are $7+\sqrt{105}$ and $7-\sqrt{105}$.

12 a $(x + 5)^2 + (y + 2)^2 = 125$, $A(a, 0)$, $B(0, b)$

At $A(a, 0)$: $(a + 5)^2 + (0 + 2)^2 = 125$

$$a^2 + 10a + 25 + 4 - 125 = 0$$
$$a^2 + 10a - 96 = 0$$
$$(a + 16)(a - 6) = 0$$
As $a > 0$, $a = 6$

At $B(0, b)$: $(0 + 5)^2 + (b + 2)^2 = 125$

$$25 + b^2 + 4b + 4 - 125 = 0$$
$$b^2 + 4b - 96 = 0$$
$$(b + 12)(b - 8) = 0$$
As $b > 0$, $b = 8$

So $a = 6$, $b = 8$

12 b $A(6, 0)$, $B(0, 8)$

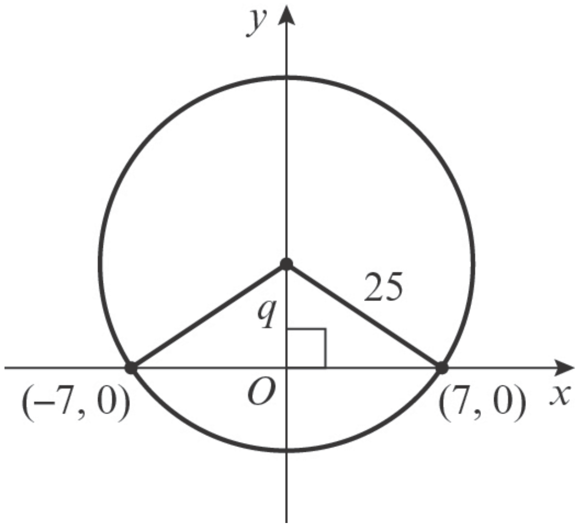
gradient = $\dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{8-0}{0-6} = -\dfrac{4}{3}$

y-intercept = 8

Equation of the line AB is $y = -\dfrac{4}{3}x + 8$

c Area of triangle $OAB = \dfrac{1}{2} \times 6 \times 8 = 24$ units^2

13 a By symmetry $p = 0$.



Using Pythagoras' theorem

$$q^2 + 7^2 = 25^2$$
$$q^2 + 49 = 625$$
$$q^2 = 576$$
$$q = \pm\sqrt{576}$$
$$q = \pm 24$$

As $q > 0$, $q = 24$.

b The circle meets the y-axis at $q \pm r$; i.e.

at $24 + 25 = 49$

and $24 - 25 = -1$

So the coordinates are (0, 49) and (0, –1).

14 The gradient of the line joining $(-3,-7)$ and $(5,1)$ is

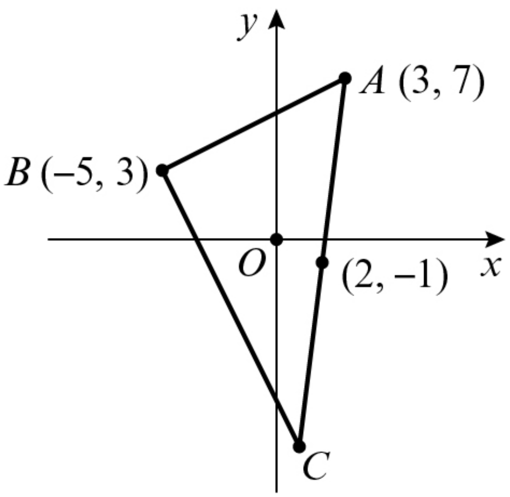
$$\frac{y_2-y_1}{x_2-x_1}=\frac{1-(-7)}{5-(-3)}=\frac{1+7}{5+3}=\frac{8}{8}=1$$

So the gradient of the tangent is $-\frac{1}{(1)}=-1$.

The equation of the tangent is

$$y-y_1=m(x-x_1)$$
$$y-(-7)=-1(x-(-3))$$
$$y+7=-1(x+3)$$
$$y+7=-x-3$$
$$y=-x-10 \text{ or } x+y+10=0$$

15



Let the coordinates of C be (p, q).
$(2, –1)$ is the mid-point of $(3, 7)$ and (p, q)

So $\frac{3+p}{2}=2$ and $\frac{7+q}{2}=-1$

$$\frac{3+p}{2}=2$$
$$3+p=4$$
$$p=1$$
$$\frac{7+q}{2}=-1$$
$$7+q=-2$$
$$q=-9$$

15 So the coordinates of C are (1, –9).

The length of AB is
$$\sqrt{\left(x_2-x_1\right)^2+\left(y_2-y_1\right)^2}=\sqrt{\left(-5-3\right)^2+\left(3-7\right)^2}=\sqrt{\left(-8\right)^2+\left(-4\right)^2}=\sqrt{64+16}=\sqrt{80}$$

The length of BC is
$$\sqrt{\left(x_2-x_1\right)^2+\left(y_2-y_1\right)^2}=\sqrt{\left(-5-1\right)^2+\left(3-\left(-9\right)\right)^2}=\sqrt{\left(-6\right)^2+\left(12\right)^2}=\sqrt{36+144}=\sqrt{180}$$

The area of ΔABC is
$$\frac{1}{2}\sqrt{180}\sqrt{80}=\frac{1}{2}\sqrt{14400}=\frac{1}{2}\sqrt{144\times100}=\frac{1}{2}\sqrt{144}\times\sqrt{100}=\frac{1}{2}\times12\times10=60$$

16 $(x-6)^2+(y-5)^2=17$
Centre of the circle is (6, 5).

Equation of the line touching the circle is $y = mx + 12$

Substitute the equation of the line into the equation of the circle:
$$(x-6)^2+(mx+7)^2=17$$
$$x^2-12x+36+m^2x^2+14mx+49-17=0$$
$$(1+m^2)x^2+(14m-12)x+68=0$$

There is one solution so using the discriminant $b^2-4ac=0$:
$$(14m-12)^2-4(1+m^2)(68)=0$$
$$196m^2-336m+144-272m^2-272=0$$
$$76m^2+336m+128=0$$
$$19m^2+84m+32=0$$
$$(19m+8)(m+4)=0$$

$m=-\frac{8}{19}$ or $m=-4$

$y=-\frac{8}{19}x+12$ and $y=-4x+12$

17 a Gradient of $AB=\frac{y_2-y_1}{x_2-x_1}=\frac{1-7}{5-3}=-3$

Midpoint of $AB=\left(\frac{3+5}{2},\frac{7+1}{2}\right)=(4, 4)$

$M(4, 4)$

Line l is perpendicular to AB, so gradient of line $l=\frac{1}{3}$

$y-y_1=m(x-x_1)$

$y-4=\frac{1}{3}(x-4)$

17 a $y = \frac{1}{3}x + \frac{8}{3}$

b $C(-2, c)$

$y = \frac{1}{3}(-2) + \frac{8}{3} = 2$

$C(-2, 2)$

Radius of the circle = distance CA

$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(-2-3)^2 + (2-7)^2} = \sqrt{50}$

Equation of the circle is $(x + 2)^2 + (y - 2)^2 = 50$

c Base of triangle = distance $AB = \sqrt{(5-3)^2 + (1-7)^2} = \sqrt{40}$

Height of triangle = distance $CM = \sqrt{(4+2)^2 + (4-2)^2} = \sqrt{40}$

Area of triangle $ABC = \frac{1}{2} \times \sqrt{40} \times \sqrt{40} = 20$ units^2

18 a $(x - 3)^2 + (y + 3)^2 = 52$

The equations of the lines l_1 and l_2 are $y = \frac{3}{2}x + c$

Diameter of the circle that touches l_1 and l_2 has gradient $-\frac{2}{3}$ and passes through the centre of the circle $(3, -3)$

$y = -\frac{2}{3}x + d$

$-3 = -\frac{2}{3}(3) + d$

$d = -1$

$y = -\frac{2}{3}x - 1$ is the equation of the diameter that touches l_1 and l_2.

Solve the equation of the diameter and circle simultaneously:

$$(x - 3)^2 + (-\frac{2}{3}x + 2)^2 = 52$$

$$x^2 - 6x + 9 + \frac{4}{9}x^2 - \frac{8}{3}x + 4 - 52 = 0$$

$$\frac{13}{9}x^2 - \frac{26}{3}x - 39 = 0$$

$$13x^2 - 78x - 351 = 0$$

$$x^2 - 6x - 27 = 0$$

$$(x - 9)(x + 3) = 0$$

$$x = 9 \text{ or } x = -3$$

18 a When $x = 9$, $y = -\frac{2}{3}(9) - 1 = -7$

When $x = -3$, $y = -\frac{2}{3}(-3) - 1 = 1$

$(9, -7)$ and $(-3, 1)$ are the coordinates where the diameter touches lines l_1 and l_2.
$P(9, -7)$ and $Q(-3, 1)$

b The equations of the lines l_1 and l_2 are $y = \frac{3}{2}x + c$

l_1 touches the circle at $(-3, 1)$:

$1 = \frac{3}{2}(-3) + c$, $c = \frac{11}{2}$, so $y = \frac{3}{2}x + \frac{11}{2}$

l_2 touches the circle at $(9, -7)$:

$-7 = \frac{3}{2}(9) + c$, $c = -\frac{41}{2}$, so $y = \frac{3}{2}x - \frac{41}{2}$

19 a $x^2 + 6x + y^2 - 2y = 7$
Equation of the lines are $y = mx + 6$

Substitute $y = mx + 6$ into the equation of the circle:

$$x^2 + 6x + (mx + 6)^2 - 2(mx + 6) = 7$$
$$x^2 + 6x + m^2x^2 + 12mx + 36 - 2mx - 12 - 7 = 0$$
$$(1 + m^2)x^2 + (6 + 10m)x + 17 = 0$$

There is one solution so using the discriminant $b^2 - 4ac = 0$:

$$(6 + 10m)^2 - 4(1 + m^2)(17) = 0$$
$$100m^2 + 120m + 36 - 68m^2 - 68 = 0$$
$$32m^2 + 120m - 32 = 0$$
$$4m^2 + 15m - 4 = 0$$
$$(4m - 1)(m + 4) = 0$$

$m = \frac{1}{4}$ or $m = -4$

$y = \frac{1}{4}x + 6$ and $y = -4x + 6$

b The gradient of $l_1 = \frac{1}{4}$ and the gradient of $l_2 = -4$, so the two lines are perpendicular,

Therefore, $APRQ$ is a square.
$x^2 + 6x + y^2 - 2y = 7$
Completing the square:

$$(x + 3)^2 - 9 + (y - 1)^2 - 1 = 7$$
$$(x + 3)^2 + (y - 1)^2 = 17$$

Radius = $\sqrt{17}$

19 b Let point P have the coordinates (x, y)
Using Pythagoras' theorem:
l_2: $(0 - x)^2 + (6 - y)^2 = 17$

Using the equation for l_2, $y = \frac{1}{4}x + 6$, $(0-x)^2 + \left(6-\left(\frac{1}{4}x+6\right)\right)^2 = 17$

$$x^2 + \frac{1}{16}x^2 = 17$$
$$\frac{17}{16}x^2 = 17$$
$$x^2 = 16$$
$$x = \pm 4$$

From the diagram we know that x is negative, so $x = -4$, $y = \frac{1}{4}(-4) + 6 = 5$

$P(-4, 5)$

Now let point Q have the coordinates (x, y).
Using the equation for l_1, $y = -4x + 6$, $(0 - x)^2 + (6 - (-4x + 6))^2 = 17$

$$x^2 + 16x^2 = 17$$
$$17x^2 = 17$$
$$x^2 = 1$$
$$x = \pm 1$$

From the diagram x is positive, so $x = 1$, $y = -4(1) + 6 = 2$
$Q(1, 2)$

c Area of the square = radius^2 = 17 units^2

20 a Equation of the circle: $(x - 6)^2 + (y - 9)^2 = 50$

Equation of l_1: $y = -x + 21$

Substitute the equation of the line into the equation of the circle:
$$(x - 6)^2 + (-x + 12)^2 = 50$$
$$x^2 - 12x + 36 + x^2 - 24x + 144 - 50 = 0$$
$$2x^2 - 36x + 130 = 0$$
$$x^2 - 18x + 65 = 0$$
$$(x - 13)(x - 5) = 0$$
$x = 13$ or $x = 5$
When $x = 13$, $y = -13 + 21 = 8$
When $x = 5$, $y = -5 + 21 = 16$
$P(5, 16)$ and $Q(13, 8)$

20 b The gradient of the line $AP = \frac{y_2 - y_1}{x_2 - x_1} = \frac{9-16}{6-5} = -7$

So the gradient of the line perpendicular to AP, l_2, is $\frac{1}{7}$.

The equation of the perpendicular line is

$y - y_1 = m(x - x_1)$

$m = \frac{1}{7}$ and $(x_1, y_1) = P(5, 16)$

So $y - 16 = \frac{1}{7}(x - 5)$

$y = \frac{1}{7}x + \frac{107}{7}$

The gradient of the line $AQ = \frac{y_2 - y_1}{x_2 - x_1} = \frac{9-8}{6-13} = -\frac{1}{7}$

So the gradient of the line perpendicular to AQ, l_3, is 7.
The equation of the perpendicular line is

$y - y_1 = m(x - x_1)$

$m = 7$ and $(x_1, y_1) = Q(13, 8)$

So $y - 8 = 7(x - 13)$
$y = 7x - 83$

l_2: $y = \frac{1}{7}x + \frac{107}{7}$ and l_3: $y = 7x - 83$

c The gradient of the line $PQ = \frac{y_2 - y_1}{x_2 - x_1} = \frac{8-16}{13-5} = -1$

So the gradient of the line perpendicular to PQ, l_4, is 1.
The equation of the perpendicular line is

$y - y_1 = m(x - x_1)$

$m = 1$ and $(x_1, y_1) = A(6, 9)$

So $y - 9 = 1(x - 6)$
l_4: $y = x + 3$

d l_2: $y = \frac{1}{7}x + \frac{107}{7}$, l_3: $y = 7x - 83$ and l_4: $y = x + 3$

Solve these equations simultaneously one pair at a time:

l_2 and l_3: $\frac{1}{7}x + \frac{107}{7} = 7x - 83$

$x + 107 = 49x - 581$

$48x = 688$

$x = \frac{43}{3}$, so $y = 7\left(\frac{43}{3}\right) - 83 = \frac{52}{3}$

20 d l_2 and l_3 intersect at $\left(\frac{43}{3}, \frac{52}{3}\right)$.

l_3 and l_4: $7x - 83 = x + 3$

$6x = 86$

$x = \frac{43}{3}$, so $y = \frac{43}{3} + 3 = \frac{52}{3}$

Therefore all three lines intersect at $R\left(\frac{43}{3}, \frac{52}{3}\right)$

e Area of kite $APRQ = \frac{1}{2} \times AR \times PQ$

$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(13-5)^2 + (8-16)^2} = \sqrt{128} = 8\sqrt{2}$

$AR = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{\left(\frac{43}{3} - 6\right)^2 + \left(\frac{52}{3} - 9\right)^2} = \sqrt{\left(\frac{25}{3}\right)^2 + \left(\frac{25}{3}\right)^2} = \sqrt{\frac{1250}{9}} = \frac{25\sqrt{2}}{3}$

Area $= \frac{1}{2} \times \frac{25\sqrt{2}}{3} \times 8\sqrt{2} = \frac{200}{3}$

21 a $y = -3x + 12$

Substitute $x = 0$ into $y = -3x + 12$

$y = -3(0) + 12 = 12$

So A is (0, 12)

Substitute $y = 0$ into $y = -3x + 12$

$0 = -3x + 12$

$3x = 12$

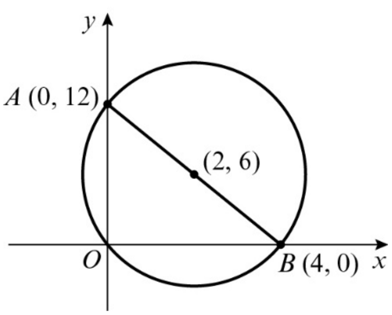
$x = 4$

So B is (4, 0).

b The mid-point of AB is

$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{0+4}{2}, \frac{12+0}{2}\right) = (2, 6)$

c



21 c $\angle AOB = 90°$, so AB is a diameter of the circle.
The centre of the circle is the mid-point of AB, i.e. (2, 6).
The length of the diameter AB is

$$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}=\sqrt{(4-0)^2+(0-12)^2}=\sqrt{4^2+(-12)^2}=\sqrt{16+144}=\sqrt{160}$$

So the radius of the circle is $\dfrac{\sqrt{160}}{2}$.

The equation of the circle is

$$(x-2)^2+(y-6)^2=\left(\frac{\sqrt{160}}{2}\right)^2$$

$$(x-2)^2+(y-6)^2=\frac{160}{4}$$

$$(x-2)^2+(y-6)^2=40$$

22 a $A(-3, -2)$, $B(-6, 0)$ and $C(1, q)$

$AB = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} = \sqrt{(-6+3)^2+(0+2)^2} = \sqrt{13}$

Diameter = $BC = \sqrt{(1+6)^2+(q-0)^2} = \sqrt{49+q^2}$

$AC = \sqrt{(1+3)^2+(q+2)^2} = \sqrt{16+q^2+4q+4} = \sqrt{q^2+4q+20}$

Using Pythagoras' theorem $AC^2 + AB^2 = BC^2$:

$$(q^2+4q+20)+13=49+q^2$$
$$4q-16=0$$
$$q=4$$

b The centre of the circle is the midpoint of $B(-6, 0)$ and $C(1, 4)$

Midpoint $BC = \left(\dfrac{-6+1}{2}, \dfrac{0+4}{2}\right) = \left(-\dfrac{5}{2}, 2\right)$

The radius is half of $BC = \dfrac{1}{2}$ of $\sqrt{49+q^2} = \dfrac{1}{2}$ of $\sqrt{49+4^2} = \dfrac{1}{2}$ of $\sqrt{65} = \dfrac{\sqrt{65}}{2}$

Equation of the circle is $\left(x+\dfrac{5}{2}\right)^2+(y-2)^2=\left(\dfrac{\sqrt{65}}{2}\right)^2$

$$\left(x+\frac{5}{2}\right)^2+(y-2)^2=\frac{65}{4}$$

23 a $R(-4, 3)$, $S(7, 4)$ and $T(8, -7)$

$RT = \sqrt{(8+4)^2+(-7-3)^2} = \sqrt{244}$

$RS = \sqrt{(7+4)^2+(4-3)^2} = \sqrt{122}$

$ST = \sqrt{(8-7)^2+(-7-4)^2} = \sqrt{122}$

Using Pythagoras' theorem, $ST^2 + RS^2 = 122 + 122 = 244 = RT^2$, therefore, RT is the diameter of the circle.

b The radius of the circle is

$\frac{1}{2}\times\text{diameter} = \frac{1}{2}\sqrt{244} = \frac{1}{2}\sqrt{4\times 61} = \frac{1}{2}\sqrt{4}\times\sqrt{61} = \frac{1}{2}\times 2\sqrt{61} = \sqrt{61}$

The centre of the circle is the mid-point of RT:

$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = \left(\frac{-4+8}{2}, \frac{3+(-7)}{2}\right) = \left(\frac{4}{2}, \frac{-4}{2}\right) = (2, -2)$

So the equation of the circle is

$(x-2)^2+(y+2)^2 = \left(\sqrt{61}\right)^2$ or $(x-2)^2+(y+2)^2 = 61$

24 $A(-4, 0)$, $B(4, 8)$ and $C(6, 0)$

The gradient of the line $AB = \frac{y_2-y_1}{x_2-x_1} = \frac{8-0}{4+4} = 1$

So the gradient of the line perpendicular to AB, is -1.

Midpoint of $AB = \left(\frac{-4+4}{2}, \frac{0+8}{2}\right) = (0, 4)$

The equation of the perpendicular line through the midpoint of AB is

$y - y_1 = m(x - x_1)$

$m = -1$ and $(x_1, y_1) = (0, 4)$

So $y - 4 = -(x - 0)$

$y = -x + 4$

The gradient of the line $BC = \frac{y_2-y_1}{x_2-x_1} = \frac{0-8}{6-4} = -4$

So the gradient of the line perpendicular to BC, is $\frac{1}{4}$.

Midpoint of $BC = \left(\frac{4+6}{2}, \frac{8+0}{2}\right) = (5, 4)$

24 The equation of the perpendicular line through the midpoint of BC is

$y - y_1 = m(x - x_1)$

$m = \frac{1}{4}$ and $(x_1, y_1) = (5, 4)$

So $y - 4 = \frac{1}{4}(x - 5)$

$y = \frac{1}{4}x + \frac{11}{4}$

Solving these two equations simultaneously will give the centre of the circle:

$-x + 4 = \frac{1}{4}x + \frac{11}{4}$

$-4x + 16 = x + 11$

$5x = 5$

$x = 1$, so $y = -1 + 4 = 3$

The centre of the circle is (1, 3).

The radius is the distance from the centre of the circle (1, 3) to a point on the circumference $C(6, 0)$:

Radius = $\sqrt{(6-1)^2 + (0-3)^2} = \sqrt{34}$

The equation of the circle is $(x - 1)^2 + (y - 3)^2 = 34$

25 a i $A(-7, 7)$ and $B(1, 9)$

The gradient of the line $AB = \frac{y_2 - y_1}{x_2 - x_1} = \frac{9-7}{1+7} = \frac{1}{4}$

So the gradient of the line perpendicular to AB, is -4.

Midpoint of $AB = \left(\frac{-7+1}{2}, \frac{7+9}{2}\right) = (-3, 8)$

The equation of the perpendicular line is

$y - y_1 = m(x - x_1)$

$m = -4$ and $(x_1, y_1) = (-3, 8)$

So $y - 8 = -4(x + 3)$

$y = -4x - 4$

ii $C(3, 1)$ and $D(-7, 1)$

The line CD is $y = 1$

Midpoint of $CD = \left(\frac{3-7}{2}, \frac{1+1}{2}\right) = (-2, 1)$

The equation of the perpendicular line is $x = -2$

25 b The two perpendicular bisectors cross at the centre of the circle
Solve $y = -4x - 4$ and $x = -2$ simultaneously:
$y = -4(-2) - 4 = 4$
The centre of the circle = $(-2, 4)$
The radius is the distance from the centre of the circle $(-2, 4)$ to a point on the circumference $C(3, 1)$:
Radius = $\sqrt{(3+2)^2+(1-4)^2} = \sqrt{34}$
The equation of the circle is $(x + 2)^2 + (y - 4)^2 = 34$

Challenge

a Solve $(x - 5)^2 + (y - 3)^2 = 20$ and $(x - 10)^2 + (y - 8)^2 = 10$ simultaneously:
$x^2 - 10x + 25 + y^2 - 6y + 9 = 20$ and $x^2 - 20x + 100 + y^2 - 16y + 64 = 10$
$x^2 - 10x + y^2 - 6y + 14 = 0$ and $x^2 - 20x + y^2 - 16y + 154 = 0$
$x^2 - 10x + y^2 - 6y + 14 = x^2 - 20x + y^2 - 16y + 154$
$-10x - 6y + 14 = -20x - 16y + 154$
$10x + 10y = 140$
$x + y = 14$
$x + y - 14 = 0$

b Solve $(x - 5)^2 + (y - 3)^2 = 20$ and $x + y = 14$ simultaneously:
$(9 - y)^2 + (y - 3)^2 = 20$
$81 - 18y + y^2 + y^2 - 6y + 9 = 20$
$2y^2 - 24y + 70 = 0$
$y^2 - 12y + 35 = 0$
$(y - 5)(y - 7) = 0$
$y = 5$ or $y = 7$

When $y = 5$, $x = 14 - 5 = 9$
When $y = 7$, $x = 14 - 7 = 7$

$P(7, 7)$ and $Q(9, 5)$

c Area of kite $APBQ = \frac{1}{2} \times PQ \times AB$
$PQ = \sqrt{(9-7)^2+(5-7)^2} = \sqrt{8}$

$A(5, 3)$ and $B(10, 8)$

$AB = \sqrt{(10-5)^2+(8-3)^2} = \sqrt{50}$

Area $= \frac{1}{2}\times\sqrt{8}\times\sqrt{50} = \frac{1}{2}\times\sqrt{400} = 10$ units^2

Algebraic methods 7A

1 a $\dfrac{4x^4+5x^2-7x}{x}$

$= \dfrac{4x^4}{x}+\dfrac{5x^2}{x}-\dfrac{7x}{x}$

$= 4x^3+5x-7$

b $\dfrac{7x^5-5x^5+9x^3+x^2}{x}$

$= \dfrac{7x^5}{x}-\dfrac{5x^5}{x}+\dfrac{9x^3}{x}+\dfrac{x^2}{x}$

$= 7x^4-5x^4+9x^2+x$

c $\dfrac{-x^4+4x^2+6}{x}$

$= \dfrac{-x^4}{x}+\dfrac{4x^2}{x}+\dfrac{6}{x}$

$= -x^3+4x+\dfrac{6}{x}$

d $\dfrac{7x^5-x^3-4}{x}$

$= \dfrac{7x^5}{x}-\dfrac{x^3}{x}-\dfrac{4}{x}$

$= 7x^4-x^2-\dfrac{4}{x}$

e $\dfrac{8x^4-4x^3+6x}{2x}$

$= \dfrac{8x^4}{2x}-\dfrac{4x^3}{2x}+\dfrac{6x}{2x}$

$= 4x^3-2x^2+3$

f $\dfrac{9x^2-12x^3-3x}{3x}$

$= \dfrac{9x^2}{3x}-\dfrac{12x^3}{3x}-\dfrac{3x}{3x}$

$= 3x-4x^2-1$

g $\dfrac{7x^3-x^4-2}{5x}$

$= \dfrac{7x^3}{5x}-\dfrac{x^4}{5x}-\dfrac{2}{5x}$

$= \dfrac{7x^2}{5}-\dfrac{x^3}{5}-\dfrac{2}{5x}$

h $\dfrac{-4x^2+6x^4-2x}{-2x}$

$= \dfrac{-4x^2}{-2x}+\dfrac{6x^4}{-2x}-\dfrac{2x}{-2x}$

$= 2x-3x^3+1$

i $\dfrac{-x^8+9x^4-4x^3+6}{-2x}$

$= -\dfrac{x^8}{-2x}+\dfrac{9x^4}{-2x}-\dfrac{4x^3}{-2x}+\dfrac{6}{-2x}$

$= \dfrac{x^7}{2}-\dfrac{9x^3}{2}+2x^2-\dfrac{3}{x}$

j $\dfrac{-9x^9-6x^6+4x^4-2}{-3x}$

$= \dfrac{-9x^9}{-3x}-\dfrac{6x^6}{-3x}+\dfrac{4x^4}{-3x}-\dfrac{2}{-3x}$

$= 3x^8+2x^5-\dfrac{4x^3}{3}+\dfrac{2}{3x}$

2 a $\dfrac{(x+3)(x-2)}{(x-2)} = x+3$

b $\dfrac{(x+4)(3x-1)}{(3x-1)} = x+4$

c $\dfrac{(x+3)^2}{(x+3)} = \dfrac{(x+3)(x+3)}{(x+3)}$

$= x+3$

d $\dfrac{x^2+10x+21}{(x+3)} = \dfrac{(x+7)(x+3)}{(x+3)}$

$= x+7$

e $\dfrac{x^2+9x+20}{(x+4)} = \dfrac{(x+4)(x+5)}{(x+4)}$

$= x+5$

f $\dfrac{x^2+x-12}{(x-3)} = \dfrac{(x-3)(x+4)}{(x-3)}$

$= x+4$

2 **g** $\frac{x^2+x-20}{x^2+2x-15}=\frac{(x+5)(x-4)}{(x+5)(x-3)}$
$=\frac{x-4}{x-3}$

h $\frac{x^2+3x+2}{x^2+5x+4}=\frac{(x+2)(x+1)}{(x+4)(x+1)}$
$=\frac{x+2}{x+4}$

i $\frac{x^2+x-12}{x^2-9x+18}=\frac{(x+4)(x-3)}{(x-6)(x-3)}$
$=\frac{x+4}{x-6}$

j $\frac{2x^2+7x+6}{(x-5)(x+2)}=\frac{(2x+3)(x+2)}{(x-5)(x+2)}$
$=\frac{2x+3}{x-5}$

k $\frac{2x^2+9x-18}{(x+6)(x+1)}=\frac{(2x-3)(x+6)}{(x+6)(x+1)}$
$=\frac{2x-3}{x+1}$

l $\frac{3x^2-7x+2}{(3x-1)(x+2)}=\frac{(3x-1)(x-2)}{(3x-1)(x+2)}$
$=\frac{x-2}{x+2}$

m $\frac{2x^2+3x+1}{x^2-x-2}=\frac{(2x+1)(x+1)}{(x-2)(x+1)}$
$=\frac{2x+1}{x-2}$

n $\frac{x^2+6x+8}{3x^2+7x+2}=\frac{(x+4)(x+2)}{(3x+1)(x+2)}$
$=\frac{x+4}{3x+1}$

o $\frac{2x^2-5x-3}{2x^2-9x+9}=\frac{(2x+1)(x-3)}{(2x-3)(x-3)}$
$=\frac{2x+1}{2x-3}$

3 $6x^3+3x^2-84x=3x(2x^2+x-28)$
$=3x(2x-7)(x+4)$

$6x^2-33x+42=3(2x^2-11x+14)$
$=3(x-2)(2x-7)$

$\frac{6x^3+3x^2-84x}{6x^2-33x+42}=\frac{3x(2x-7)(x+4)}{3(x-2)(2x-7)}$
$=\frac{x(x+4)}{(x-2)}$

$a=1$, $b=4$, $c=-2$

Algebraic methods 7B

1 a
$$\begin{array}{r} x^2+5x+3 \\ x+1\,\overline{\smash{)}\,x^3+6x^2+8x+3} \\ \underline{x^3+x^2} \\ 5x^2+8x \\ \underline{5x^2+5x} \\ 3x+3 \\ \underline{3x+3} \\ 0 \end{array}$$

So $\dfrac{x^3+6x^2+8x+3}{x+1}$

$= (x+1)(x^2+5x+3)$

b
$$\begin{array}{r} x^2+6x+1 \\ x+4\,\overline{\smash{)}\,x^3+10x^2+25x+4} \\ \underline{x^3+4x^2} \\ 6x^2+25x \\ \underline{6x^2+24x} \\ x+4 \\ \underline{x+4} \\ 0 \end{array}$$

So $\dfrac{x^3+10x^2+25x+4}{x+4}$

$= (x+4)(x^2+6x+1)$

c
$$\begin{array}{r} x^2-3x+7 \\ x+2\,\overline{\smash{)}\,x^3-x^2+x+14} \\ \underline{x^3+2x^2} \\ -3x^2+x \\ \underline{-3x^2-6x} \\ 7x+14 \\ \underline{7x+14} \\ 0 \end{array}$$

So $\dfrac{x^3-x^2+x+14}{x+2}$

$= (x+2)(x^2-3x+7)$

d
$$\begin{array}{r} x^2+4x+5 \\ x-3\,\overline{\smash{)}\,x^3+x^2-7x-15} \\ \underline{x^3-3x^2} \\ 4x^2-7x \\ \underline{4x^2-12x} \\ 5x-15 \\ \underline{5x-15} \\ 0 \end{array}$$

So $\dfrac{x^3+x^2-7x-15}{x-3}$

$= (x-3)(x^2+4x+5)$

e
$$\begin{array}{r} x^2-3x-2 \\ x-5\,\overline{\smash{)}\,x^3-8x^2+13x+10} \\ \underline{x^3-5x^2} \\ -3x^2+13x \\ \underline{-3x^2+15x} \\ -2x+10 \\ \underline{-2x+10} \\ 0 \end{array}$$

So $\dfrac{x^3-8x^2+13x+10}{x-5}$

$= (x-5)(x^2-3x-2)$

f
$$\begin{array}{r} x^2+2x+8 \\ x-7\,\overline{\smash{)}\,x^3-5x^2-6x-56} \\ \underline{x^3-7x^2} \\ 2x^2-6x \\ \underline{2x^2-14x} \\ 8x-56 \\ \underline{8x-56} \\ 0 \end{array}$$

So $\dfrac{x^3-5x^2-6x-56}{x-7}$

$= (x-7)(x^2+2x+8)$

2 a

$$
\begin{array}{rl}
 & 6x^2+3x+2 \\
x+4 & \overline{)\;6x^3+27x^2+14x+8} \\
 & \underline{6x^3+24x^2} \\
 & 3x^2+14x \\
 & \underline{3x^2+12x} \\
 & 2x+8 \\
 & \underline{2x+8} \\
 & 0
\end{array}
$$

So $6x^3+27x^2+14x+8$
$= (x+4)(6x^2+3x+2)$

b

$$
\begin{array}{rl}
 & 4x^2+x-5 \\
x+2 & \overline{)\;4x^3+9x^2-3x-10} \\
 & \underline{4x^3+8x^2} \\
 & x^2-3x \\
 & \underline{x^2+2x} \\
 & -5x-10 \\
 & \underline{-5x-10} \\
 & 0
\end{array}
$$

So $4x^3+9x^2-3x-10$
$= (x+2)(4x^2+x-5)$

c

$$
\begin{array}{rl}
 & 2x^2-2x-3 \\
x+3 & \overline{)\;2x^3+4x^2-9x-9} \\
 & \underline{2x^3+6x^2} \\
 & -2x^2-9x \\
 & \underline{-2x^2-6x} \\
 & -3x-9 \\
 & \underline{-3x-9} \\
 & 0
\end{array}
$$

So $2x^3+4x^2-9x-9$
$= (x+3)(2x^2-2x-3)$

d

$$
\begin{array}{rl}
 & 2x^2-3x-4 \\
x-6 & \overline{)\;2x^3-15x^2+14x+24} \\
 & \underline{2x^3-12x^2} \\
 & -3x^2+14x \\
 & \underline{-3x^2+18x} \\
 & -4x+24 \\
 & \underline{-4x+24} \\
 & 0
\end{array}
$$

So $2x^3-15x^2+14x+24$
$= (x-6)(2x^2-3x-4)$

e

$$
\begin{array}{rl}
 & -5x^2+3x+5 \\
x+6 & \overline{)\;-5x^3-27x^2+23x+30} \\
 & \underline{-5x^3-30x^2} \\
 & 3x^2+23x \\
 & \underline{3x^2+18x} \\
 & 5x+30 \\
 & \underline{5x+30} \\
 & 0
\end{array}
$$

So $-5x^3-27x^2+23x+30$
$= (x+6)(-5x^2+3x+5)$

f

$$
\begin{array}{rl}
 & -4x^2+x-1 \\
x-2 & \overline{)\;-4x^3+9x^2-3x+2} \\
 & \underline{-4x^3+8x^2} \\
 & x^2-3x \\
 & \underline{x^2-2x} \\
 & -x+2 \\
 & \underline{-x+2} \\
 & 0
\end{array}
$$

So $-4x^3+9x^2-3x+2$
$= (x-2)(-4x^2+x-1)$

3 a

$$\begin{array}{r} x^3+3x^2-4x+1 \\ x+2 \overline{\big)\, x^4+5x^3+2x^2-7x+2} \\ \underline{x^4+2x^3} \qquad\qquad\qquad \\ 3x^3+2x^2 \qquad\qquad \\ \underline{3x^3+6x^2} \qquad\qquad \\ -4x^2-7x \qquad \\ \underline{-4x^2-8x} \qquad \\ x+2 \\ \underline{x+2} \\ 0 \end{array}$$

So $\dfrac{x^4+5x^3+2x^2-7x+2}{x+2}$

$= x^3+3x^2-4x+1$

b

$$\begin{array}{r} 4x^3+2x^2-3x-5 \\ x+3 \overline{\big)\, 4x^4+14x^3+3x^2-14x-15} \\ \underline{4x^4+12x^3} \qquad\qquad\qquad \\ 2x^3+3x^2 \qquad\qquad \\ \underline{2x^3+6x^2} \qquad\qquad \\ -3x^2-14x \qquad \\ \underline{-3x^2-9x} \qquad \\ -5x-15 \\ \underline{-5x-15} \\ 0 \end{array}$$

So $\dfrac{4x^4+14x^3+3x^2-14x-15}{x+3}$

$= 4x^3+2x^2-3x-5$

c

$$\begin{array}{r} -3x^3+3x^2-4x-7 \\ x-2 \overline{\big)\, -3x^4+9x^3-10x^2+x+14} \\ \underline{-3x^4+6x^3} \qquad\qquad\qquad \\ 3x^3-10x^2 \qquad\qquad \\ \underline{3x^3-6x^2} \qquad\qquad \\ -4x^2+x \qquad \\ \underline{-4x^2+8x} \qquad \\ -7x+14 \\ \underline{-7x+14} \\ 0 \end{array}$$

c So $\dfrac{-3x^4+9x^3-10x^2+x+14}{x-2}$

$= -3x^3+3x^2-4x-7$

d

$$\begin{array}{r} -5x^4+2x^3+4x^2-3x+7 \\ x-1 \overline{\big)\, -5x^5+7x^4+2x^3-7x^2+10x-7} \\ \underline{-5x^5+5x^4} \qquad\qquad\qquad\qquad \\ 2x^4+2x^3 \qquad\qquad\qquad \\ \underline{2x^4-2x^3} \qquad\qquad\qquad \\ 4x^3-7x^2 \qquad\qquad \\ \underline{4x^3-4x^2} \qquad\qquad \\ -3x^2+10x \qquad \\ \underline{-3x^2+3x} \qquad \\ 7x-7 \\ \underline{7x-7} \\ 0 \end{array}$$

So $\dfrac{-5x^5+7x^4+2x^3-7x^2+10x-7}{x-1}$

$= -5x^4+2x^3+4x^2-3x+7$

4 a

$$\begin{array}{r} x^3+2x^2-5x+4 \\ 3x+2 \overline{\big)\, 3x^4+8x^3-11x^2+2x+8} \\ \underline{3x^4+2x^3} \qquad\qquad\qquad \\ 6x^3-11x^2 \qquad\qquad \\ \underline{6x^3+4x^2} \qquad\qquad \\ -15x^2+2x \qquad \\ \underline{-15x^2-10x} \qquad \\ 12x+8 \\ \underline{12x+8} \\ 0 \end{array}$$

So $\dfrac{3x^4+8x^3-11x^2+2x+8}{3x+2}$

$= x^3+2x^2-5x+4$

4 b

$$
\begin{array}{r|l}
 & x^3 - x^2 + 3x - 1 \\
\hline
4x+1 & 4x^4 - 3x^3 + 11x^2 - x - 1 \\
 & \underline{4x^4 + x^3} \\
 & \quad -4x^3 + 11x^2 \\
 & \quad \underline{-4x^3 - x^2} \\
 & \qquad\quad 12x^2 - x \\
 & \qquad\quad \underline{12x^2 + 3x} \\
 & \qquad\qquad\quad -4x - 1 \\
 & \qquad\qquad\quad \underline{-4x - 1} \\
 & \qquad\qquad\qquad\quad 0
\end{array}
$$

So $\dfrac{4x^4 - 3x^3 + 11x^2 - x - 1}{4x+1}$

$= x^3 - x^2 + 3x - 1$

c

$$
\begin{array}{r|l}
 & 2x^3 + 5x + 2 \\
\hline
2x-3 & 4x^4 - 6x^3 + 10x^2 - 11x - 6 \\
 & \underline{4x^4 - 6x^3} \\
 & \qquad 0 + 10x^2 - 11x \\
 & \qquad\quad \underline{10x^2 - 15x} \\
 & \qquad\qquad\qquad 4x - 6 \\
 & \qquad\qquad\qquad \underline{4x - 6} \\
 & \qquad\qquad\qquad\quad 0
\end{array}
$$

So $\dfrac{4x^4 - 6x^3 + 10x^2 - 11x - 6}{2x-3}$

$= 2x^3 + 5x + 2$

d

$$
\begin{array}{r|l}
 & 3x^4 + 2x^3 - 5x^2 + 3x + 6 \\
\hline
2x+3 & 6x^5 + 13x^4 - 4x^3 - 9x^2 + 21x + 18 \\
 & \underline{6x^5 + 9x^4} \\
 & \quad 4x^4 - 4x^3 \\
 & \quad \underline{4x^4 + 6x^3} \\
 & \qquad -10x^3 - 9x^2 \\
 & \qquad\quad \underline{10x^3 - 15x^2} \\
 & \qquad\qquad\quad 6x^2 + 21x \\
 & \qquad\qquad\quad \underline{6x^2 + 9x} \\
 & \qquad\qquad\qquad\quad 12x + 18 \\
 & \qquad\qquad\qquad\quad \underline{12x + 18} \\
 & \qquad\qquad\qquad\qquad\quad 0
\end{array}
$$

d So $\dfrac{6x^5 + 13x^4 - 4x^3 - 9x^2 + 21x + 18}{2x+3}$

$= 3x^4 + 2x^3 - 5x^2 + 3x + 6$

e

$$
\begin{array}{r|l}
 & 2x^4 - 2x^3 + 3x^2 + 4x - 7 \\
\hline
3x-1 & 6x^5 - 8x^4 + 11x^3 + 9x^2 - 25x + 7 \\
 & \underline{6x^5 - 2x^4} \\
 & \quad -6x^4 + 11x^3 \\
 & \quad \underline{-6x^4 + 2x^3} \\
 & \qquad\qquad 9x^3 + 9x^2 \\
 & \qquad\qquad \underline{9x^3 - 3x^2} \\
 & \qquad\qquad\quad 12x^2 - 25x \\
 & \qquad\qquad\quad \underline{12x^2 - 4x} \\
 & \qquad\qquad\qquad\quad -21x + 7 \\
 & \qquad\qquad\qquad\quad \underline{-21x + 7} \\
 & \qquad\qquad\qquad\qquad\quad 0
\end{array}
$$

So $\dfrac{6x^5 - 8x^4 + 11x^3 + 9x^2 - 25x + 7}{3x-1}$

$= 2x^4 - 2x^3 + 3x^2 + 4x - 7$

f

$$
\begin{array}{r|l}
 & 4x^4 - 3x^3 - 2x^2 + 6x - 5 \\
\hline
2x-5 & 8x^5 - 26x^4 + 11x^3 + 22x^2 - 40x + 25 \\
 & \underline{8x^5 - 20x^4} \\
 & \quad -6x^4 + 11x^3 \\
 & \quad \underline{-6x^4 + 15x^3} \\
 & \qquad\quad -4x^3 + 22x^2 \\
 & \qquad\quad \underline{-4x^3 + 10x^2} \\
 & \qquad\qquad\quad 12x^2 - 40x \\
 & \qquad\qquad\quad \underline{12x^2 - 30x} \\
 & \qquad\qquad\qquad\quad -10x + 25 \\
 & \qquad\qquad\qquad\quad \underline{-10x + 25} \\
 & \qquad\qquad\qquad\qquad\quad 0
\end{array}
$$

So $\dfrac{8x^5 - 26x^4 + 11x^3 + 22x^2 - 40x + 25}{2x-5}$

$= 4x^4 - 3x^3 - 2x^2 + 6x - 5$

4 g

$$
\begin{array}{r}
5x^3+12x^2-6x-2 \\
5x+3\overline{)25x^4+75x^3+6x^2-28x-6} \\
\underline{25x^4+15x^3} \\
60x^3+6x^2 \\
\underline{60x^3+36x^2} \\
-30x^2-28x \\
\underline{-30x^2-18x} \\
-10x-6 \\
\underline{-10x-6} \\
0
\end{array}
$$

So $\dfrac{25x^4+75x^3+6x^2-28x-6}{5x+3}$

$= 5x^3 + 12x^2 - 6x - 2$

h

$$
\begin{array}{r}
3x^4+5x^3+6 \\
7x-2\overline{)21x^5+29x^4-10x^3+42x-12} \\
\underline{21x^5-6x^4} \\
35x^4-10x^3 \\
\underline{35x^4-10x^3} \\
0+42x-12 \\
\underline{42x-12} \\
0
\end{array}
$$

So $\dfrac{21x^5+29x^4-10x^3+42x-12}{7x-2}$

$= 3x^4 + 5x^3 + 6$

5 a

$$
\begin{array}{r}
x^2-2x+5 \\
x+2\overline{)x^3+0x^2+x+10} \\
\underline{x^3+2x^2} \\
-2x^2+x \\
\underline{-2x^2-4x} \\
5x+10 \\
\underline{5x+10} \\
0
\end{array}
$$

So $\dfrac{x^3+x+10}{x+2} = x^2-2x+5$

5 b

$$
\begin{array}{r}
2x^2-6x+1 \\
x+3\overline{)2x^3+0x^2-17x+3} \\
\underline{2x^3+6x^2} \\
-6x^2-17x \\
\underline{-6x^2-18x} \\
x+3 \\
\underline{x+3} \\
0
\end{array}
$$

So $\dfrac{2x^3-17x+3}{x+3} = 2x^2-6x+1$

c

$$
\begin{array}{r}
-3x^2-12x+2 \\
x-4\overline{)-3x^3+0x^2+50x-8} \\
\underline{-3x^3+12x^2} \\
-12x^2+50x \\
\underline{-12x^2+48x} \\
2x-8 \\
\underline{2x-8} \\
0
\end{array}
$$

So $\dfrac{-3x^3+50x-8}{x-4} = -3x^2-12x+2$

6 a

$$
\begin{array}{r}
x^2+4x+12 \\
x-3\overline{)x^3+x^2+0x-36} \\
\underline{x^3-3x^2} \\
4x^2+0x \\
\underline{4x^2-12x} \\
12x-36 \\
\underline{12x-36} \\
0
\end{array}
$$

So $\dfrac{x^3+x^2-36}{x-3} = x^2+4x+12$

6 b

$$\begin{array}{r l} & 2x^2-x+5 \\ x+5 & \overline{)\,2x^3+9x^2+0x+25} \\ & \underline{2x^3+10x^2} \\ & -x^2+0x \\ & \underline{-x^2-5x} \\ & 5x+25 \\ & \underline{5x+25} \\ & 0 \end{array}$$

So $\dfrac{2x^3+9x^2+25}{x+5} = 2x^2-x+5$

c

$$\begin{array}{r l} & -3x^2+5x+10 \\ x-2 & \overline{)\,-3x^3+11x^2+0x-20} \\ & \underline{-3x^3+6x^2} \\ & 5x^2+0x \\ & \underline{5x^2-10x} \\ & 10x-20 \\ & \underline{10x-20} \\ & 0 \end{array}$$

So $\dfrac{-3x^3+11x^2-20}{x-2} = -3x^2+5x+10$

7 RHS $= (x+2)(x^2-5)$
$= x^3+2x^2-5x-10$
$=$ LHS

8 a

$$\begin{array}{r l} & x^2-x+2 \\ x+5 & \overline{)\,x^3+4x^2-3x+2} \\ & \underline{x^3+5x^2} \\ & -x^2-3x \\ & \underline{-x^2-5x} \\ & 2x+2 \\ & \underline{2x+10} \\ & -8 \end{array}$$

So the remainder is -8.

8 b

$$\begin{array}{r l} & 3x^2-2x-2 \\ x-6 & \overline{)\,3x^3-20x^2+10x+5} \\ & \underline{3x^3-18x^2} \\ & -2x^2+10x \\ & \underline{-2x^2+12x} \\ & -2x+5 \\ & \underline{-2x+12} \\ & -7 \end{array}$$

So the remainder is -7.

c

$$\begin{array}{r l} & -2x^2-5x-8 \\ x-4 & \overline{)\,-2x^3+3x^2+12x+20} \\ & \underline{-2x^3+8x^2} \\ & -5x^2+12x \\ & \underline{-5x^2+20x} \\ & -8x+20 \\ & \underline{-8x+32} \\ & -12 \end{array}$$

So the remainder is -12.

9

$$\begin{array}{r l} & 3x^2+x+1 \\ x-1 & \overline{)\,3x^3-2x^2+0x+4} \\ & \underline{3x^3-3x^2} \\ & x^2+0x \\ & \underline{x^2-x} \\ & x+4 \\ & \underline{x-1} \\ & 5 \end{array}$$

So the remainder is 5.

10

$$\begin{array}{r l} & 3x^3-11x^2+21x-24 \\ x+1 & \overline{)\,3x^4-8x^3+10x^2-3x-25} \\ & \underline{3x^4+3x^3} \\ & -11x^3+10x^2 \\ & \underline{-11x^3-11x^2} \\ & 21x^2-3x \\ & \underline{21x^2+21x} \\ & -24x-25 \\ & \underline{-24x-24} \\ & -1 \end{array}$$

10 So the remainder is −1.

11
$$
\begin{array}{r}
5x^2-20x+7 \\
x+4\overline{\big)\,5x^3+0x^2-73x+28} \\
\underline{5x^3+20x^2} \qquad\qquad\quad \\
-20x^2-73x \qquad\; \\
\underline{-20x^2-80x} \qquad\; \\
7x+28 \\
\underline{7x+28} \\
0
\end{array}
$$

The remainder is 0, so $(x + 4)$ is a factor of $5x^3 - 73x + 28$.

12
$$
\begin{array}{r}
3x^2+6x+4 \\
x-2\overline{\big)\,3x^3+0x^2-8x-8} \\
\underline{3x^3-6x^2} \qquad\qquad \\
6x^2-8x \qquad \\
\underline{6x^2-12x} \qquad \\
4x-8 \\
\underline{4x-8} \\
0
\end{array}
$$

So $\dfrac{3x^3-8x-8}{x-2} = 3x^2+6x+4$

13
$$
\begin{array}{r}
x^2+x+1 \\
x-1\overline{\big)\,x^3+0x^2+0x-1} \\
\underline{x^3-x^2} \qquad\qquad \\
x^2+0x \qquad \\
\underline{x^2-x} \qquad \\
x-1 \\
\underline{x-1} \\
0
\end{array}
$$

So $\dfrac{x^3-1}{x-1} = x^2+x+1$

14
$$
\begin{array}{r}
x^3-2x^2+4x-8 \\
x+2\overline{\big)\,x^4+0x^3+0x^2+0x-16} \\
\underline{x^4+2x^3} \qquad\qquad\qquad\quad \\
-2x^3+0x^2 \qquad\qquad \\
\underline{-2x^3-4x^2} \qquad\qquad \\
4x^2+0x \qquad \\
\underline{4x^2+8x} \qquad \\
-8x-16 \\
\underline{-8x-16} \\
0
\end{array}
$$

So $\dfrac{x^4-16}{x+2} = x^3-2x^2+4x-8$

15
$$
\begin{array}{r}
2x^2+7x-6 \\
5x+4\overline{\big)\,10x^3+43x^2-2x-10} \\
\underline{10x^3+8x^2} \qquad\qquad \\
35x^2-2x \qquad \\
\underline{35x^2+28x} \qquad \\
-30x-10 \\
\underline{-30x-24} \\
14
\end{array}
$$

So the remainder is 14.

16 a
$$
\begin{array}{r}
3x^2-5x-62 \\
x-3\overline{\big)\,3x^3-14x^2-47x-14} \\
\underline{3x^3-9x^2} \qquad\qquad\quad \\
-5x^2-47x \qquad \\
\underline{-5x^2+15x} \qquad \\
-62x-14 \\
\underline{-62x+186} \\
-200
\end{array}
$$

So the remainder is −200.

16 b

$$
\begin{array}{rl}
 & 3x^2-20x-7 \\
x+2 & \overline{)\,3x^3-14x^2-47x-14} \\
 & \underline{3x^3+6x^2}\qquad\qquad\quad \\
 & -20x^2-47x\qquad\;\; \\
 & \underline{-20x^2-40x}\qquad\;\; \\
 & -7x-14 \\
 & \underline{-7x-14} \\
 & 0
\end{array}
$$

$\mathrm{f}(x) = 3x^3 - 14x^2 - 47x - 14$
$= (x + 2)(3x^2 - 20x - 7)$
$= (x + 2)(3x + 1)(x - 7)$

17 a i

$$
\begin{array}{rl}
 & x^2+8x+21 \\
x-2 & \overline{)\,x^3+6x^2+5x-12} \\
 & \underline{x^3-2x^2}\qquad\qquad\quad \\
 & 8x^2+5x\qquad\;\; \\
 & \underline{8x^2-16x}\qquad\;\; \\
 & 21x-12 \\
 & \underline{21x-42} \\
 & 30
\end{array}
$$

So the remainder is 30.

ii

$$
\begin{array}{rl}
 & x^2+3x-4 \\
x+3 & \overline{)\,x^3+6x^2+5x-12} \\
 & \underline{x^3+3x^2}\qquad\qquad\quad \\
 & 3x^2+5x\qquad\;\; \\
 & \underline{3x^2+9x}\qquad\;\; \\
 & -4x-12 \\
 & \underline{-4x-12} \\
 & 0
\end{array}
$$

So the remainder is 0.

b $\mathrm{f}(x) = x^3 + 6x^2 + 5x - 12$
$= (x + 3)(x^2 + 3x - 4)$
$= (x + 3)(x + 4)(x - 1)$
So $x = -3$, $x = -4$ or $x = 1$

18 a

$$
\begin{array}{rl}
 & x^2+2x-3 \\
2x-1 & \overline{)\,2x^3+3x^2-8x+3} \\
 & \underline{2x^3-x^2}\qquad\qquad\quad \\
 & 4x^2-8x\qquad\;\; \\
 & \underline{4x^2-2x}\qquad\;\; \\
 & -6x+3 \\
 & \underline{-6x+3} \\
 & 0
\end{array}
$$

$\mathrm{f}(x) = 2x^3 + 3x^2 - 8x + 3$
$= (2x - 1)(x^2 + 2x - 3)$
$a = 1$, $b = 2$, $c = -3$

b $\mathrm{f}(x) = 2x^3 + 3x^2 - 8x + 3$
$= (2x - 1)(x^2 + 2x - 3)$
$= (2x - 1)(x - 1)(x + 3)$

c $(2x - 1)(x - 1)(x + 3) = 0$
$x = \frac{1}{2}$, $x = 1$ and $x = -3$

19 a

$$
\begin{array}{rl}
 & 3x^2+2x+1 \\
4x-1 & \overline{)\,12x^3+5x^2+2x-1} \\
 & \underline{12x^3-3x^2}\qquad\qquad\quad \\
 & 8x^2+2x\qquad\;\; \\
 & \underline{8x^2-2x}\qquad\;\; \\
 & 4x-1 \\
 & \underline{4x-1} \\
 & 0
\end{array}
$$

$\mathrm{f}(x) = (4x - 1)(3x^2 + 2x + 1)$
$a = 3$, $b = 2$, $c = 1$

b $(4x - 1)(3x^2 + 2x + 1) = 0$
Using the discriminant for $3x^2 + 2x + 1$:
$b^2 - 4ac = 2^2 - 4(3)(1) = -8 < 0$ so there are no real solutions.
So $\mathrm{f}(x)$ has only one real solution, $x = \frac{1}{4}$.

Algebraic methods 7C

1 a $f(x) = 4x^3 - 3x^2 - 1$
$f(1) = 4(1)^3 - 3(1)^2 - 1$
$= 4 - 3 - 1$
$= 0$
So $(x - 1)$ is a factor of $4x^3 - 3x^2 - 1$.

b $f(x) = 5x^4 - 45x^2 - 6x - 18$
$f(-3) = 5(-3)^4 - 45(-3)^2 - 6(-3) - 18$
$= 5(81) - 45(9) + 18 - 18$
$= 405 - 405$
$= 0$
So $(x + 3)$ is a factor of $5x^4 - 45x^2 - 6x - 18$.

c $f(x) = -3x^3 + 13x^2 - 6x + 8$
$f(4) = -3(4)^3 + 13(4)^2 - 6(4) + 8$
$= -192 + 208 - 24 + 8$
$= 0$
So $(x - 4)$ is a factor of $-3x^3 + 13x^2 - 6x + 8$.

2 $f(x) = x^3 + 6x^2 + 5x - 12$
$f(-1) = (1)^3 + 6(1)^2 + 5(1) - 12$
$= 1 + 6 + 5 - 12$
$= 0$
So $(x - 1)$ is a factor of $x^3 + 6x^2 + 5x - 12$.

$$
\begin{array}{r l}
 & x^2 + 7x + 12 \\
x - 1 \,\big) & x^3 + 6x^2 + 5x - 12 \\
 & \underline{x^3 - x^2} \\
 & \quad 7x^2 + 5x \\
 & \quad \underline{7x^2 - 7x} \\
 & \qquad 12x - 12 \\
 & \qquad \underline{12x - 12} \\
 & \qquad\qquad 0
\end{array}
$$

$x^3 + 6x^2 + 5x - 12 = (x - 1)(x^2 + 7x + 12)$
$= (x - 1)(x + 3)(x + 4)$

3 $f(x) = x^3 + 3x^2 - 33x - 35$
$f(-1) = (-1)^3 + 3(-1)^2 - 33(-1) - 35$
$= -1 + 3 + 33 - 35$
$= 0$
So $(x + 1)$ is a factor of $x^3 + 3x^2 - 33x - 35$.

3

$$
\begin{array}{r l}
 & x^2 + 2x - 35 \\
x + 1 \,\big) & x^3 + 3x^2 - 33x - 35 \\
 & \underline{x^3 + x^2} \\
 & \quad 2x^2 - 33x \\
 & \quad \underline{2x^2 + 2x} \\
 & \qquad -35x - 35 \\
 & \qquad \underline{-35x - 35} \\
 & \qquad\qquad 0
\end{array}
$$

$x^3 + 3x^2 - 33x - 35 = (x + 1)(x^2 + 2x - 35)$
$= (x + 1)(x + 7)(x - 5)$

4 $f(x) = x^3 + 7x^2 + 2x + 40$
$f(5) = (5)^3 + 7(5)^2 + 2(5) + 40$
$= 125 - 175 + 10 + 40$
$= 0$
So $(x - 5)$ is a factor of $x^3 + 7x^2 + 2x + 40$.

$$
\begin{array}{r l}
 & x^2 - 2x - 8 \\
x - 5 \,\big) & x^3 - 7x^2 + 2x + 40 \\
 & \underline{x^3 - 5x^2} \\
 & \quad -2x^2 + 2x \\
 & \quad \underline{-2x^2 + 10x} \\
 & \qquad -8x + 40 \\
 & \qquad \underline{-8x + 40} \\
 & \qquad\qquad 0
\end{array}
$$

$x^3 - 7x^2 + 2x + 40 = (x - 5)(x^2 - 2x - 8)$
$= (x - 5)(x - 4)(x + 2)$

5 $f(x) = 2x^3 + 3x^2 - 18x + 8$
$f(2) = 2(2)^3 + 3(2)^2 - 18(2) + 8$
$= 16 + 12 - 36 + 8$
$= 0$
So $(x - 2)$ is a factor of $2x^3 + 3x^2 - 18x + 8$.

5

$$\begin{array}{r}
2x^2+7x-4 \\
x-2\overline{)\,2x^3+3x^2-18x+8} \\
\underline{2x^3-4x^2}\qquad\qquad \\
7x^2-18x\quad \\
\underline{7x^2-14x}\quad \\
-4x+8 \\
\underline{-4x+8} \\
0
\end{array}$$

$2x^3 + 3x^2 - 18x + 8$
$= (x - 2)(2x^2 + 7x - 4)$
$= (x - 2)(2x - 1)(x + 4)$

6 a $f(x) = x^3 - 10x^2 + 19x + 30$
$f(-1) = (-1)^3 - 10(-1)^2 + 19(-1) + 30$
$= -1 - 10 - 19 + 30$
So $(x + 1)$ is a factor of
$x^3 - 10x^2 + 19x + 30$.

$$\begin{array}{r}
x^2-11x+30 \\
x+1\overline{)\,x^3-10x^2+19x+30} \\
\underline{x^3+x^2}\qquad\qquad\qquad \\
-11x^2+19x\quad \\
\underline{-11x^2-11x}\quad \\
30x+30 \\
\underline{30x+30} \\
0
\end{array}$$

$x^3 - 10x^2 + 19x + 30$
$= (x + 1)(x^2 - 11x + 30)$
$= (x + 1)(x - 5)(x - 6)$

b $f(x) = x^3 + x^2 - 4x - 4$
$f(-1) = (-1)^3 + (-1)^2 - 4(-1) - 4$
$= -1 + 1 + 4 - 4$
$= 0$
So $(x + 1)$ is a factor of $x^3 + x^2 - 4x - 4$.

$$\begin{array}{r}
x^2-4 \\
x+1\overline{)\,x^3+x^2-4x-4} \\
\underline{x^3+x^2}\qquad\quad \\
0\;\;-4x-4 \\
\underline{-4x-4} \\
0
\end{array}$$

$x^3 + x^2 - 4x - 4 = (x + 1)(x^2 - 4)$
$= (x + 1)(x - 2)(x + 2)$

6 c $f(x) = x^3 - 4x^2 - 11x + 30$
$f(2) = (2)^3 - 4(2)^2 - 11(2) + 30$
$= 8 - 16 - 22 + 30$
$= 0$
So $(x - 2)$ is a factor of
$x^3 - 4x^2 - 11x + 30$.

$$\begin{array}{r}
x^2-2x-15 \\
x-2\overline{)\,x^3-4x^2-11x+30} \\
\underline{x^3-2x^2}\qquad\qquad \\
-2x^2-11x\quad \\
\underline{-2x^2+4x}\quad \\
-15x+30 \\
\underline{-15x+30} \\
0
\end{array}$$

$x^3 - 4x^2 - 11x + 30 = (x - 2)(x^2 - 2x - 15)$
$= (x - 2)(x + 3)(x - 5)$

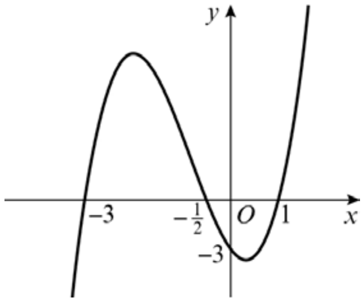
7 a i $f(x) = 2x^3 + 5x^2 - 4x - 3$
$f(1) = 2(1)^3 + 5(1)^2 - 4(1) - 3$
$= 2 + 5 - 4 - 3$
$= 0$
So $(x - 1)$ is a factor of
$2x^3 + 5x^2 - 4x - 3$.

$$\begin{array}{r}
2x^2+7x+3 \\
x-1\overline{)\,2x^3+5x^2-4x-3} \\
\underline{2x^3-2x^2}\qquad\quad \\
7x^2-4x\quad \\
\underline{7x^2-7x}\quad \\
3x-3 \\
\underline{3x-3} \\
0
\end{array}$$

$y = 2x^3 + 5x^2 - 4x - 3$
$= (x - 1)(2x^2 + 7x + 3)$
$= (x - 1)(2x + 1)(x + 3)$

ii $0 = (x - 1)(2x + 1)(x + 3)$
So the curve crosses the x-axis at $(1, 0)$, $(-\frac{1}{2}, 0)$ and $(-3, 0)$.
When $x = 0$, $y = (-1)(1)(3) = -3$
The curve crosses the y-axis at $(0, -3)$.
$x \to \infty, y \to \infty$
$x \to -\infty, y \to -\infty$

7 a ii

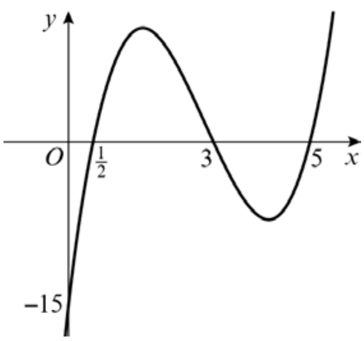


b i $f(x) = 2x^3 - 17x^2 + 38x - 15$
$f(3) = 2(3)^3 - 17(3)^2 + 38(3) - 15$
$= 54 - 153 + 114 - 15$
$= 0$
So $(x - 3)$ is a factor of
$2x^3 - 17x^2 + 38x - 15$.

$$\begin{array}{r l}
 & 2x^2 - 11x + 5 \\
x-3 \big) & 2x^3 - 17x^2 + 38x - 15 \\
 & \underline{2x^3 - 6x^2} \\
 & \quad -11x^2 + 38x \\
 & \quad \underline{-11x^2 + 33x} \\
 & \qquad\qquad 5x - 15 \\
 & \qquad\qquad \underline{5x - 15} \\
 & \qquad\qquad\qquad 0
\end{array}$$

$y = 2x^3 - 17x^2 + 38x - 15$
$= (x - 3)(2x^2 - 11x + 5)$
$= (x - 3)(2x - 1)(x - 5)$

ii $0 = (x - 3)(2x - 1)(x - 5)$
So the curve crosses the x-axis at $(3, 0)$, $(\frac{1}{2}, 0)$ and $(5, 0)$.
When $x = 0$, $y = (-3)(-1)(-5) = -15$
The curve crosses the y-axis at $(0, -15)$.
$x \to \infty, y \to \infty$
$x \to -\infty, y \to -\infty$

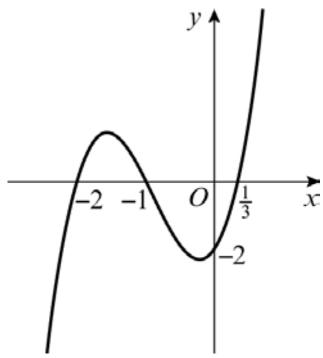


c i $f(x) = 3x^3 + 8x^2 + 3x - 2$
$f(-1) = 3(-1)^3 + 8(-1)^2 + 3(-1) - 2$
$= -3 + 8 - 3 - 2$
$= 0$
So $(x + 1)$ is a factor of
$3x^3 + 8x^2 + 3x - 2$.

$$\begin{array}{r l}
 & 3x^2 + 5x - 2 \\
x+1 \big) & 3x^3 + 8x^2 + 3x - 2 \\
 & \underline{3x^3 + 3x^2} \\
 & \qquad 5x^2 + 3x \\
 & \qquad \underline{5x^2 + 5x} \\
 & \qquad\qquad -2x - 2 \\
 & \qquad\qquad \underline{-2x - 2} \\
 & \qquad\qquad\qquad 0
\end{array}$$

$y = 3x^3 + 8x^2 + 3x - 2$
$= (x + 1)(3x^2 + 5x - 2)$
$= (x + 1)(3x - 1)(x + 2)$

ii $0 = (x + 1)(3x - 1)(x + 2)$
So the curve crosses the x-axis at
$(-1, 0)$, $(\frac{1}{3}, 0)$ and $(-2, 0)$.
When $x = 0$, $y = (1)(-1)(2) = -2$
The curve crosses the y-axis at $(0, -2)$.
$x \to \infty, y \to \infty$
$x \to -\infty, y \to -\infty$



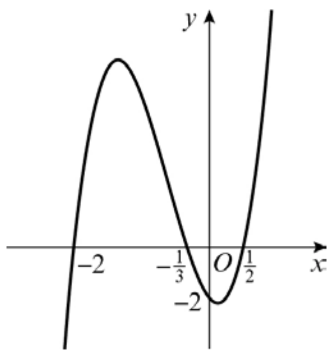
d i $f(x) = 6x^3 + 11x^2 - 3x - 2$
$f(-2) = 6(-2)^3 + 11(-2)^2 - 3(-2) - 2$
$= -48 + 44 + 6 - 2$
$= 0$
So $(x + 2)$ is a factor of
$6x^3 + 11x^2 - 3x - 2$.

7 d i

$$
\begin{array}{r|l}
 & 6x^2 - x - 1 \\
\hline
x+2 & 6x^3 + 11x^2 - 3x - 2 \\
 & \underline{6x^3 + 12x^2} \\
 & \quad -x^2 - 3x \\
 & \quad \underline{-x^2 - 2x} \\
 & \qquad -x - 2 \\
 & \qquad \underline{-x - 2} \\
 & \qquad\quad 0
\end{array}
$$

$y = 6x^3 + 11x^2 - 3x - 2$
$= (x + 2)(6x^2 - x - 1)$
$= (x + 2)(3x + 1)(2x - 1)$

ii $0 = (x + 2)(3x + 1)(2x - 1)$
So the curve crosses the x-axis at $(-2, 0)$, $(-\frac{1}{3}, 0)$ and $(\frac{1}{2}, 0)$.
When $x = 0$, $y = (2)(1)(-1) = -2$
The curve crosses the y-axis at $(0, -2)$.
$x \to \infty, y \to \infty$
$x \to -\infty, y \to -\infty$

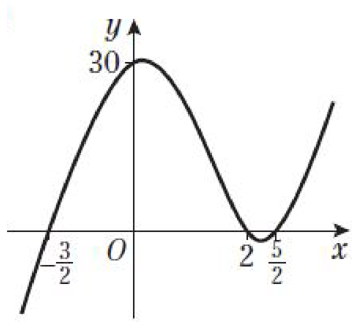


e i $f(x) = 4x^3 - 12x^2 - 7x + 30$
$f(2) = 4(2)^3 - 12(2)^2 - 7(2) + 30$
$= 32 - 48 - 14 + 30$
$= 0$
So $(x - 2)$ is a factor of $4x^3 - 12x^2 - 7x + 30$.

$$
\begin{array}{r|l}
 & 4x^2 - 4x - 15 \\
\hline
x-2 & 4x^3 - 12x^2 - 7x + 30 \\
 & \underline{4x^3 - 8x^2} \\
 & \quad -4x^2 - 7x \\
 & \quad \underline{-4x^2 + 8x} \\
 & \qquad -15x + 30 \\
 & \qquad \underline{-15x + 30} \\
 & \qquad\quad 0
\end{array}
$$

e i $y = 4x^3 - 12x^2 - 7x + 30$
$= (x - 2)(4x^2 - 4x - 15)$
$= (x - 2)(2x + 3)(2x - 5)$

ii $0 = (x - 2)(2x + 3)(2x - 5)$
So the curve crosses the x-axis at $(2, 0)$, $(-\frac{3}{2}, 0)$ and $(\frac{5}{2}, 0)$.
When $x = 0$, $y = (-2)(3)(-5) = 30$
The curve crosses the y-axis at $(0, 30)$.
$x \to \infty, y \to \infty$
$x \to -\infty, y \to -\infty$



8 $f(x) = 5x^3 - 9x^2 + 2x + a$
$f(1) = 0$
$5(1)^3 - 9(1)^2 + 2(1) + a = 0$
$5 - 9 + 2 + a = 0$
$a = 2$

9 $f(x) = 6x^3 - bx^2 + 18$
$f(-3) = 0$
$6(-3)^3 - b(-3)^2 + 18 = 0$
$-162 - 9b + 18 = 0$
$9b = -144$
$b = -16$

10 $f(x) = px^3 + qx^2 - 3x - 7$
$f(1) = 0$
$p(1)^3 + q(1)^2 - 3(1) - 7 = 0$
$p + q - 3 - 7 = 0$
$p + q = 10$ **(1)**

$f(-1) = 0$
$p(-1)^3 + q(-1)^2 - 3(-1) - 7 = 0$
$-p + q + 3 - 7 = 0$
$-p + q = 4$ **(2)**

(1) + **(2)**:
$2q = 14$
$q = 7$
Substituting in **(1)**:
$p + 7 = 10$
$p = 3$
So $p = 3$, $q = 7$

11 $f(x) = cx^3 + dx^2 - 9x - 10$
$f(-1) = 0$
$c(-1)^3 + d(-1)^2 - 9(-1) - 10 = 0$
$-c + d + 9 - 10 = 0$
$d = c + 1 \quad$ (**1**)

$f(2) = 0$
$c(2)^3 + d(2)^2 - 9(2) - 10 = 0$
$8c + 4d - 18 - 10 = 0$
$8c + 4d - 28 = 0$
$8c + 4d = 28 \quad$ (**2**)

Substituting (**1**) in (**2**):
$8c + 4(c + 1) = 28$
$12c + 4 = 28$
$c = 2$
Substituting in (**1**):
$d = 2 + 1 = 3$
So $c = 2$, $d = 3$

12 $f(x) = gx^3 + hx^2 - 14x + 24$
$f(-2) = 0$
$g(-2)^3 + h(-2)^2 - 14(-2) + 24 = 0$
$-8g + 4h + 28 + 24 = 0$
$-8g + 4h + 52 = 0$
$h = 2g - 13 \quad$ (**1**)

$f(3) = 0$
$g(3)^3 + h(3)^2 - 14(3) + 24 = 0$
$27g + 9h - 42 + 24 = 0$
$27g + 9h = 18 \quad$ (**2**)

Substituting (**1**) in (**2**):
$27g + 9(2g - 13) = 18$
$45g = 135$
$g = 3$
Substituting in (**1**):
$h = 2(3) - 13 = -7$
So $g = 3$, $h = -7$

13 a $f(x) = 3x^3 - 12x^2 + 6x - 24$
$f(4) = 3(4)^3 - 12(4)^2 + 6(4) - 24$
$= 192 - 192 + 24 - 24$
$= 0$
So $(x - 4)$ is a factor of $f(x)$.

b
$$\begin{array}{r|l} & 3x^2 + 6 \\ \hline x-4 & 3x^3 - 12x^2 + 6x - 24 \\ & \underline{3x^3 - 12x^2} \\ & 0 + 6x - 24 \\ & \underline{6x - 24} \\ & 0 \end{array}$$

13 b $f(x) = (x - 4)(3x^2 + 6)$
$(x - 4)(3x^2 + 6) = 0$
Using the discriminant for $3x^2 + 6$:
$b^2 - 4ac = 0 - 4(3)(6) = -72 < 0$.
Therefore $3x^2 + 6$ has no real roots, so $f(x)$ only has one real root of $x = 4$.

14 a $f(x) = 4x^3 + 4x^2 - 11x - 6$
$f(-2) = 4(-2)^3 + 4(-2)^2 - 11(-2) - 6$
$= -32 + 16 + 22 - 6$
$= 0$
So $(x + 2)$ is a factor of $f(x)$.

b
$$\begin{array}{r|l} & 4x^2 - 4x - 3 \\ \hline x+2 & 4x^3 + 4x^2 - 11x - 6 \\ & \underline{4x^3 + 8x^2} \\ & -4x^2 - 11x \\ & \underline{-4x^2 - 8x} \\ & -3x - 6 \\ & \underline{-3x - 6} \\ & 0 \end{array}$$

$f(x) = (x + 2)(4x^2 - 4x - 3)$
$= (x + 2)(2x - 3)(2x + 1)$

c $0 = (x + 2)(2x - 3)(2x + 1)$
The solutions are $x = -2$, $x = \frac{3}{2}$ and $x = -\frac{1}{2}$.

15 a $f(x) = 9x^4 - 18x^3 - x^2 + 2x$
$f(2) = 9(2)^4 - 18(2)^3 - (2)^2 + 2(2)$
$= 144 - 144 - 4 + 4$
$= 0$
So $(x - 2)$ is a factor of $9x^4 - 18x^3 - x^2 + 2x$.

b
$$\begin{array}{r|l} & 9x^3 - x \\ \hline x-2 & 9x^4 - 18x^3 - x^2 + 2x \\ & \underline{9x^4 - 18x^3} \\ & 0 - x^2 + 2x \\ & \underline{-x^2 + 2x} \\ & 0 \end{array}$$

$9x^4 - 18x^3 - x^2 + 2x$
$= (x - 2)(9x^3 - x)$
$= x(x - 2)(9x^2 - 1)$
$= x(x - 2)(3x + 1)(3x - 1)$
$0 = x(x - 2)(3x + 1)(3x - 1)$

15 b The solutions are $x = 0$, $x = 2$, $x = -\frac{1}{3}$ and $x = \frac{1}{3}$.

Challenge

a $f(x) = 2x^4 - 5x^3 - 42x^2 - 9x + 54$
$f(1) = 2(1)^4 - 5(1)^3 - 42(1)^2 - 9(1) + 54$
$= 0$
$f(-3) = 2(-3)^4 - 5(-3)^3 - 42(-3)^2 - 9(-3) + 54$
$= 162 + 135 - 378 + 27 + 54$
$= 0$

b
$$\begin{array}{r}
2x^3 - 3x^2 - 45x - 54 \\
x-1\,\big)\,\overline{2x^4 - 5x^3 - 42x^2 - 9x + 54} \\
\underline{2x^4 - 2x^3} \\
-3x^3 - 42x^2 \\
\underline{-3x^3 + 3x^2} \\
-45x^2 - 9x \\
\underline{-45x^2 + 45x} \\
-54x + 54 \\
\underline{-54x + 54} \\
0
\end{array}$$

$$\begin{array}{r}
2x^2 - 9x - 18 \\
x+3\,\big)\,\overline{2x^3 - 3x^2 - 45x - 54} \\
\underline{2x^3 + 6x^2} \\
-9x^2 - 45x \\
\underline{-9x^2 - 27x} \\
-18x - 54 \\
\underline{-18x - 54} \\
0
\end{array}$$

$f(x) = (x - 1)(x + 3)(2x^2 - 9x - 18)$
$= (x - 1)(x + 3)(2x + 3)(x - 6)$
$0 = (x - 1)(x + 3)(2x + 3)(x - 6)$
The solutions are $x = 1$, $x = -3$, $x = -\frac{3}{2}$ and $x = 6$.

Algebraic methods 7D

1 $n^2 - n = n(n - 1)$
If n is even, $n - 1$ is odd
and even × odd = even
If n is odd, $n - 1$ is even
and odd × even = even
So $n^2 - n$ is even for all values of n.

2 LHS $= \dfrac{x}{(1+\sqrt{2})} \times \dfrac{(1-\sqrt{2})}{(1-\sqrt{2})}$

$= \dfrac{x(1-\sqrt{2})}{(1-2)}$

$= \dfrac{x - x\sqrt{2}}{-1}$

$= x\sqrt{2} - x$

= RHS

So $\dfrac{x}{(1+\sqrt{2})} \equiv x\sqrt{2} - x$

3 LHS $= (x+\sqrt{y})(x-\sqrt{y})$

$= x^2 - x\sqrt{y} + x\sqrt{y} - y$

$= x^2 - y$

= RHS

So $(x+\sqrt{y})(x-\sqrt{y}) \equiv x^2 - y$

4 LHS $= (2x - 1)(x + 6)(x - 5)$
$= (2x - 1)(x^2 + x - 30)$
$= 2x^3 + x^2 - 61x + 30$
= RHS
So $(2x - 1)(x + 6)(x - 5) \equiv 2x^3 + x^2 - 61x + 30$

5 Completing the square:

$x^2 + bx = \left(x + \dfrac{b}{2}\right)^2 - \left(\dfrac{b}{2}\right)^2$

So $x^2 + bx \equiv \left(x + \dfrac{b}{2}\right)^2 - \left(\dfrac{b}{2}\right)^2$

6 $x^2 + 2bx + c = 0$
Completing the square:
$(x + b)^2 - b^2 + c = 0$
$(x + b)^2 = b^2 - c$
$x + b = \pm\sqrt{b^2 - c}$
$x = -b \pm \sqrt{b^2 - c}$
So the solutions of $x^2 + 2bx + c = 0$ are $x = -b \pm \sqrt{b^2 - c}$.

7 LHS $= \left(x - \dfrac{2}{x}\right)^3$

$= \left(x - \dfrac{2}{x}\right)\left(x^2 - 4 + \dfrac{4}{x^2}\right)$

$= x^3 - 6x + \dfrac{12}{x} - \dfrac{8}{x^3}$

= RHS

So $\left(x - \dfrac{2}{x}\right)^3 \equiv x^3 - 6x + \dfrac{12}{x} - \dfrac{8}{x^3}$

8 LHS $= \left(x^3 - \dfrac{1}{x}\right)\left(x^{\frac{3}{2}} + x^{-\frac{5}{2}}\right)$

$= x^{\frac{9}{2}} + x^{\frac{1}{2}} - x^{\frac{1}{2}} - x^{-\frac{7}{2}}$

$= x^{\frac{9}{2}} - x^{-\frac{7}{2}}$

$= x^{\frac{1}{2}}\left(x^4 - \dfrac{1}{x^4}\right)$

= RHS

So $\left(x^3 - \dfrac{1}{x}\right)\left(x^{\frac{3}{2}} + x^{-\frac{5}{2}}\right) \equiv x^{\frac{1}{2}}\left(x^4 - \dfrac{1}{x^4}\right)$

9 $3n^2 - 4n + 10 = 3\left(n^2 - \dfrac{4}{3}n + \dfrac{10}{3}\right)$

$= 3\left(\left(n - \dfrac{2}{3}\right)^2 - \dfrac{4}{9} + \dfrac{10}{3}\right)$

$= 3\left(n - \dfrac{2}{3}\right)^2 + \dfrac{26}{3}$

The minimum value is $\frac{26}{3}$ so
$3n^2 - 4n + 10$ is always positive.

10 $-n^2 - 2n - 3 = -(n^2 + 2n + 3)$
$= -((n + 1)^2 - 1 + 3)$
$= -(n + 1)^2 - 2$
The maximum value is -2,
so $-n^2 - 2n - 3$ is always negative.

11 $x^2 + 8x + 20$
Complete the square
$(x + 4)^2 - 16 + 20 = (x + 4)^2 + 4$

11 The minimum value of $(x + 4)^2 + 4$ is 4
So $(x + 4)^2 + 4 \geq 4$
Therefore, $x^2 + 8x + 20 \geq 4$

12 $kx^2 + 5kx + 3 = 0$ has no real roots,
so $b^2 - 4ac < 0$
$(5k)^2 - 4k(3) < 0$
$25k^2 - 12k < 0$
$k(25k - 12) < 0$
$0 < k < \frac{12}{25}$
When $k = 0$:
$(0)x^2 + 5(0)x + 3 = 0$
$3 = 0$
which is impossible, so no real roots.
So combining these:
$0 \leq k < \frac{12}{25}$

13 $px^2 - 5x - 6 = 0$ has two distinct real roots, so
$b^2 - 4ac > 0$
$25 + 24p > 0$
$p > -\frac{25}{24}$

14 $A(1, 2)$, $B(1, 2)$ and $C(2, 4)$
The gradient of line $AB = \frac{2-1}{1-3} = -\frac{1}{2}$

The gradient of line $BC = \frac{4-2}{2-1} = 2$

The gradient of line $AC = \frac{4-1}{2-3} = -3$

The gradients are different so the three points are not collinear.
Hence ABC is a triangle.
Gradient of AB × gradient of BC
$= -\frac{1}{2} \times 2$
$= -1$
So AB is perpendicular to BC,
and the triangle is a right-angled triangle.

15 $A(1, 1)$, $B(2, 4)$, $C(6, 5)$ and $D(5, 2)$
The gradient of line $AB = \frac{4-1}{2-1} = 3$

The gradient of line $BC = \frac{5-4}{6-2} = \frac{1}{4}$

The gradient of line $CD = \frac{2-5}{5-6} = 3$

The gradient of line $AD = \frac{2-1}{5-1} = \frac{1}{4}$

15 Gradient of AB = gradient of CD, so AB and CD are parallel.
Gradient of BC = gradient of AD, so BC and AD are parallel.

So $ABCD$ can be a parallelogram or a rectangle and we need to check further.
Since there is not a pair of gradients which multiply to give −1 there is no right angle.
Hence $ABCD$ is a parallelogram.

16 $A(2, 1)$, $B(5, 2)$, $C(4, -1)$ and $D(1,-2)$

The gradient of line $AB = \frac{2-1}{5-2} = \frac{1}{3}$

The gradient of line $BC = \frac{-1-2}{4-5} = 3$

The gradient of line $CD = \frac{-2+1}{1-4} = \frac{1}{3}$

The gradient of line $AD = \frac{-2-1}{1-2} = 3$

Gradient of AB = gradient of CD,
so AB and CD are parallel.
Gradient of BC = gradient of AD,
so BC and AD are parallel.

Distance $AB = \sqrt{(5-2)^2 + (2-1)^2}$
$= \sqrt{10}$

Distance $BC = \sqrt{(4-5)^2 + (-1-2)^2}$
$= \sqrt{10}$

Distance $CD = \sqrt{(1-4)^2 + (-2+1)^2}$
$= \sqrt{10}$

Distance $AD = \sqrt{(1-2)^2 + (-2-1)^2}$
$= \sqrt{10}$

All four sides are equal. Since no pairs of gradients multiply to give −1 there are no right angles at a vertex so this is not a square. Hence $ABCD$ is a rhombus.

17 $A(-5,2)$, $B(-3,-4)$ and $C(3,-2)$

The gradient of line $AB = \frac{-4-2}{-3+5} = -3$

17 The gradient of line $BC = \frac{-2+4}{3+3} = \frac{1}{3}$

The gradient of line $AC = \frac{-2-2}{3+5} = -\frac{1}{2}$

The gradients are different so the three points are not collinear. Hence ABC is a triangle.

Gradient of AB × gradient of BC
$= -3 \times \frac{1}{3}$
$= -1$
So AB is perpendicular to BC.

Distance $AB = \sqrt{(-3+5)^2 + (-4-2)^2}$
$= \sqrt{40}$

Distance $BC = \sqrt{(3+3)^2 + (-2+4)^2}$
$= \sqrt{40}$
$AB = BC$

As two sides are equal and an angle is right-angled, ABC is an isosceles right-angled triangle.

18 Substituting $y = ax$ into $(x-1)^2 + y^2 = k$:

$$(x-1)^2 + a^2x^2 = k$$
$$x^2 - 2x + 1 + a^2x^2 - k = 0$$
$$x^2(1 + a^2) - 2x + 1 - k = 0$$

The straight line cuts the circle at two distinct points, so this equation has two distinct real roots, so

$$b^2 - 4ac > 0$$
$$(-2)^2 - 4(1 + a^2)(1 - k) > 0$$
$$4 - 4(1 - k + a^2 - ka^2) > 0$$
$$4k - 4a^2 + 4ka^2 > 0$$
$$-a^2 + k + ka^2 > 0$$
$$-a^2 + k(1 + a^2) > 0$$
$$k > \frac{a^2}{1+a^2}$$

19 $4y - 3x + 26 = 0$

$$4y = 3x - 26$$
$$y = \frac{3}{4}x - \frac{13}{2}$$

Substituting $y = \frac{3}{4}x - \frac{13}{2}$ into $(x + 4)^2 + (y - 3)^2 = 100$:

$$(x + 4)^2 + \left(\frac{3}{4}x - \frac{19}{2}\right)^2 = 100$$
$$x^2 + 8x + 16 + \frac{9}{16}x^2 - \frac{57}{4}x + \frac{361}{4} - 100 = 0$$
$$16x^2 + 128x + 256 + 9x^2 - 228x + 1444 - 1600 = 0$$
$$25x^2 - 100x + 100 = 0$$
$$x^2 - 4x + 4 = 0$$
$$(x - 2)^2 = 0$$
$$x = 2$$

There is only one solution so the line $4y - 3x + 26 = 0$ only touches the circle in one place, so it is a tangent to the circle.

20 Area of square $= (a + b)^2 = a^2 + 2ab + b^2$

Shaded area $= 4\left(\frac{1}{2}ab\right) = 2ab$

Area of smaller square
$= a^2 + 2ab + b^2 - 2ab$
$= a^2 + b^2$
$= c^2$

Challenge

1 Find the equations of the perpendicular bisectors to the chords AB and BC:
$A(7, 8)$ and $B(-1, 8)$

Midpoint $= \left(\frac{7-1}{2}, \frac{8+8}{2}\right) = (3, 8)$

The gradient of the line segment AB
$= \frac{8-8}{-1-7}$
$= 0$

So the line perpendicular to AB is a vertical line $x = 3$.

$B(-1, 8)$ and $C(6, 1)$

Midpoint $= \left(\frac{-1+6}{2}, \frac{8+1}{2}\right)$
$= \left(\frac{5}{2}, \frac{9}{2}\right)$

The gradient of the line segment BC
$= \frac{1-8}{6+1}$
$= -1$

Challenge

1 So the gradient of the line perpendicular to BC is 1.

The equation of the perpendicular line is
$y - y_1 = m(x - x_1)$
$m = 1$ and $(x_1, y_1) = \left(\frac{5}{2}, \frac{9}{2}\right)$

$$\text{So } y - \frac{9}{2} = x - \frac{5}{2}$$
$$y = x + 2$$

AB and BC intersect at the centre of the circle, so solving $x = 3$ and $y = x + 2$ simultaneously:
$x = 3$, $y = 5$
Centre of the circle, X, is (3, 5).

$$\text{Distance } AX = \sqrt{(7-3)^2 + (8-5)^2}$$
$$= \sqrt{25}$$
$$= 5$$

$$\text{Distance } BX = \sqrt{(-1-3)^2 + (8-5)^2}$$
$$= \sqrt{25}$$
$$= 5$$

$$\text{Distance } CX = \sqrt{(6-3)^2 + (1-5)^2}$$
$$= \sqrt{25}$$
$$= 5$$

$$\text{Distance } DX = \sqrt{(0-3)^2 + (9-5)^2}$$
$$= \sqrt{25}$$
$$= 5$$

The distance from the centre of the circle to all four points is 5 units, so all four points lie on a circle with centre (3, 5).

2 $3 = 2^2 - 1^2$
$5 = 3^2 - 2^2$
$7 = 4^2 - 3^2$
$11 = 6^2 - 5^2$

Let p be a prime number greater than 2.

$$\left(\frac{1}{2}(p+1)\right)^2 - \left(\frac{1}{2}(p-1)\right)^2$$
$$= \frac{1}{4}\left((p+1)^2 - (p-1)^2\right)$$
$$= \frac{1}{4}(4p)$$
$$= p$$

So any odd prime number can be written as the difference of two squares.

Algebraic methods 7E

1 Example: when $n = 1$, $m = 3$
and 3 is not divisible by 10.
So the statement is not true.

2 3, 5, 7, 11, 13, 17, 19, 23 are the prime numbers between 2 and 26.
The other odd numbers between 2 and 26 are 9, 15, 21, 25.
$9 = 3 \times 3$
$15 = 5 \times 3$
$21 = 7 \times 3$
$25 = 5 \times 5$
So every odd integer between 2 and 26 is either prime or the product of two primes.

3 $1^2 + 2^2 = \text{odd}$
$2^2 + 3^2 = \text{odd}$
$3^2 + 4^2 = \text{odd}$
$4^2 + 5^2 = \text{odd}$
$5^2 + 6^2 = \text{odd}$
$6^2 + 7^2 = \text{odd}$
$7^2 + 8^2 = \text{odd}$
So the sum of two consecutive square numbers between 1^2 and 8^2 is always an odd number.

4 Break down the integers into numbers divisible by 3 and numbers giving a remainder of 1 or 2 when divided by 3.

$(3n)^3 = 27n^3 = 9n(3n^2)$ which is a multiple of 9.

$$(3n+1)^3 = 27n^3 + 27n^2 + 9n + 1$$
$$= 9n(3n^2 + 3n + 1) + 1$$
which is one more than a multiple of 9.

$$(3n+2)^3 = 27n^3 + 54n^2 + 36n + 8$$
$$= 9n(3n^2 + 6n + 4) + 8$$
which is one less than a multiple of 9.

So all cube numbers are either a multiple of 9 or 1 more or 1 less than a multiple of 9.

5 a Example: when $n = 2$, $2^4 - 2 = 14$
14 is not divisible by 4.

b Any square number has an odd number of factors, for example 25 has 3 factors.

c Example: when $n = \frac{1}{2}$,
$$2\left(\frac{1}{2}\right)^2 - 6\left(\frac{1}{2}\right) + 1 = 2\left(\frac{1}{4}\right) - 3 + 1$$
$$= \frac{1}{2} - 2$$
$$= -\frac{3}{2}$$
which is negative.

d Example: when $n = 1$,
$2(1)^2 - 2(1) - 4 = 2 - 2 - 4 = -4$
which is not a multiple of 3.

6 a The error lies in the last stage. We can only write this statement if $3(x^2)y + 3x(y^2)$ is greater than zero. No work has been done to prove or disprove this.

b Example, when $x = 0$ and $y = 0$,
$0^3 + 0^3 = (0 + 0)^3$

7 $(x+5)^2 \geq 0$ for all real values of x
As $(x+5)^2 = x^2 + 10x + 25$
and $(x+6)^2 = x^2 + 12x + 36$
$(x+5)^2 + 2x + 11 = (x+6)^2$
So $(x+6)^2 \geq 2x + 11$

8 As a is positive, multiplying both sides by a does not reverse the inequality
So $a^2 + 1 \geq 2a$
Then $a^2 - 2a + 1 \geq 0$
Factorising gives
$(a-1)^2 \geq 0$ which we know is true.

9 a By squaring both sides, consider $(p+q)^2$
$(p+q)^2 = p^2 + 2pq + q^2$
$= (p-q)^2 + 4pq$
$(p-q)^2 \geq 0$ since it is a square
so $(p+q)^2 \geq 4pq$
p and q are both positive
so $p > 0$ and $q > 0$
Therefore, $p + q > 0$
So $p + q \geq \sqrt{4pq}$

9 b When $p = q = -1$, $p + q = -2$
and $\sqrt{4pq} = 2$
but $-2 < 2$, i.e $p + q < \sqrt{4pq}$
which is inconsistent.

10 a The student had forgotten the significance of x and y both being negative i.e the left hand side is negative while the right hand side can be positive. In this case the inequality could not be true.

10 b When $x = y = -1$, $x + y = -2$
and $\sqrt{x^2 + y^2} = \sqrt{2}$
$-2 < \sqrt{2}$

c $(x + y)^2 = x^2 + 2xy + y^2$
As $x > 0$ and $y > 0$ then $2xy > 0$.
So $x^2 + 2xy + y^2 \geq x^2 + y^2$
As $x + y > 0$, square root both sides
$x + y \geq \sqrt{x^2 + y^2}$

Algebraic methods, 7 Mixed Exercise

1 a $\dfrac{3x^4-21x}{3x}=\dfrac{3x^4}{3x}-\dfrac{21x}{3x}$
$=x^3-7$

b $\dfrac{x^2-2x-24}{x^2-7x+6}$
$=\dfrac{(x-6)(x+4)}{(x-6)(x-1)}$
$=\dfrac{x+4}{x-1}$

c $\dfrac{2x^2+7x-4}{2x^2+9x+4}$
$=\dfrac{(2x-1)(x+4)}{(2x+1)(x+4)}$
$=\dfrac{2x-1}{2x+1}$

2
$$\begin{array}{r l}
 & 3x^2+5 \\
x+4 \,\big) & 3x^3+12x^2+5x+20 \\
 & \underline{3x^3+12x^2} \\
 & 0+5x+20 \\
 & \underline{5x+20} \\
 & 0
\end{array}$$

So $\dfrac{3x^3+12x^2+5x+20}{x+4}=3x^2+5$

3
$$\begin{array}{r l}
 & 2x^2-2x+5 \\
x+1 \,\big) & 2x^3+0x^2+3x+5 \\
 & \underline{2x^3+2x^2} \\
 & -2x^2+3x \\
 & \underline{-2x^2-2x} \\
 & 5x+5 \\
 & \underline{5x+5} \\
 & 0
\end{array}$$

So $\dfrac{2x^3+3x+5}{x+1}=2x^2-2x+5$

4 a $f(x)=2x^3-2x^2-17x+15$
$f(3)=2(3)^3-2(3)^2-17(3)+15$
$=54-18-51+15$
$=0$
So $(x-3)$ is a factor of $2x^3-2x^2-17x+15$.

b
$$\begin{array}{r l}
 & 2x^2+4x-5 \\
x-3 \,\big) & 2x^3-2x^2-17x+15 \\
 & \underline{2x^3-6x^2} \\
 & 4x^2-17x \\
 & \underline{4x^2-12x} \\
 & -5x+15 \\
 & \underline{-5x+15} \\
 & 0
\end{array}$$

$2x^3-2x^2-17x+15$
$=(x-3)(2x^2+4x-5)$
So $A=2$, $B=4$, $C=-5$

5 a $f(x)=x^3+4x^2-3x-18$
$f(2)=(2)^3+4(2)^2-3(2)-18$
$=8+16-6-18$
$=0$
So $(x-2)$ is a factor of $x^3+4x^2-3x-18$.

b
$$\begin{array}{r l}
 & x^2+6x+9 \\
x-2 \,\big) & x^3+4x^2-3x-18 \\
 & \underline{x^3-2x^2} \\
 & 6x^2-3x \\
 & \underline{6x^2-12x} \\
 & 9x-18 \\
 & \underline{9x-18} \\
 & 0
\end{array}$$

$x^3+4x^2-3x-18=(x-2)(x^2+6x+9)$
$=(x-2)(x+3)^2$
So $p=1$, $q=3$

6 $f(x)=2x^3+3x^2-18x+8$
$f(2)=2(2)^3+3(2)^2-18(2)+8$
$=16+12-36+8$
$=0$
So $(x-2)$ is a factor of $2x^3+3x^2-18x+8$.

6

$$\begin{array}{r} 2x^2+7x-4 \\ x-2\overline{)2x^3+3x^2-18x+8} \\ \underline{2x^3-4x^2} \qquad\qquad \\ 7x^2-18x \qquad \\ \underline{7x^2-14x} \qquad \\ -4x+8 \\ \underline{-4x+8} \\ 0 \end{array}$$

$2x^3 + 3x^2 - 18x + 8 = (x - 2)(2x^2 + 7x - 4)$
$= (x - 2)(2x - 1)(x + 4)$

7 $f(x) = x^3 - 3x^2 + kx - 10$
$f(2) = 0$
$(2)^3 - 3(2)^2 + k(2) - 10 = 0$
$8 - 12 + 2k - 10 = 0$
$2k = 14$
$k = 7$

8 a $f(x) = 2x^2 + px + q$
$f(-3) = 0$
$2(-3)^2 + p(-3) + q = 0$
$18 - 3p + q = 0$
$3p - q = 18$ **(1)**
$f(4) = 21$
$2(4)^2 + p(4) + q = 21$
$32 + 4p + q = 21$
$4p + q = -11$ **(2)**
(1) + **(2)**:
$7p = 7$
$p = 1$
Substituting in **(2)**:
$4(1) + q = -11$
$q = -15$
Checking in (1):
$3p - q = 3(1) - (-15) = 3 + 15 = 18$✓
So $p = 1$, $q = -15$

b $f(x) = 2x^2 + x - 15$
$= (2x - 5)(x + 3)$

9 a $h(x) = x^3 + 4x^2 + rx + s$
$h(-1) = 0$
$(-1)^3 + 4(-1)^2 + r(-1) + s = 0$
$-1 + 4 - r + s = 0$
$r - s = 3$ **(1)**
$h(2) = 30$
$(2)^3 + 4(2)^2 + r(2) + s = 30$
$8 + 16 + 2r + s = 30$
$2r + s = 6$ **(2)**

9 a **(1)** + **(2)**:
$3r = 9$
$r = 3$
Substituting in **(1)**
$3 - s = 3$
$s = 0$
Checking in **(2)**:
$2r + s = 2(3) + (0) = 6$✓
So $r = 3$, $s = 0$

b $h(x) = x^3 + 4x^2 + 3x$
$= x(x^2 + 4x + 3)$
$= x(x + 3)(x + 1)$

10 a $g(x) = 2x^3 + 9x^2 - 6x - 5$
$g(1) = 2(1)^3 + 9(1)^2 - 6(1) - 5$
$= 2 + 9 - 6 - 5$
$= 0$
So $(x - 1)$ is a factor of $2x^3 + 9x^2 - 6x - 5$.

$$\begin{array}{r} 2x^2+11x+5 \\ x-1\overline{)2x^3+9x^2-6x-5} \\ \underline{2x^3-2x^2} \qquad\qquad \\ 11x^2-6x \qquad \\ \underline{11x^2-11x} \qquad \\ 5x-5 \\ \underline{5x-5} \\ 0 \end{array}$$

$g(x) = 2x^3 + 9x^2 - 6x - 5$
$= (x - 1)(2x^2 + 11x + 5)$
$= (x - 1)(2x + 1)(x + 5)$

b $g(x) = 0$
$(x - 1)(2x + 1)(x + 5) = 0$
So $x = 1$, $x = -\frac{1}{2}$ or $x = -5$

11 a $f(x) = x^3 + x^2 - 5x - 2$
$f(2) = (2)^3 + (2)^2 - 5(2) - 2$
$= 8 + 4 - 10 - 2$
$= 0$
So $(x - 2)$ is a factor of $x^3 + x^2 - 5x - 2$.

11 b

$$\begin{array}{r}
x^2+3x+1 \\
x-2\overline{)\,x^3+x^2-5x-2} \\
\underline{x^3-2x^2} \\
3x^2-5x \\
\underline{3x^2-6x} \\
x-2 \\
\underline{x-2} \\
0
\end{array}$$

$f(x) = x^3 + x^2 - 5x - 2$
$= (x - 2)(x^2 + 3x + 1)$

$f(x) = 0$ when $x = 2$
or $x^2 + 3x + 1 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-3 \pm \sqrt{(3)^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{-3 \pm \sqrt{5}}{2}$$

So the solutions are $x = 2$, $x = \dfrac{-3+\sqrt{5}}{2}$ and $x = \dfrac{-3-\sqrt{5}}{2}$.

12

$$\begin{array}{r}
2x^2-7x+3 \\
x+1\overline{)\,2x^3-5x^2-4x+3} \\
\underline{2x^3+2x^2} \\
-7x^2-4x \\
\underline{-7x^2-7x} \\
3x+3 \\
\underline{3x+3} \\
0
\end{array}$$

$2x^3 - 5x^2 - 4x + 3 = (x + 1)(2x^2 - 7x + 3)$
$= (x + 1)(2x - 1)(x - 3)$

The roots are $x = -1$, $x = \frac{1}{2}$ and $x = 3$.

So the positive roots are $x = \frac{1}{2}$ and $x = 3$.

13 a $f(x) = x^3 - 2x^2 - 19x + 20$
$f(-4) = (-4)^3 - 2(-4)^2 - 19(-4) + 20$
$= -64 - 32 + 76 + 20$
$= 0$

The remainder is 0.

13 b

$$\begin{array}{r}
x^2-6x+5 \\
x+4\overline{)\,x^3-2x^2-19x+20} \\
\underline{x^3+4x^2} \\
-6x^2-19x \\
\underline{-6x^2-24x} \\
5x+20 \\
\underline{5x+20} \\
0
\end{array}$$

$f(x) = x^3 - 2x^2 - 19x + 20$
$= (x + 4)(x^2 - 6x + 5)$
$= (x + 4)(x - 5)(x - 1)$

$f(x) = 0$ when
$x = -4$, $x = 5$ or $x = 1$

14 a $f(x) = 6x^3 + 17x^2 - 5x - 6$

$$f\left(\frac{2}{3}\right) = 6\left(\frac{2}{3}\right)^3 + 17\left(\frac{2}{3}\right)^2 - 5\left(\frac{2}{3}\right) - 6$$

$$= 6\left(\frac{8}{27}\right) + 17\left(\frac{4}{9}\right) - 5\left(\frac{2}{3}\right) - 6$$

$$= \frac{16}{9} + \frac{68}{9} - \frac{10}{3} - 6$$

$= 0$

So $(3x - 2)$ is a factor of $f(x)$.

$$\begin{array}{r}
2x^2+7x+3 \\
3x-2\overline{)\,6x^3+17x^2-5x-6} \\
\underline{6x^3-4x^2} \\
21x^2-5x \\
\underline{21x^2-14x} \\
9x-6 \\
\underline{9x-6} \\
0
\end{array}$$

$f(x) = 6x^3 + 17x^2 - 5x - 6$
$= (3x - 2)(2x^2 + 7x + 3)$

So $a = 2$, $b = 7$, $c = 3$

b $f(x) = (3x - 2)(2x^2 + 7x + 3)$
$= (3x - 2)(2x + 1)(x + 3)$

c $(3x - 2)(2x + 1)(x + 3) = 0$

The real roots are $x = \frac{2}{3}$, $x = -\frac{1}{2}$ and $x = -3$.

15 LHS $= \dfrac{x-y}{\left(\sqrt{x}-\sqrt{y}\right)} \times \dfrac{\left(\sqrt{x}+\sqrt{y}\right)}{\left(\sqrt{x}+\sqrt{y}\right)}$

$$= \frac{(x-y)\left(\sqrt{x}+\sqrt{y}\right)}{x-y}$$

$$= \sqrt{x}+\sqrt{y}$$

= RHS

So $\dfrac{x-y}{\sqrt{x}-\sqrt{y}} \equiv \sqrt{x}+\sqrt{y}$

16 Completing the square:
$n^2 - 8n + 20 = (n-4)^2 + 4$
The minimum value is 4, so $n^2 - 8n + 20$ is always positive.

17 $A(1,1)$, $B(3,2)$, $C(4,0)$ and $D(2,-1)$

The gradient of line $AB = \dfrac{2-1}{3-1} = \dfrac{1}{2}$

The gradient of line $BC = \dfrac{0-2}{4-3} = -2$

The gradient of line $CD = \dfrac{-1-0}{2-4} = \dfrac{1}{2}$

The gradient of line $AD = \dfrac{-1-1}{2-1} = -2$

AB and *BC*, *BC* and *CD*, *CD* and *AD* and *AB* and *AD* are all perpendicular.

Distance $AB = \sqrt{(3-1)^2+(2-1)^2}$
$= \sqrt{5}$

Distance $BC = \sqrt{(4-3)^2+(0-2)^2}$
$= \sqrt{5}$

Distance $CD = \sqrt{(2-4)^2+(-1-0)^2}$
$= \sqrt{5}$

Distance $AD = \sqrt{(2-1)^2+(-1-1)^2}$
$= \sqrt{5}$

All four sides are equal and all four angles are right angles, therefore *ABCD* is a square.

18 $1 + 3 =$ even
$3 + 5 =$ even
$5 + 7 =$ even
$7 + 9 =$ even
So the sum of two consecutive positive odd numbers is always even.

19 To show something is untrue you only need to find one counter example.
Example: when $n = 6$,
$n^2 - n + 3 = 6^2 - 6 + 3 = 33$
which is not a prime number.
So the statement is untrue.

20 LHS $= \left(x - \dfrac{1}{x}\right)\left(x^{\frac{4}{3}} + x^{-\frac{2}{3}}\right)$

$$= x^{\frac{7}{3}} + x^{\frac{1}{3}} - x^{\frac{1}{3}} - x^{-\frac{5}{3}}$$

$$= x^{\frac{7}{3}} - x^{-\frac{5}{3}}$$

$$= x^{\frac{1}{3}}\left(x^2 - \frac{1}{x^2}\right)$$

= RHS

So $\left(x - \dfrac{1}{x}\right)\left(x^{\frac{4}{3}} + x^{-\frac{2}{3}}\right) \equiv x^{\frac{1}{3}}\left(x^2 - \dfrac{1}{x^2}\right)$

21 Remember, in an identity you can start from the RHS or the LHS. Here it is easier to start from the RHS.
RHS $= (x+4)(x-5)(2x+3)$
$= (x+4)(2x^2 - 7x - 15)$
$= 2x^3 + x^2 - 43x - 60$
= LHS
So $2x^3 + x^2 - 43x - 60$
$\equiv (x+4)(x-5)(2x+3)$

22 $x^2 - kx + k = 0$ has two equal roots,
so $b^2 - 4ac = 0$
$k^2 - 4k = 0$
$k(k-4) = 0$
$k = 4$ or 0.
So $k = 4$ is a solution.

23 Using Pythagoras' theorem:
The distance between opposite edges

$$= 2\left(\left(\sqrt{3}\right)^2 - \left(\frac{\sqrt{3}}{2}\right)^2\right)$$

$$= 2\left(3 - \frac{3}{4}\right)$$

$$= \frac{9}{2}$$

$\dfrac{9}{2}$ is rational.

24 a Let the first even number be $2n$.
The next even number is $2n + 2$.
$(2n + 2)^2 - (2n)^2 = 4n^2 + 8n + 4 - 4n^2$
$= 8n + 4$
$= 4(2n + 1)$
$4(2n + 1)$ is a multiple of 4 so is always divisible by 4.
So the difference of the squares of two consecutive even numbers is always divisible by 4.

b Let the first odd number be $2n - 1$.
The next odd number is $2n + 1$.
$(2n + 1)^2 - (2n - 1)^2$
$= (4n^2 + 4n + 1) - (4n^2 - 4n + 1)$
$= 8n$
$8n$ is a multiple of 8, which is always divisible by 4, so the statement is also true for odd numbers.

25 a The assumption is that x is positive.

b When $x = 0$, $1 + 0^2 = (1 + 0)^2$

Challenge

1 **a** Diameter of circle = 1,
so side of outside square = 1
Using Pythagoras' theorem:

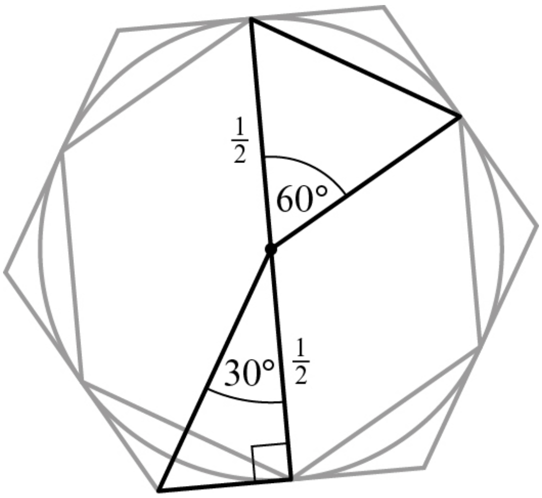
Perimeter of the inside square = $4\left(\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}\right)=\frac{4}{\sqrt{2}}=2\sqrt{2}$

Perimeter of the outside square = 4 × 1 = 4
The circumference of the circle is between the perimeters of the two squares, so $2\sqrt{2} < \pi < 4$.

b Perimeter of inside hexagon = $6\times\frac{1}{2}=3$ because the triangles with 60° angles are equilateral.

Perimeter of outside hexagon = $6\times\frac{\sqrt{3}}{3}=2\sqrt{3}$

The circumference of the circle is between the perimeters of the two hexagons, so $3 < \pi < 2\sqrt{3}$



2
$$
\begin{array}{r}
ax^2+(b+ap)x+(c+bp+ap^2) \\
x-p\overline{\big)\, ax^3+bx^2 \quad +cx \quad +d} \\
\underline{ax^3-apx^2} \\
(b+ap)x^2+cx \\
\underline{(b+ap)x^2-(bp+ap^2)x} \\
(c+bp+ap^2)x+d \\
\underline{(c+bp+ap^2)x-(cp+bp^2+ap^3)} \\
d+cp+bp^2+ap^3
\end{array}
$$

So $\frac{ax^3+bx^2+cx+d}{x-p}=ax^2+(b+ap)x+(c+bp+ap^2)$ with remainder.
So, $d + cp + bp^2 + ap^3$
$f(p) = ap^3 + bp^2 + cp + d = 0$, which matches the remainder $d + cp + bp^2 + ap^3 = 0$
Therefore $(x - p)$ is a factor of $f(x)$.

The binomial expansion 8A

1 a $(x + y)^3$
The $(n + 1)$th row of Pascal's triangle gives the coefficients in the expansion of $(a + b)^n$.
$3 + 1 = 4$th row

b $(3x - 7)^{15}$
$15 + 1 = 16$th row

c $\left(2x + \frac{1}{2}\right)^n$
$n + 1 = (n + 1)$th row

d $(y - 2x)^{n+4}$
$n + 4 + 1 = (n + 5)$th row

2 a $(x + y)^4$ has coefficients and terms

1	4	6	4	1
x^4	x^3y	x^2y^2	xy^3	y^4

$$\begin{aligned}(x + y)^4 &= 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4\\ &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4\end{aligned}$$

b $(p + q)^5$ has coefficients and terms

1	5	10	10	5	1
p^5	p^4q	p^3q^2	p^2q^3	pq^4	q^5

$$\begin{aligned}(p + q)^5 &= 1p^5 + 5p^4q + 10p^3q^2 + 10p^2q^3 + 5pq^4 + 1q^5\\ &= p^5 + 5p^4q + 10p^3q^2 + 10p^2q^3 + 5pq^4 + q^5\end{aligned}$$

c $(a - b)^3$ has coefficients and terms

1	3	3	1
a^3	$a^2(-b)$	$a(-b)^2$	$(-b)^3$

$$\begin{aligned}(a - b)^3 &= 1a^3 - 3a^2b + 3ab^2 - 1b^3\\ &= a^3 - 3a^2b + 3ab^2 - b^3\end{aligned}$$

d $(x + 4)^3$ has coefficients and terms

1	3	3	1
x^3	x^24	$x4^2$	4^3

$$\begin{aligned}(x + 4)^3 &= 1x^3 + 3x^2(4) + 3x(4)^2 + 4^3\\ &= x^3 + 12x^2 + 48x + 64\end{aligned}$$

e $(2x - 3)^4$ has coefficients and terms

1	4	6	4	1
$(2x)^4$	$(2x)^3(-3)$	$(2x)^2(-3)^2$	$(2x)(-3)^3$	$(-3)^4$

$$\begin{aligned}(2x - 3)^4 &= 1(2x)^4 + 4(2x)^3(-3) + 6(2x)^2(-3)^2 + 4(2x)(-3)^3 + 1(-3)^4\\ &= 16x^4 - 96x^3 + 216x^2 - 216x + 81\end{aligned}$$

2 f $(a + 2)^5$ has coefficients and terms

1	5	10	10	5	1
a^2	$a^4 2$	$a^3 2^2$	$a^2 2^3$	$a 2^4$	2^5

$$\begin{aligned}(a + 2)^5 &= 1a^5 + 5a^4(2) + 10a^3(2)^2 + 10a^2(2)^3 + 5a(2)^4 + 32 \\ &= a^5 + 10a^4 + 40a^3 + 80a^2 + 80a + 32\end{aligned}$$

g $(3x - 4)^4$ has coefficients and terms

1	4	6	4	1
$(3x)^4$	$(3x)^3(-4)$	$(3x)^2(-4)^2$	$(3x)(-4)^3$	$(-4)^4$

$$\begin{aligned}(3x - 4)^4 &= 1(3x)^4 + 4(3x)^3(-4) + 6(3x)^2(-4)^2 + 4(3x)(-4)^3 + 1(-4)^4 \\ &= 81x^4 - 432x^3 + 864x^2 - 768x + 256\end{aligned}$$

h $(2x - 3y)^4$ has coefficients and terms

1	4	6	4	1
$(2x)^4$	$(2x)^3(-3y)$	$(2x)^2(-3y)^2$	$(2x)(-3y)^3$	$(-3y)^4$

$$\begin{aligned}(2x - 3y)^4 &= 1(2x)^4 + 4(2x)^3(-3y) + 6(2x)^2(-3y)^2 + 4(2x)(-3y)^3 + 1(-3y)^4 \\ &= 16x^4 - 96x^3y + 216x^2y^2 - 216xy^3 + 81y^4\end{aligned}$$

3 a $(4 + x)^4$ has coefficients 1 4 6 **4** 1
The bold number is the coefficient of the term $4x^3$.
Term is $4 \times 4x^3 = 16x^3$.
Coefficient = 16

b $(1 - x)^5$ has coefficients 1 5 10 **10** 5 1
The bold number is the coefficient of the term $1^2(-x)^3$.
Term is $10 \times 1^2(-x)^3 = -10x^3$.
Coefficient = −10

c $(3 + 2x)^3$ has coefficients 1 3 3 **1**
The bold number is the coefficient of the term $(2x)^3$.
Term is $1 \times (2x)^3 = 8x^3$.
Coefficient = 8

d $(4 + 2x)^5$ has coefficients 1 5 10 **10** 5 1
The bold number is the coefficient of the term $4^2(2x)^3$.
Term is $10 \times 4^2(2x)^3 = 1280x^3$.
Coefficient = 1280

e $(2 + x)^6$ has coefficients 1 6 15 **20** 15 6 1
The bold number is the coefficient of the term 2^3x^3.
Term is $20 \times 2^3x^3 = 160x^3$.
Coefficient = 160

f $\left(4 - \frac{1}{2}x\right)^4$ has coefficients 1 4 6 **4** 1

The bold number is the coefficient of the term $4\left(-\frac{1}{2}x\right)^3$

3 **f** Term is $4 \times 4\left(-\frac{1}{2}x\right)^3 = -2x^3$

Coefficient = −2

g $(x + 2)^5$ has coefficients 1 5 **10** 10 5 1
The bold number is the coefficient of the term $x^3 2^2$.
Term is $10 \times x^3 2^2 = 40x^3$.
Coefficient = 40

h $(3 - 2x)^4$ has coefficients 1 4 6 **4** 1
The bold number is the coefficient of the term $3(-2x)^3$.
Term is $4 \times 3(-2x)^3 = -96x^3$.
Coefficient = −96

4 $(1 + 2x)^3$ has coefficients and terms

1	3	3	1
1^3	$1^2(2x)$	$1(2x)^2$	$(2x)^3$

Hence $(1 + 2x)^3 = 1 + 6x + 12x^2 + 8x^3$

$(1 + 3x)(1 + 2x)^3$
$= (1 + 3x)(1 + 6x + 12x^2 + 8x^3)$
$= 1(1 + 6x + 12x^2 + 8x^3) + 3x(1 + 6x + 12x^2 + 8x^3)$
$= 1 + 6x + 12x^2 + 8x^3 + 3x + 18x^2 + 36x^3 + 24x^4$
$= 1 + 9x + 30x^2 + 44x^3 + 24x^4$

5 $(2 + y)^3$ has coefficients and terms

1	3	3	1
2^3	2^2y	$2y^2$	y^3

Hence $(2 + y)^3 = 8 + 12y + 6y^2 + y^3$

Substitute $y = x - x^2$
$(2 + x - x^2)^3 = 8 + 12(x - x^2) + 6(x - x^2)^2 + (x - x^2)^3$
$= 8 + 12x(1 - x) + 6x^2(1 - x^2)^2 + x^3(1 - x)^3$

Now
$(1 - x)^3 = (1 - x)(1 - x)^2$
$(1 - x)^3 = (1 - x)(1 - 2x + x^2)$
$(1 - x)^3 = 1 - 2x + x^2 - x + 2x^2 - x^3$
$(1 - x)^3 = 1 - 3x + 3x^2 - x^3$

Or, using Pascal's triangle
$(1 - x)^3 = 1(1)^3 + 3(1)^2(-x) + 3(1)(-x)^2 + 1(-x)^3$
$(1 - x)^3 = 1 - 3x + 3x^2 - x^3$

So $(2 + x - x^2) = 8 + 12x(1 - x) + 6x^2(1 - 2x + x^2) + x^3(1 - 3x + 3x^2 - x^3)$
$= 8 + 12x - 12x^2 + 6x^2 - 12x^3 + 6x^4 + x^3 - 3x^4 + 3x^5 - x^6$
$= 8 + 12x - 6x^2 - 11x^3 + 3x^4 + 3x^5 - x^6$

6 $(2 + ax)^3$ has coefficients 1 3 **3** 1
The bold number is the coefficient of the term $2(ax)^2$
Term in x^2 is $3 \times 2(ax)^2 = 6a^2x^2$
Coefficient of x^2 is $6a^2$

So $6a^2 = 54$
$a^2 = 9$
$a = \pm 3$

7 $(3 + bx)^3$ has coefficients and terms

1	3	3	1
3^3	$3^2(bx)$	$3(bx)^2$	$(bx)^3$

$(3 + bx)^3 = 1 \times 3^3 + 3 \times 3^2bx + 3 \times 3(bx)^2 + 1 \times (bx)^3$
$= 27 + 27bx + 9b^2x^2 + b^3x^3$

So $(2 - x)(3 + bx)^3 = (2 - x)(27 + 27bx + 9b^2x^2 + b^3x^3)$

Term in x^2 is $2 \times 9b^2x^2 - x \times 27bx = 18b^2x^2 - 27bx^2$
Coefficient of x^2 is $18b^2 - 27b$

So $18b^2 - 27b = 45$
$2b^2 - 3b = 5$
$2b^2 - 3b - 5 = 0$
$(2b - 5)(b + 1) = 0$
$b = \frac{5}{2}, -1$

8 The coefficients are 1, 3, **3**, 1
The term in x^2 is $3(p)(-2x)^2 = 12px^2$
So the coefficient of the term x^2 is $12p$

9 $500\left(1+\dfrac{X}{100}\right)^5$

The coefficients are 1, 5, 10, 10, 5, 1

$$500\left(1+\frac{X}{100}\right)^5 = 500\left(1(1)^5+5(1)^4\left(\frac{X}{100}\right)+10(1)^3\left(\frac{X}{100}\right)^2+\dots\right)$$
$$= 500\left(1+\frac{X}{20}+\frac{X^2}{1000}+\dots\right)$$
$$\approx 500 + 25X + \frac{X^2}{2}$$
$$A = 500,\ B = 25,\ C = \tfrac{1}{2}$$

Challenge

$(x^2 - \dfrac{1}{2x})^3$

Coefficients are 1 3 **3** 1

Third term is $3(x^2)^1(-\dfrac{1}{2x})^2 = \dfrac{3x^2}{4x^2} = \dfrac{3}{4}$

The binomial expansion 8B

1 a $4! = 4 \times 3 \times 2 \times 1$
$= 24$

b $9! = 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$
$= 362\,880$

c $\frac{10!}{7!} = \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}$

$= \frac{3\,628\,800}{5040}$ or $10 \times 9 \times 8$ since the 7! on numerator and denominator cancel

$= 720$

d $\frac{15!}{13!} = \frac{15 \times 14 \times 13 \times 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{13 \times 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}$

$= \frac{1\,307\,674\,368\,000}{6\,227\,020\,800}$ or 15×14 because the 13! cancels

$= 210$

2 a $\binom{4}{2} = \frac{4!}{2!2!}$

$= \frac{4 \times 3 \times 2 \times 1}{2 \times 1 \times 2 \times 1}$

$= \frac{4 \times 3}{2 \times 1}$

$= 6$

b $\binom{6}{4} = \frac{6!}{4!2!}$

$= \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1 \times 2 \times 1}$

$= \frac{6 \times 5}{2 \times 1}$

$= 15$

c ${}^6C_3 = \frac{6!}{3!3!}$

$= \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1 \times 3 \times 2 \times 1}$

$= \frac{6 \times 5 \times 4}{3 \times 2 \times 1}$

$= 20$

2 d $\begin{pmatrix}5\\4\end{pmatrix} = \dfrac{5!}{4!1!}$

$= \dfrac{5\times4\times3\times2\times1}{4\times3\times2\times1\times1}$

$= \dfrac{5}{1}$

$= 5$

e ${}^{10}C_8 = \dfrac{10!}{8!2!}$

$= \dfrac{10\times9\times8\times7\times6\times5\times4\times3\times2\times1}{8\times7\times6\times5\times4\times3\times2\times1\times2\times1}$

$= \dfrac{10\times9}{2\times1}$

$= 45$

f $\begin{pmatrix}9\\5\end{pmatrix} = \dfrac{9!}{5!4!}$

$= \dfrac{9\times8\times7\times6\times5\times4\times3\times2\times1}{5\times4\times3\times2\times1\times4\times3\times2\times1}$

$= \dfrac{9\times8\times7\times6}{4\times3\times2\times1}$

$= 126$

3 a $\begin{pmatrix}15\\6\end{pmatrix} = 5005$

b ${}^{10}C_7 = 120$

c $\begin{pmatrix}20\\10\end{pmatrix} = 184\,756$

d $\begin{pmatrix}20\\17\end{pmatrix} = 1140$

e ${}^{14}C_9 = 2002$

f ${}^{18}C_5 = 8568$

4 The rth entry in the nth row of Pascal's triangle is given by ${}^{n-1}C_{r-1}$.

a ${}^{5-1}C_{2-1} = {}^{4}C_1$

b ${}^{6-1}C_{3-1} = {}^{5}C_2$

c $= {}^{6}C_2$

d ${}^{7-1}C_{4-1} = {}^{6}C_3$

5 5th number on the 12th row = ${}^{12-1}C_{5-1} = {}^{11}C_4 = 330$

6 a ${}^{11-1}C_{4-1} = {}^{10}C_3 = 120$
${}^{11-1}C_{5-1} = {}^{10}C_4 = 210$

b The coefficients are 1, 10, 45, 120, 210, ...
The term in x^3 is $120(1)^7(2x)^3 = 960x^3$.
Coefficient = 960

7 a ${}^{14-1}C_{4-1} = {}^{13}C_3 = 286$
${}^{14-1}C_{5-1} = {}^{13}C_4 = 715$

b The coefficients are 1, 13, 78, 286, 715, ...
The term in x^4 is $715(1)^9(3x)^4 = 57\,915x^4$.
Coefficient = 57 915

8 $\binom{20}{10}0.5^{20} = {}^{20}C_{10}0.5^{20}$
$= 184\,756 \times 0.5^{20}$
$= 0.1762$ (to 4 s.f.)
Whilst this seems a low probability, there is more chance of the coin landing on 10 heads than any other number of heads.

9 a $${}^{n}C_1 = \frac{n!}{1!(n-1)!} = \frac{n\times(n-1)\times(n-2)\times\ldots\times 2\times 1}{1\times(n-1)\times(n-2)\times(n-3)\times\ldots\times 2\times 1} = n$$

b $${}^{n}C_2 = \frac{n!}{2!(n-2)!} = \frac{n\times(n-1)\times(n-2)\times\ldots\times 2\times 1}{1\times 2\times(n-2)\times(n-3)\times\ldots\times 2\times 1} = \frac{n(n-1)}{2}$$

10 $\binom{50}{13} = \frac{50!}{13!a!}$
$\binom{50}{13} = \frac{50!}{13!37!}$
$a = 37$

11 $\binom{35}{p} = \frac{35!}{p!18!}$
$\binom{35}{17} = \frac{35!}{17!18!}$
$p = 17$

Challenge

a $\quad {}^{10}C_3 = \frac{10!}{3!7!} = 120$

$\quad {}^{10}C_7 = \frac{10!}{7!3!} = 120$

b $\quad {}^{14}C_5 = \frac{14!}{5!9!} = 2002$

$\quad {}^{14}C_9 = \frac{14!}{9!5!} = 2002$

c $\quad$ The two answers for part **a** are the same and the two answers for part **b** are the same.

d $\quad {}^{n}C_r = \frac{n!}{r!(n-r)!}$ and ${}^{n}C_{n-r} = \frac{n!}{(n-r)!r!}$ because $\frac{n!}{(n-r)!\big(n-(n-r)\big)!}$ and $(n-(n-r)) = r$

As $\frac{n!}{r!(n-r)!} = \frac{n!}{(n-r)!r!}$, ${}^{n}C_r = {}^{n}C_{n-r}$

The binomial expansion 8C

1 a $(1+x)^4 = 1^4 + \binom{4}{1}1^3x + \binom{4}{2}1^2x^2 + \binom{4}{3}1x^3 + x^4$
$= 1 + 4x + 6x^2 + 4x^3 + x^4$

b $(3+x)^4 = 3^4 + \binom{4}{1}3^3x + \binom{4}{2}3^2x^2 + \binom{4}{3}3x^3 + x^4$
$= 81 + 108x + 54x^2 + 12x^3 + x^4$

c $(4-x)^4 = 4^4 + \binom{4}{1}4^3(-x) + \binom{4}{2}4^2(-x)^2 + \binom{4}{3}4(-x)^3 + (-x)^4$
$= 256 - 256x + 96x^2 - 16x^3 + x^4$

d $(x+2)^6 = x^6 + \binom{6}{1}x^5 2 + \binom{6}{2}x^4 2^2 + \binom{6}{3}x^3 2^3 + \binom{6}{4}x^2 2^4 + \binom{6}{5}x2^5 + 2^6$
$= x^6 + 12x^5 + 60x^4 + 160x^3 + 240x^2 + 192x + 64$

e $(1+2x)^4 = 1^4 + \binom{4}{1}1^3(2x) + \binom{4}{2}1^2(2x)^2 + \binom{4}{3}1(2x)^3 + (2x)^4$
$= 1 + 8x + 24x^2 + 32x^3 + 16x^4$

f $\left(1-\frac{1}{2}x\right)^4 = 1^4 + \binom{4}{1}1^3(-\frac{1}{2}x) + \binom{4}{2}1^2(-\frac{1}{2}x)^2 + \binom{4}{3}1(-\frac{1}{2}x)^3 + (-\frac{1}{2}x)^4$
$= 1 - 2x + \frac{3}{2}x^2 - \frac{1}{2}x^3 + \frac{1}{16}x^4$

2 a $(1+x)^{10} = 1^{10} + \binom{10}{1}1^9x + \binom{10}{2}1^8x^2 + \binom{10}{3}1^7x^3 + \ldots$
$= 1 + 10 \times 1x + 45 \times 1x^2 + 120 \times 1x^3 + \ldots$
$= 1 + 10x + 45x^2 + 120x^3 + \ldots$

b $(1-2x)^5 = 1^5 + \binom{5}{1}1^4(-2x) + \binom{5}{2}1^3(-2x)^2 + \binom{5}{3}1^2(-2x)^3 + \ldots$
$= 1 \times 1 + 5 \times (-2x) + 10 \times 4x^2 + 10 \times (-8x^3) + \ldots$
$= 1 - 10x + 40x^2 - 80x^2 + \ldots$

c $(1+3x)^6 = 1^6 + \binom{6}{1}1^5(3x) + \binom{6}{2}1^4(3x)^2 + \binom{6}{3}1^3(3x)^3 + \ldots$
$= 1 \times 1 + 6 \times 3x + 15 \times 9x^2 + 20 \times 27x^3 + \ldots$
$= 1 + 18x + 135x^2 + 540x^3 + \ldots$

d $(2-x)^8 = 2^8 + \binom{8}{1}2^7(-x) + \binom{8}{2}2^6(-x)^2 + \binom{8}{3}2^5(-x)^3 + \ldots$
$= 1 \times 256 + 8 \times (-128x) + 28 \times 64x^2 + 56 \times (-32x^3) + \ldots$
$= 256 - 1024x + 1792x^2 - 1792x^3 + \ldots$

2 e $\left(2-\frac{1}{2}x\right)^{10} = 2^{10}+\binom{10}{1}2^9\left(-\frac{1}{2}x\right)+\binom{10}{2}2^8\left(-\frac{1}{2}x\right)^2+\binom{10}{3}2^7\left(-\frac{1}{2}x\right)^3+\ldots$

$= 1 \times 1024 + 10 \times (-256x) + 45 \times 64x^2 + 120 \times (-16x^3) + \ldots$

$= 1024 - 2560x + 2880x^2 - 1920x^3 + \ldots$

f $(3-x)^7 = 3^7 + \binom{7}{1}3^6(-x) + \binom{7}{2}3^5(-x)^2 + \binom{7}{3}3^4(-x)^3 + \ldots$

$= 1 \times 2187 + 7 \times (-729x) + 21 \times 243x^2 + 35 \times (-81x^3) + \ldots$

$= 2187 - 5103x + 5103x^2 - 2835x^3 + \ldots$

3 a $(2x+y)^6 = (2x)^6+\binom{6}{1}(2x)^5y+\binom{6}{2}(2x)^4y^2+\binom{6}{3}(2x)^3y^3+\ldots$

$= 64x^6 + 192x^5y + 240x^4y^2 + 160x^3y^3$

b $(2x+3y)^5 = (2x)^5+\binom{5}{1}(2x)^4(3y)+\binom{5}{2}(2x)^3(3y)^2+\binom{5}{3}(2x)^2(3y)^3+\ldots$

$= 32x^5 + 240x^4y + 720x^3y^2 + 1080x^2y^3 + \ldots$

c $(p-q)^8 = p^8+\binom{8}{1}p^7(-q)+\binom{8}{2}p^6(-q)^2+\binom{8}{3}p^5(-q)^3+\ldots$

$= p^8 - 8p^7q + 28p^6q^2 - 56p^5q^3 + \ldots$

d $(3x-y)^6 = (3x)^6+\binom{6}{1}(3x)^5(-y)+\binom{6}{2}(3x)^4(-y)^2+\binom{6}{3}(3x)^3(-y)^3+\ldots$

$= 729x^6 - 1458x^5y + 1215x^4y^2 - 540x^3y^3 + \ldots$

e $(x+2y)^8 = x^8+\binom{8}{1}x^7(2y)+\binom{8}{2}x^6(2y)^2+\binom{8}{3}x^5(2y)^3+\ldots$

$= x^8 + 16x^7y + 112x^6y^2 + 448x^5y^3 + \ldots$

f $(2x-3y)^9 = (2x)^9+\binom{9}{1}(2x)^8(-3y)+\binom{9}{2}(2x)^7(-3y)^2+\binom{9}{3}(2x)^6(-3y)^3+\ldots$

$= 512x^9 - 6912x^8y + 41\,472x^7y^2 - 145\,152x^6y^3 + \ldots$

4 a $(1+x)^8 = 1^8 + \binom{8}{1}1^7x + \binom{8}{2}1^6x^2 + \binom{8}{3}1^5x^3 + \ldots$

$= 1 + 8x + 28x^2 + 56x^3 + \ldots$

b $(1-2x)^6 = 1^6 + \binom{6}{1}1^5(-2x) + 1^4\binom{6}{2}(-2x)^2 + \binom{6}{3}1^3(-2x)^3 + \ldots$

$= 1 - 12x + 60x^2 - 160x^3 + \ldots$

c $\left(1+\frac{x}{2}\right)^{10} = 1^{10}+\binom{10}{1}1^9\left(\frac{x}{2}\right)+\binom{10}{2}1^8\left(\frac{x}{2}\right)^2+\binom{10}{3}1^7\left(\frac{x}{2}\right)^3+\ldots$

$= 1 + 5x + \frac{45}{4}x^2 + 15x^3 + \ldots$

4 d $(1-3x)^5 = 1^5 + \binom{5}{1}1^4(-3x) + \binom{5}{2}1^3(-3x)^2 + \binom{5}{3}1^2(-3x)^3 + \ldots$
$= 1 - 15x + 90x^2 - 270x^3 + \ldots$

e $(2+x)^7 = 2^7 + \binom{7}{1}2^6x + \binom{7}{2}2^5x^2 + \binom{7}{3}2^4x^3 + \ldots$
$= 128 + 448x + 672x^2 + 560x^3 + \ldots$

f $(3-2x)^3 = 3^3 + \binom{3}{1}3^2(-2x) + \binom{3}{2}3(-2x)^2 + (-2x)^3$
$= 27 - 54x + 36x^2 - 8x^3$

g $(2-3x)^6 = 2^6 + \binom{6}{1}2^5(-3x) + \binom{6}{2}2^4(-3x)^2 + \binom{6}{3}2^3(-3x)^3 + \ldots$
$= 64 - 576x + 2160x^2 - 4320x^3 + \ldots$

h $(4+x)^4 = 4^4 + \binom{4}{1}4^3x + \binom{4}{2}4^2x^2 + \binom{4}{3}4x^3 + \ldots$
$= 256 + 256x + 96x^2 + 16x^3 + \ldots$

i $(2+5x)^7 = 2^7 + \binom{7}{1}2^6(5x) + \binom{7}{2}2^5(5x)^2 + \binom{7}{3}2^4(5x)^3 + \ldots$
$= 128 + 2240x + 16\,800x^2 + 70\,000x^3 + \ldots$

5 $(2-x)^6 = 2^6 + \binom{6}{1}2^5(-x) + \binom{6}{2}2^4(-x)^2 + \ldots$

$= 64 - 192x + 240x^2 + \ldots$

6 $(3-2x)^5 = 3^5 + \binom{5}{1}3^4(-2x) + \binom{5}{2}3^3(-2x)^2 + \ldots$
$= 243 - 810x + 1080x^2 + \ldots$

7 $\left(x+\frac{1}{x}\right)^5 = x^5 + \binom{5}{1}x^4\left(\frac{1}{x}\right) + \binom{5}{2}x^3\left(\frac{1}{x}\right)^2 + \binom{5}{3}x^2\left(\frac{1}{x}\right)^3 + \binom{5}{4}x\left(\frac{1}{x}\right)^4 + \left(\frac{1}{x}\right)^5$
$= x^5 + 5x^3 + 10x + \frac{10}{x} + \frac{5}{x^3} + \frac{1}{x^5}$

Challenge

a $(a+b)^4 = a^4 + \binom{4}{1}a^3b + \binom{4}{2}a^2b^2 + \binom{4}{3}ab^3 + b^4$
$= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

$(a-b)^4 = a^4 + \binom{4}{1}a^3(-b) + \binom{4}{2}a^2(-b)^2 + \binom{4}{3}a(-b)^3 + (-b)^4$
$= a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$

$(a+b)^4 - (a-b)^4 = (a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4) - (a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4)$
$= 8a^3b + 8ab^3$
$= 8ab(a^2 + b^2)$

b $82\,896 = 17^4 - 5^4$
$a = 11$ and $b = 6$
$= 8 \times 11 \times 6 \times (11^2 + 6^2)$
$= 8 \times 11 \times 6 \times 157$
$= 2^3 \times 11 \times 2 \times 3 \times 157$
$= 2^4 \times 3 \times 11 \times 157$

The binomial expansion 8D

1 **a** $(3+x)^5$

x^3 term $= \binom{5}{3} 3^2(x)^3 = 10 \times 9x^3 = 90x^3$

Coefficient = 90

b $(1+2x)^5$

x^3 term $= \binom{5}{3} 1^2\,(2x)^3 = 10 \times 8x^3 = 80x^3$

Coefficient = 80

c $(1-x)^6$

x^3 term $= \binom{6}{3}\,(1)^3(-x)^3 = 20 \times (-x^3) = -20x^3$

Coefficient = −20

d $(3x+2)^5$

x^3 term $= \binom{5}{2}\,(3x)^3 2^2 = 10 \times 108x^3 = 1080x^3$

Coefficient = 1080

e $(1+x)^{10}$

x^3 term $= \binom{10}{3} 1^7(x)^3 = 120 \times 1x^3 = 120x^3$

Coefficient = 120

f $(3-2x)^6$

x^3 term $= \binom{6}{3} 3^3(-2x)^3 = 20 \times (-216x^3) = -4320x^3$

Coefficient = −4320

g $(1+x)^{20}$

x^3 term $= \binom{20}{3} 1^{17}x^3 = 1140 \times x^3 = 1140x^3$

Coefficient = 1140

h $(4-3x)^7$

x^3 term $= \binom{7}{3} 4^4(-3x)^3 = 35 \times (-6912x^3) = -241\,920x^3$

Coefficient = −241 920

i $\left(1-\frac{1}{2}x\right)^6$

x^3 term $= \binom{6}{3} 1^3\left(-\frac{1}{2}x\right)^3 = 20 \times -\frac{1}{8}x^3 = -2.5x^3$ or $-\frac{5}{2}x^3$

Coefficient = −2.5 or $-\frac{5}{2}$

1 **j** $\left(3+\frac{1}{2}x\right)^7$

x^3 term $= \binom{7}{3}3^4\left(\frac{1}{2}x\right)^3 = 35 \times 81 \times \frac{1}{8}x^3 = 354.375x^3$ or $\frac{2835}{8}x^3$

Coefficient $= 354.375$ or $\frac{2835}{8}$

k $\left(2-\frac{1}{2}x\right)^8$

x^3 term $= \binom{8}{3}2^5\left(-\frac{1}{2}x\right)^3 = 56 \times 32 \times -\frac{1}{8}x^3 = -224x^3$

Coefficient $= -224$

l $\left(5+\frac{1}{4}x\right)^5$

x^3 term $= \binom{5}{3}5^2\left(\frac{1}{4}x\right)^3 = 10 \times 25 \times \frac{1}{64}x^3 = 3.90625x^3$ or $\frac{125}{32}$

Coefficient $= 3.90625$ or $\frac{125}{32}$

2 $(2 + ax)^6$

x^2 term $= \binom{6}{2}2^4(ax)^2 = 15 \times 16a^2x^2 = 240a^2x^2$

$240a^2 = 60$

$a^2 = \frac{1}{4}$

$a = \pm\frac{1}{2}$

3 $(3 + bx)^5$

x^3 term $= \binom{5}{3}3^2(bx)^3 = 10 \times 9b^3x^3 = 90b^3x^3$

$90b^3 = -720$

$b^3 = -8$

$b = -2$

4 $(3 - ax)^4 = 3^4 + \binom{4}{1}3^3(-ax) + \binom{4}{2}3^2(-ax)^2 + \binom{4}{3}3^1(-ax)^3 + (-ax)^4$

$= 81 + 4 \times (-27ax) + 6 \times 9a^2x^2 + 4 \times (-3a^3x^3) + a^4x^4$

$= 81 - 108ax + 54a^2x^2 - 12a^3x^3 + a^4x^4$

$(2 + x)(3 - ax)^4 = (2 + x)(81 + 108ax + 54a^2x^2 - 12a^3x^3 + a^4x^4)$

x^3 term $= 2 \times (-12a^3x^3) + x \times 54a^2x^2$

$= -24a^3x^3 + 54a^2x^3$

4 $-24a^3 + 54a^2 = 30$

$-4a^3 + 9a^2 = 5$

$0 = 4a^3 - 9a^2 + 5$

$0 = (a-1)(4a^2 - 5a - 5)$

Either $a = 1$ or $4a^2 - 5a - 5 = 0 \Rightarrow a = \dfrac{5 \pm \sqrt{25+80}}{8} = \dfrac{5 \pm \sqrt{105}}{8}$

Possible values of a are 1, $\dfrac{5+\sqrt{105}}{8}$ and $\dfrac{5-\sqrt{105}}{8}$.

5 a $(1-2x)^p = 1^p + \binom{p}{1}1^{p-1}(-2x) + \binom{p}{2}1^{p-2}(-2x)^2 + \ldots$

$\binom{p}{1} = \dfrac{p!}{1!(p-1)!} = p$

$\binom{p}{2} = \dfrac{p!}{2!(p-2)!} = \dfrac{p(p-1)}{2}$

So $(1-2x)^p = 1 + p(-2x) + \dfrac{p(p-1)}{2}(-2x)^2 + \ldots$

$= 1 - 2px + 2p(p-1)x^2 + \ldots$

x^2 term $= 2p(p-1)x^2$

$2p(p-1) = 40$

$p^2 - p - 20 = 0$

$(p-5)(p+4) = 0$

$p > 0$, so $p = 5$

b Coefficient of $x = -2p = -10$

c x^3 term $= \binom{p}{3}1^{p-3}(-2x)^3$

$= \dfrac{p(p-1)(p-2)}{3!}(-2x)^3$

$= \dfrac{5 \times 4 \times 3}{3!}(-8x^3)$

$= -80x^3$

Coefficient of $x^3 = -80$

6 a $(5+px)^{30} = 5^{30} + \binom{30}{1}5^{29}(px) + \binom{30}{2}5^{28}(px)^2 + \ldots$

$= 5^{30} + 5^{29}(30px) + 5^{28}(435p^2x^2) + \ldots$

b $5^{28}(435p^2) = 29 \times 5^{29}(30p)$

$435p^2 = 29 \times 5(30p)$

$435p^2 = 4350p$

$p^2 = 10p$

$p^2 - 10p = 0$

$p(p-10) = 0$

$p = 0$ or $p = 10$

p is a non-zero constant, so $p = 10$

7 a $(1+qx)^{10} = 1^{10} + \binom{10}{1}1^9(qx) + \binom{10}{2}1^8(qx)^2 + \binom{10}{3}1^7(qx)^3 + \ldots$
$= 1 + 10qx + 45q^2x^2 + 120q^3x^3 + \ldots$

b Coefficient of x^3 is $120q^3$
Coefficient of x is $10q$
So $120q^3 = 108 \times 10q$
$\Rightarrow 120q^3 - 1080q = 0$
$\Rightarrow 120q(q^2 - 9) = 0$
$\Rightarrow 120q(q+3)(q-3) = 0$
$q = 0$, $q = -3$ or $q = 3$
But as q is non-zero, $q = \pm 3$.

8 a $(1+px)^{11} = 1^{11} + \binom{11}{1}1^{10}(px) + \binom{11}{2}1^9(px)^2 + \ldots$
$= 1 + 11px + 55p^2x^2 + \ldots$

b $11p = 77$ and $55p^2 = q$
$p = 7$
$q = 55 \times 7^2 = 2695$
$p = 7$, $q = 2695$

9 a $(1+px)^{15} = 1^{15} + \binom{15}{1}1^{14}(px)^1 + \binom{15}{2}1^{13}(px)^2 + \ldots$
$= 1 + 15px + 105p^2x^2 + \ldots$

b $15p = -q$ and $105p^2 = 5q$
$21p^2 = q$
Substituting $15p = -q$ into $21p^2 = q$:
$21p^2 = -15p$
$21p^2 + 15p = 0$
$3p(7p + 5) = 0$
$p = 0$ or $-\frac{5}{7}$
p is a non-zero constant, so $p = -\frac{5}{7}$
$q = -15 \times -\frac{5}{7} = \frac{75}{7} = 10\frac{5}{7}$
$p = -\frac{5}{7}$, $q = 10\frac{5}{7}$

10 $(1+x)^{30}$
x^9 term $= \binom{30}{9}1^{21}x^9 = 14\,307\,150x^9$
x^{10} term $= \binom{30}{10}1^{20}x^{10} = 30\,045\,015x^{10}$
$p = 14\,307\,150$ and $q = 30\,045\,015$
$\frac{q}{p} = \frac{30\,045\,015}{14\,307\,150} = 2.1$ (to 2 s.f.)

Challenge

a $(3-2x^2)^9$

x^4 term $= \binom{9}{2}3^7\left(-2x^2\right)^2$

$= 36 \times 2187 \times 4x^4$

$= 314\,928x^4$

The coefficient of x^4 in the binomial expansion of $(3-2x^2)^9$ is 314 928.

b $\left(\frac{5}{x}+x^2\right)^8$

x^4 term $= \binom{8}{4}\left(\frac{5}{x}\right)^4\left(x^2\right)^4$

$= 70 \times \left(\frac{625}{x^4}\right) \times x^8$

$= 43\,750x^4$

The coefficient of x^4 in the binomial expansion of $\left(\frac{5}{x}+x^2\right)^8$ is 43 750.

The binomial expansion 8E

1 a $\left(1-\frac{x}{10}\right)^6 = 1^6 + \binom{6}{1}1^5\left(-\frac{x}{10}\right) + \binom{6}{2}1^4\left(-\frac{x}{10}\right)^2 + \binom{6}{3}1^3\left(-\frac{x}{10}\right)^3 + \ldots$

$= 1 - 0.6x + 0.15x^2 - 0.02x^3 + \ldots$

b We want $(1 - \frac{x}{10}) = 0.99$

$\frac{x}{10} = 0.01$

$x = 0.1$

Substituting $x = 0.1$ into the expansion for $\left(1-\frac{x}{10}\right)^6$:

$0.99^6 \approx 1 - 0.6(0.1) + 0.15(0.1)^2 - 0.02(0.1)^3$

$\approx 0.941\,48$

2 a $\left(2+\frac{x}{5}\right)^{10} = 2^{10} + \binom{10}{1}2^9\left(\frac{x}{5}\right) + \binom{10}{2}2^8\left(\frac{x}{5}\right)^2 + \binom{10}{3}2^7\left(\frac{x}{5}\right)^3 + \ldots$

$= 1024 + 1024x + 460.8x^2 + 122.88x^3 + \ldots$

b We want $\left(2+\frac{x}{5}\right)^{10} = 2.1$

$\frac{x}{5} = 0.1$

$x = 0.5$

Substituting $x = 0.5$ into the expansion for $\left(2+\frac{x}{5}\right)^{10}$:

$2.1^{10} \approx 1024 + 1024(0.5) + 460.8(0.5)^2 + 122.88(0.5)^3$

≈ 1666.56

3 $(1 - 3x)^5 = 1^5 + \binom{5}{1}1^4(-3x) + \binom{5}{2}1^3(-3x)^2 + \ldots$

$= 1 - 15x + 90x^2 + \ldots$

$(2 + x)(1 - 3x)^5 = (2 + x)(1 - 15x + 90x^2 + \ldots)$

$= 2 - 30x + 180x^2 + \ldots + x - 15x^2 + \ldots$

$\approx 2 - 29x + 165x^2$

4 $(3 + x)^4 = 3^4 + \binom{4}{1}3^3x + \binom{4}{2}3^2x^2 + \ldots$

$= 81 + 108x + 54x^2 + \ldots$

$(2 - x)(3 + x)^4 = (2 - x)(81 + 108x + 54x^2 + \ldots)$

$= 162 + 216x + 108x^2 + \ldots - 81x - 108x^2 + \ldots$

$\approx 162 + 135x + 0x^2 + \ldots$

$a = 162$, $b = 135$, $c = 0$

5 **a** $(1+2x)^8 = 1^8 + \binom{8}{1}1^7(2x) + \binom{8}{2}1^6(2x)^2 + \binom{8}{3}1^5(2x)^3 + \ldots$
$= 1 + 16x + 112x^2 + 448x^3 + \ldots$

b We want $(1 + 2x) = 1.02$
$2x = 0.02$
$x = 0.01$

Substituting $x = 0.01$ into the expansion for $(1 + 2x)^8$:
$1.02^8 \approx 1 + 16(0.01) + 112(0.01)^2 + 448(0.01)^3$
$\approx 1.171\,648$

6 **a** $(1-5x)^{30} = 1^{30} + \binom{30}{1}1^{29}(-5x) + \binom{30}{2}1^{28}(-5x)^2 + \binom{30}{3}1^{27}(-5x)^3 + \ldots$
$= 1 - 150x + 10\,875x^2 - 507\,500x^3 + \ldots$

b We want $(1 - 5x) = 0.995$
$5x = 0.005$
$x = 0.001$

Substituting $x = 0.001$ into the expansion for $(1 - 5x)^{30}$
$0.995^{30} \approx 1 - 150(0.001) + 10\,875(0.001)^2 - 507\,500(0.001)^3$
$\approx 0.860\,368$

c $0.995^{30} = 0.860\,384$ (to 6 d.p.)

Percentage error $= \dfrac{0.860\,384 - 0.860\,368}{0.860\,384} \times 100 = 0.0019\%$

7 **a** $\left(3-\frac{x}{5}\right)^{10} = 3^{10} + \binom{10}{1}3^9\left(-\frac{x}{5}\right) + \binom{10}{2}3^8\left(-\frac{x}{5}\right)^2 + \ldots$
$= 59\,049 - 39\,366x + 11\,809.8x^2 + \ldots$

b We want $\left(3-\frac{x}{5}\right)^{10} = 2.98$
$\frac{x}{5} = 0.02$
$x = 0.1$

Substitute $x = 0.1$ into the expansion for $\left(3-\frac{x}{5}\right)^{10}$.

8 **a** $(1-3x)^5 = 1^5 + \binom{5}{1}1^4(-3x) + \binom{5}{2}1^3(-3x)^2 + \binom{5}{3}1^2(-3x)^3 + \ldots$
$= 1 - 15x + 90x^2 - 270x^3 + \ldots$

b For the expansion $(1 - 3x)^5$, only use the first two terms as x^2 and higher powers can be ignored.
$(1 + x)(1 - 3x)^5 \approx (1 + x)(1 - 15x)$
$\approx 1 - 15x + x - 15x^2$
$\approx 1 - 14x$

9 **a** So that higher powers of p can be ignored as they tend to 0.

b $(1-p)^{200} = 1^{200} + \binom{200}{1}1^{199}(-p) + \binom{200}{2}1^{198}(-p)^2 + \ldots$

$\approx 1 - 200p + 19\,900p^2$

c $1 - 200p + 19\,900p^2 = 0.92$

$19\,900p^2 - 200p + 0.08 = 0$

$p = \dfrac{-(-200) \pm \sqrt{(-200)^2 - 4(19\,900)(0.08)}}{2(19\,900)}$

$= \dfrac{200 \pm \sqrt{33\,632}}{39\,800}$

$p = 0.009\,63$ or $p = 0.000\,417$ (to 3 s.f.)

As $p < 0.001$, the maximum value for p would be $0.000\,417$.

The binomial expansion, Mixed Exercise 8

1 a ${}^{16-1}C_{4-1} = {}^{15}C_3 = 455$
${}^{16-1}C_{5-1} = {}^{15}C_4 = 1365$

b The coefficients are 1, 15, 105, 455, 1365, ...
x^3 term of $(1+2x)^{15} = 455(1)^{12}(2x)^3 = 3640x^3$
Coefficient = 3640

2 $\binom{45}{17} = \dfrac{45!}{17!a!}$
$\binom{45}{17} = \dfrac{45!}{17!28!}$
$a = 28$

3 a When $n = 5$ and $p = 0.5$,
$\binom{20}{n} p^n (1-p)^{20-n} = \binom{20}{5} 0.5^5 (1-0.5)^{20-15}$
$= 0.0148$ (to 3 s.f.)

b When $n = 0$ and $p = 0.7$,
$\binom{20}{n} p^n (1-p)^{20-n} = \binom{20}{0} 0.7^0 (1-0.7)^{20}$
$= 0.000\,000\,000\,034\,9$ (to 3 s.f.)

c When $n = 13$ and $p = 0.6$,
$\binom{20}{n} p^n (1-p)^{20-n} = \binom{20}{13} 0.6^{13} (1-0.6)^{7}$
$= 0.166$ (to 3 s.f.)

4 $$\left(1-\frac{3x}{2}\right)^p = 1 + \binom{p}{1}1^{p-1}\left(-\frac{3x}{2}\right) + \binom{p}{2}1^{p-2}\left(-\frac{3x}{2}\right) + \binom{p}{3}1^{p-3}\left(-\frac{3x}{2}\right) + \ldots$$
$$= 1 + p\left(-\frac{3x}{2}\right) + \frac{p(p-1)}{2!}\left(-\frac{3x}{2}\right)^2 + \frac{p(p-1)(p-2)}{3!}\left(-\frac{3x}{2}\right)^3 + \ldots$$

a Coefficient of x is $-\dfrac{3p}{2}$.

$-\dfrac{3p}{2} = -24$
$p = 16$

b Coefficient of $x^2 = \dfrac{p(p-1)}{2} \times \dfrac{9}{4} = \dfrac{16\times 15}{2} \times \dfrac{9}{4} = 270$

c Coefficient of $x^3 = -\dfrac{p(p-1)(p-2)}{3!} \times \dfrac{27}{8} = -\dfrac{16\times 15\times 14}{3\times 2} \times \dfrac{27}{8} = -1890$

5 $(2-x)^{13} = 2^{13} + \binom{13}{1}2^{12}(-x) + \binom{13}{2}2^{11}(-x)^2 + \ldots$

$= 8192 + 13 \times (-4096x) + 78 \times 2048x^2 + \ldots$

$= 8192 - 53\,248x + 159\,744x^2 + \ldots$

$= A + Bx + Cx^2 + \ldots$

So $A = 8192$, $B = -53\,248$, $C = 159\,744$

6 a $(1-2x)^{10} = 1 + \binom{10}{1}1^9(-2x) + \binom{10}{2}1^8(-2x)^2 + \binom{10}{3}1^7(-2x)^3 + \ldots$

$= 1 + 10 \times (-2x) + 45 \times (-2x)^2 + 120 \times (-2x)^3 + \ldots$

$= 1 - 20x + 180x^2 - 960x^3 + \ldots$

b We need $(1-2x) = 0.98$

$2x = 0.02$

$x = 0.01$

Substituting $x = 0.01$ into the expansion for $(1-2x)^{10}$:

$0.98^{10} \approx 1 - 20 \times 0.01 + 180 \times 0.01^2 - 960 \times 0.01^3$

$= 0.81\,704 + \ldots$

7 a $(2-3x)^{10} = 2^{10} + \binom{10}{1}2^9(-3x) + \binom{10}{2}2^8(-3x)^2 + \binom{10}{3}2^7(-3x)^3 + \ldots$

$= 1024 + 10 \times (-1536x) + 45 \times 2304x^2 + 120 \times (-3456x^3) + \ldots$

$= 1024 - 15\,360x + 103\,680x^2 - 414\,720x^3 + \ldots$

b We require $(2-3x) = 1.97$

$3x = 0.03$

$x = 0.01$

Substituting $x = 0.01$ in the expansion for $(2-3x)^{10}$:

$1.97^{10} \approx 1024 - 15\,360 \times 0.01 + 103\,680 \times 0.01^2 - 414\,720 \times 0.01^3$

$= 1024 - 153.6 + 10.368 - 0.414\,72$

$= 880.35$ (to 2 d.p.)

8 a $(3+2x)^4 = 3^4 + \binom{4}{1}3^3(2x) + \binom{4}{2}3^2(2x)^2 + \binom{4}{3}3(2x)^3 + (2x)^4$

$= 3^4 + 4 \times 54x + 6 \times 36x^2 + 4 \times 24x^3 + 16x^4$

$= 81 + 216x + 216x^2 + 96x^3 + 16x^4$

b Substituting $x = -x$:

$(3-2x)^4 = 81 + 216(-x) + 216(-x)^2 + 96(-x)^3 + 16(-x)^4$

$= 81 - 216x + 216x^2 - 96x^3 + 16x$

c Using parts **a** and **b**:

$$\begin{array}{rl} (3+2x)^4 + (3-2x)^4 = & 81 + 216x + 216x^2 + 96x^3 + 16x^4 \\ & \underline{+\,81 - 216x + 216x^2 - 96x^3 + 16x^4} \\ & 162 \qquad\quad + 432x^2 \qquad\quad + 32x^4 \end{array}$$

Substituting $x = \sqrt{2}$ into both sides of this expansion gives:

$\left(3+2\sqrt{2}\right)^4 + \left(3-2\sqrt{2}\right)^4 = 164 + 432\left(\sqrt{2}\right)^2 + 32\left(\sqrt{2}\right)^4$

8 c $\left(3+2\sqrt{2}\right)^4+\left(3-2\sqrt{2}\right)^4 = 162 + 432\times 2 + 32\times 4$
$= 1154$

9 $\left(1+\frac{x}{2}\right)^n \ldots = 1+\binom{n}{1}1^{n-1}\left(\frac{x}{2}\right)+\binom{n}{2}1^{n-2}\left(\frac{x}{2}\right)^2+\binom{n}{3}1^{n-3}\left(\frac{x}{2}\right)^3+\ldots$

$=1+n\left(\frac{x}{2}\right)+\frac{n(n-1)}{2!}\left(\frac{x}{2}\right)^2+\frac{n(n-1)(n-2)}{3!}\left(\frac{x}{2}\right)^3+\frac{n(n-1)(n-2)(n-3)}{4!}\left(\frac{x}{2}\right)^4+\ldots$

a x^2 term $= \frac{n(n-1)}{2!\times 4}x^2$

$\frac{n(n-1)}{2!\times 4} = 7$

$n(n-1) = 56$

$n^2 - n - 56 = 0$

$(n-8)(n+7) = 0$

n is a positive integer, so $n = 8$

b Coefficient of $x^4 = \frac{n(n-1)(n-2)(n-3)}{4!}\times\frac{1}{2^4}$

$= \frac{\cancel{8}^{2}\times 7\times\cancel{6}\times 5}{\cancel{4}\times\cancel{3}\times\cancel{2}\times 1}\times\frac{1}{\cancel{16}_{8}}$

$= \frac{35}{8}$

10 a $(3+10x)^4 = 3^4+\binom{4}{1}3^3(10x)+\binom{4}{2}3^2(10x)^2+\binom{4}{3}3(10x)^3+(10x)^4$

$= 3^4 + 4\times 270x + 6\times 900x^2 + 4\times 3000x^3 + 10\,000x^4$

$= 81 + 1080x + 5400x^2 + 12\,000x^3 + 10\,000x^4$

b We require $(3 + 10x) = 1003$

$10x = 1000$

$x = 100$

Substituting $x = 100$ in the expansion of $(3 + 10x)^4$:

$1003^4 = 81 + 1080\times 100 + 5400\times 100^2 + 12\,000\times 100^3 + 10\,000\times 100^4$

$= 81 + 108\,000 + 54\,000\,000 + 12\,000\,000\,000 + 1\,000\,000\,000\,000$

$$\begin{array}{r} 1\,000\,000\,000\,000 \\ 12\,000\,000\,000 \\ 54\,000\,000 \\ 108\,000 \\ 81 \\ \hline 1\,012\,054\,108\,081 \end{array}$$

$1003^4 = 1\,012\,054\,108\,081$

11 a $(1 + 2x)^{12}$

$= 1^{12} + \binom{12}{1}1^{11}(2x) + \binom{12}{2}1^{10}(2x)^2 + \binom{12}{3}1^9(2x)^3 + \ldots$

$= 1 + 12 \times 2x + 66 \times 4x^2 + 220 \times 8x^3 + \ldots$

$= 1 + 24x + 264x^2 + 1760x^3 + \ldots$

b We want $(1 + 2x) = 1.02$

$2x = 0.02$

$x = 0.01$

Substituting $x = 0.01$ in the expansion for $(1 + 2x)^{12}$:

$1.02^{12} \approx 1 + 24 \times 0.01 + 264 \times 0.01^2 + 1760 \times 0.01^3$

$= 1.268\,16$

c Using a calculator:

$1.02^{12} = 1.268\,241\,795$

d Error $= \dfrac{1.268\ 241\ 795\ -\ 1.268\ 16}{1.268\ 241\ 795} \times 100\ = 0.006\,45\%$

12 $\left(x - \dfrac{1}{x}\right)^5$ has coefficients and terms

1	5	10	10	5	1
x^2	$x^4\left(-\frac{1}{x}\right)$	$x^3\left(-\frac{1}{x}\right)^2$	$x^2\left(-\frac{1}{x}\right)^3$	$x\left(-\frac{1}{x}\right)^4$	$\left(-\frac{1}{x}\right)^5$

Putting these together gives:

$$\left(x - \frac{1}{x}\right)^5 = 1x^5 + 5x^4\left(-\frac{1}{x}\right) + 10x^3\left(-\frac{1}{x}\right)^2 + 10x^2\left(-\frac{1}{x}\right)^3 + 5x\left(-\frac{1}{x}\right)^4 + 1\left(-\frac{1}{x}\right)^5$$

$$= x^5 - 5x^3 + 10x - \frac{10}{x} + \frac{5}{x^3} - \frac{1}{x^5}$$

13 a $(2k + x)^n = (2k)^n + \binom{n}{1}(2k)^{n-1}x + \binom{n}{2}(2k)^{n-2}x^2 + \binom{n}{3}(2k)^{n-3}x^3 + \ldots$

Coefficient of x^2 = coefficient of x^3

$$\binom{n}{2}(2k)^{n-2} = \binom{n}{3}(2k)^{n-3}$$

$$\frac{\cancel{n}!}{(n-2)!2!}(2k)^{n-2} = \frac{\cancel{n}!}{(n-3)!3!}(2k)^{n-3}$$

$$\frac{(2k)^{n-2}}{(2k)^{n-3}} = \frac{(n-2)!2!}{(n-3)!3!}$$

$$(2k)^1 = \frac{(n-2)!2!}{(n-3)!3!}$$

$$= \frac{(n-2)\times(\mathrm{n}-3)!2!}{(n-3)!3!}$$

13 a $2k = \dfrac{(n-2)\times \cancel{2}}{\cancel{6}_{3}}$

$3 \times 2k = n - 2$
$6k = n - 2$
$n = 6k + 2$

b If $k = \dfrac{2}{3}$ then $n = 6\times\dfrac{2}{3}+2=6$

$$\left(2\times\frac{2}{3}+x\right)^6 = \left(\frac{4}{3}+x\right)^6$$
$$= \left(\frac{4}{3}\right)^6 + \binom{6}{1}\left(\frac{4}{3}\right)^5 x + \binom{6}{2}\left(\frac{4}{3}\right)^4 x^2 + \binom{6}{3}\left(\frac{4}{3}\right)^3 x^3 + \ldots$$
$$= \frac{4096}{729} + \frac{2048}{81}x + \frac{1280}{27}x^2 + \frac{1280}{27}x^3 + \ldots$$

14 a $(2 + x)^6 = 2^6 + \binom{6}{1}2^5x + \binom{6}{2}2^4x^2 + \binom{6}{3}2^3x^3 + \binom{6}{4}2^2x^4 + \binom{6}{5}2x^5 + x^6$
$= 64 + 192x + 240x^2 + 160x^3 + 60x^4 + 12x^5 + x^6$

b With $x = \sqrt{3}$

$$\left(2+\sqrt{3}\right)^6 = 64+192\sqrt{3}+240\left(\sqrt{3}\right)^2+160\left(\sqrt{3}\right)^3+60\left(\sqrt{3}\right)^4+12\left(\sqrt{3}\right)^5+\left(\sqrt{3}\right)^6 \quad \textbf{(1)}$$

With $x = -\sqrt{3}$

$$\left(2-\sqrt{3}\right)^6 = 64+192\left(-\sqrt{3}\right)+240\left(-\sqrt{3}\right)^2+160\left(-\sqrt{3}\right)^3+60\left(-\sqrt{3}\right)^4+12\left(-\sqrt{3}\right)^5+\left(-\sqrt{3}\right)^6$$
$$\left(2-\sqrt{3}\right)^6 = 64-192\sqrt{3}+240\left(\sqrt{3}\right)^2-160\left(\sqrt{3}\right)^3+60\left(\sqrt{3}\right)^4-12\left(\sqrt{3}\right)^5+\left(\sqrt{3}\right)^6 \quad \textbf{(2)}$$

(**1**)− (**2**) gives:

$$\left(2+\sqrt{3}\right)^6-\left(2-\sqrt{3}\right)^6 = 384\sqrt{3}+320\left(\sqrt{3}\right)^3+24\left(\sqrt{3}\right)^5$$
$$= 384\sqrt{3}+320\times3\sqrt{3}+24\times3\times3\sqrt{3}$$
$$= 384\sqrt{3}+960\sqrt{3}+216\sqrt{3}$$
$$= 1560\sqrt{3}$$

Hence $k = 1560$

15 a The term in x^2 of $(2 + kx)^8$ is

$\binom{8}{2}2^6(kx)^2 = 28 \times 64k^2x^2 = 1792k^2x^2$

$1792k^2 = 2800$
$k^2 = 1.5625$
$k = \pm1.25$
k is positive, so $k = 1.25$

15 b Term in x^3 of $(2 + kx)^8$ is

$\binom{8}{3}2^5(kx)^3 = 56 \times 32k^3x^3 = 1792k^3x^3$

Coefficient of x^3 term is $1792k^3 = 1792 \times 1.25^3 = 3500$

16 a $(2 + x)^5$ has coefficients and terms

1	5	10	10	5	1
2^5	2^4x	2^3x^2	2^2x^3	$2x^4$	x^5

Putting these together gives:

$(2 + x)^5 = 1 \times 2^5 + 5 \times 2^4x + 10 \times 2^3x^2 + 10 \times 2^2x^3 + 5 \times 2x^4 + 1 \times x^5$

$(2 + x)^5 = 32 + 80x + 80x^2 + 40x^3 + 10x^4 + x^5$

Substituting $x = -x$:

$(2 - x)^5 = 32 - 80x + 80x^2 - 40x^3 + 10x^4 - x^5$

Adding:

$(2 + x)^5 + (2 - x)^5 = 64 + 160x^2 + 20x^4$

So $A = 64$, $B = 160$ and $C = 20$

b $(2 + x)^5 + (2 - x)^5 = 349$

$64 + 160x^2 + 20x^4 = 349$

$20x^4 + 160x^2 - 285 = 0$

$4x^4 + 32x^2 - 57 = 0$

Substituting $y = x^2$:

$4y^2 + 32y - 57 = 0$

$(2y - 3)(2y + 19) = 0$

$y = \frac{3}{2}, -\frac{19}{2}$

But $y = x^2$, so $x^2 = \frac{3}{2} \Rightarrow x = \pm\sqrt{\frac{3}{2}}$

17 a x^3 term $= \binom{5}{3}2^2(px)^3 = 10 \times 4p^3x^3 = 40p^3x^3$

$40p^3 = 135$

$p^3 = 3.375$

$p = 1.5$

b x^4 term $= \binom{5}{4}2(px)^4$

$= 5 \times 2p^4x^4$

$= 5 \times 2(1.5)^4x^4$

$= 50.625x^4$

Coefficient = 50.625

18 $\left(\frac{x^2}{2}-\frac{2}{x}\right)^9$

Constant term $= \binom{9}{6}\left(\frac{x^2}{2}\right)^3\left(-\frac{2}{x}\right)^6$

$= 84\times\left(\frac{x^6}{8}\right)\times\left(\frac{64}{x^6}\right)$

$= 672$

19 a $(2+px)^7 = 2^7+\binom{7}{1}2^6(px)^1+\binom{7}{2}2^5(px)^2+\ldots$

$= 128 + 448px + 672p^2x^2 + \ldots$

b $448p = 2240 \Rightarrow p = 5$

$672p^2 = q$

$672\times 5^2 = q$

$q = 16\,800$

$p = 5$ and $q = 16\,800$

20 a $(1-px)^{12} = 1^{12}+\binom{12}{1}1^{11}(-px)+\binom{12}{2}1^{10}(-px)^2+\ldots$

$= 1 - 12px + 66p^2x^2 + \ldots$

b $-12p = q$ and $66p^2 = 6q$

$11p^2 = q$

Substituting $-12p = q$ into $11p^2 = q$ gives:

$11p^2 = -12p$

$11p^2 + 12p = 0$

$p(11p + 12) = 0$

$p = 0$ or $-\frac{12}{11} = -1\frac{1}{11}$

p is a non-zero constant, so $p = -1\frac{1}{11}$

$q = -12\times -\frac{12}{11} = \frac{144}{11} = 13\frac{1}{11}$

$p = -1\frac{1}{11}$ and $q = 13\frac{1}{11}$

21 a $\left(2+\frac{x}{2}\right)^7 = 2^7+\binom{7}{1}2^6\left(\frac{x}{2}\right)+\binom{7}{2}2^5\left(\frac{x}{2}\right)^2+\ldots$

$= 128 + 224x + 168x^2 + \ldots$

b We want $\left(2+\frac{x}{2}\right) = 2.05$

$\frac{x}{2} = 0.05$

$x = 0.1$

Substitute $x = 0.1$ into the expansion for $\left(2+\frac{x}{2}\right)^7$ assuming the terms after x^2 are negligible.

22 $(4 + kx)^5$

x^3 term $= \binom{5}{3} 4^2 (kx)^3$

$= 10 \times 16 \times k^3x^3$

$= 160k^3x^3$

$160k^3 = 20$

$k^3 = \frac{1}{8}$

$k = \frac{1}{2}$

Challenge

a $(3 + x)^5 = 3^5 + \binom{5}{1} 3^4 x + \binom{5}{2} 3^3 x^2 + \ldots$

$= 243 + 405x + 270x^2 + \ldots$

$(2 - px)(3 + x)^5 = (2 - px)(243 + 405x + 270x^2 + \ldots)$

$= 486 + 810x + 540x^2 - 243px - 405px^2 + \ldots$

x^2 term $= (540 - 405p)x^2$

$540 - 405p = 0$

$405p = 540$

$p = \frac{540}{405} = \frac{4}{3}$

b $(1 + 2x)^8 = 1^8 + \binom{8}{1} 1^7 (2x) + \binom{8}{2} 1^6 (2x)^2 + \ldots$

$= 1 + 16x + 112x^2 + \ldots$

$(2 - 5x)^7 = 2^7 + \binom{7}{1} 2^6 (-5x) + \binom{7}{2} 2^5 (-5x)^2 + \ldots$

$= 128 - 2240x + 16\,800x^2 + \ldots$

The x^2 term in the expansion of $(1 + 2x)^8(2 - 5x)^7$

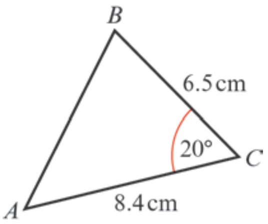
$= 1 \times 16\,800x^2 + 16x \times (-2240x) + 128 \times 112x^2$

$= -4704x^2$

The coefficient of the x^2 term is -4704.

Trigonometric ratios 9A

1 a



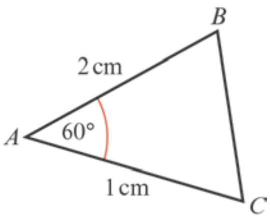
Using $c^2 = a^2 + b^2 - 2ab\cos C$

$AB^2 = 6.5^2 + 8.4^2 - 2\times 6.5\times 8.4\times \cos 20°$

$AB^2 = 10.1955\ldots$

$AB = \sqrt{10.1955\ldots} = 3.19 \text{ cm } (3 \text{ s.f.})$

b



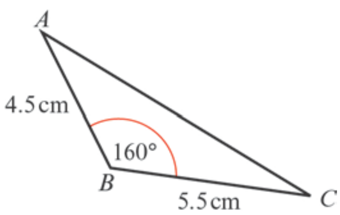
Using $a^2 = b^2 + c^2 - 2bc\cos A$

$BC^2 = 1^2 + 2^2 - 2\times 1\times 2\times \cos 60°$

$BC^2 = 3$

$BC = \sqrt{3} = 1.73 \text{ cm } (3 \text{ s.f.})$

c



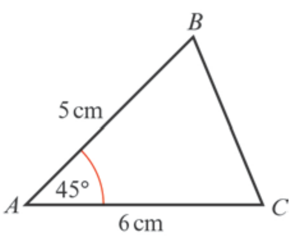
Using $b^2 = a^2 + c^2 - 2ac\cos B$

$AC^2 = 5.5^2 + 4.5^2 - 2\times 5.5\times 4.5\times \cos 160°$

$AC^2 = 97.014\ldots$

$AC = \sqrt{97.014\ldots3} = 9.85 \text{ cm } (3 \text{ s.f.})$

d



Using $a^2 = b^2 + c^2 - 2bc\cos A$

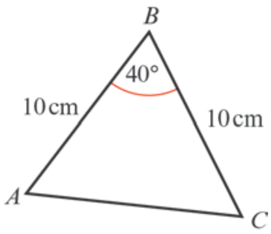
d $BC^2 = 6^2 + 5^2 - 2\times 6\times 5\times \cos 45°$

$= 18.573\ldots$

$BC = \sqrt{18.573\ldots}$

$= 4.31 \text{ cm } (3 \text{ s.f.})$

e



(This is an isosceles triangle and so you could use right-angled triangle work.)

Using $b^2 = a^2 + c^2 - 2ac\cos B$

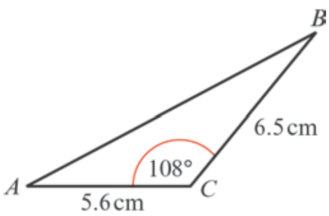
$AC^2 = 10^2 + 10^2 - 2\times 10\times 10\times \cos 40°$

$= 46.791\ldots$

$AC = \sqrt{46.791\ldots}$

$= 6.84 \text{cm } (3 \text{ s.f.})$

f



Using $c^2 = a^2 + b^2 - 2ab\cos C$

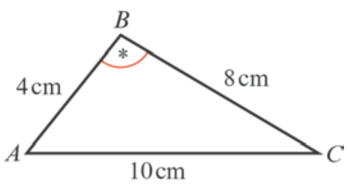
$AB^2 = 6.5^2 + 5.6^2 - 2\times 6.5\times 5.6\times \cos 108°$

$= 96.106\ldots$

$AB = \sqrt{96.106\ldots}$

$= 9.80 \text{ cm } (3 \text{ s.f.})$

2 a



Using $\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$

$\cos B = \dfrac{8^2 + 4^2 - 10^2}{2\times 8\times 4}$

2 a $\cos B = -\frac{20}{64}$

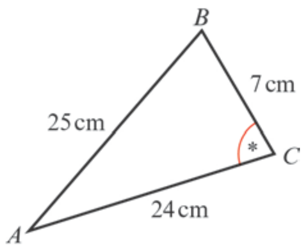
$= -\frac{5}{16}$

$B = \cos^{-1}\left(-\frac{5}{16}\right) = 108.2\ldots°$

$= 108°\ (3\text{ s.f.})$

We can use a calculator to find directly an obtuse angle with a negative cosine value.

b



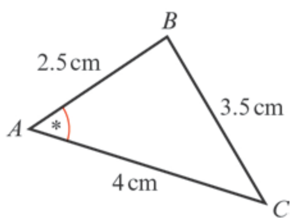
Using $\cos C = \frac{a^2+b^2-c^2}{2ab}$

$\cos C = \frac{7^2+24^2-25^2}{2\times7\times24}$

$= 0$

$\Rightarrow C = 90°$

c



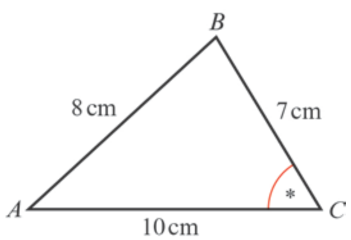
Using $\cos A = \frac{b^2+c^2-a^2}{2bc}$

$\cos A = \frac{4^2+2.5^2-3.5^2}{2\times4\times2.5}$

$= \frac{1}{2}$

$A = \cos^{-1}\left(\frac{1}{2}\right) = 60°$

d



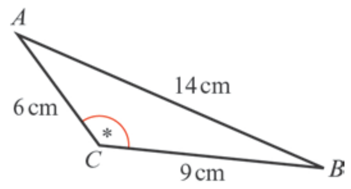
Using $\cos C = \frac{a^2+b^2-c^2}{2ab}$

d $\cos C = \frac{7^2+10^2-8^2}{2\times7\times10}$

$= 0.6071$

$\Rightarrow C = 52.6°\ (3\text{ s.f.})$

e



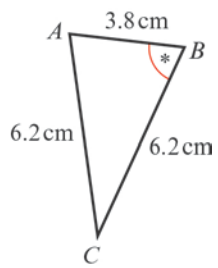
Using $\cos C = \frac{a^2+b^2-c^2}{2ab}$

$\cos C = \frac{9^2+6^2-14^2}{2\times9\times6}$

$= -0.7314\ldots$

$\Rightarrow C = 137°\ (3\text{ s.f.})$

f



(This is an isosceles triangle so you could use right-angled triangle trigonometry.)

Using $\cos B = \frac{a^2+c^2-b^2}{2ac}$

$\cos B = \frac{6.2^2+3.8^2-6.2^2}{2\times6.2\times3.8}$

$= 0.3064\ldots$

$\Rightarrow B = 72.2°\ (3\text{ s.f.})$

3 Use alternate angles to find angle of 40° and 180°−130° = 50°. Adding, this gives 90°. At this point, you can use Pythagoras' theorem or the cosine rule.

$c^2 = a^2+b^2-2ab\cos C$

$c^2 = 120^2+150^2-2\times120\times150\cos 90°$

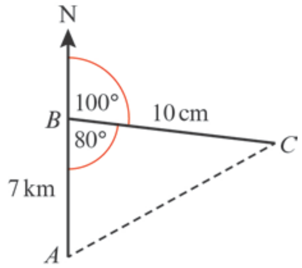
$= 14\,400 + 22\,500 - 0$

$= 36\,900$

So $c = 192.0937\ldots$

So the distance of the plane from the airport is 192 km (3 s.f.).

4



Using the cosine rule:

$b^2 = a^2 + c^2 - 2ac\cos B$

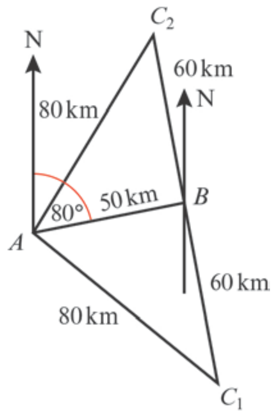
$AC^2 = 10^2 + 7^2 - 2\times 10\times 7\times \cos 80°$

$= 124.689$

$AC = \sqrt{124.689\ldots}$

$= 11.2\ \text{km}\ (3\text{ s.f.})$

5



The bearing of C from B is not given so there are two possibilities for C, using the data.

The angle A will be the same in each ΔABC.

Using $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

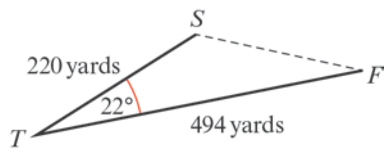
$\cos A = \dfrac{80^2 + 50^2 - 60^2}{2\times 80\times 50} = 0.6625$

$\Rightarrow A = 48.5°$

The bearing of C from A is

$80° \pm 48.5° = 128.5°$ or $031.5°$

6



Using the cosine rule:

$t^2 = f^2 + s^2 - 2fs\cos T$

6 $SF^2 = 220^2 + 494^2 - 2\times 220\times 494\cos 22°$

$= 90\,903.317$

$SF = \sqrt{90\,903.317\ldots} = 301.5\ldots$ yards

$= 302$ yards (3 s.f.)

7

$\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

$\cos A = \dfrac{5^2 + 4^2 - 6^2}{2(5)(4)}$

$\cos A = \dfrac{25 + 16 - 36}{40}$

$\cos A = \dfrac{5}{40}$

$\cos A = \dfrac{1}{8}$

8

$\cos P = \dfrac{q^2 + r^2 - p^2}{2qr}$

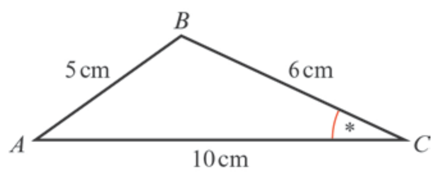
$\cos P = \dfrac{3^2 + 2^2 - 4^2}{2(3)(2)}$

$\cos P = \dfrac{9 + 4 - 16}{12}$

$\cos P = -\dfrac{3}{12}$

$\cos P = -\dfrac{1}{4}$

9



The smallest angle is C as this is opposite AB, the shortest side.

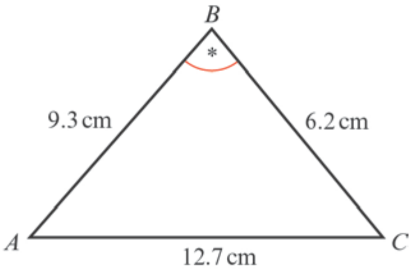
Using $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

$\cos C = \dfrac{6^2 + 10^2 - 5^2}{2\times 6\times 10}$

$= 0.925$

$C = 22.3°\ (3\text{ s.f.})$

10



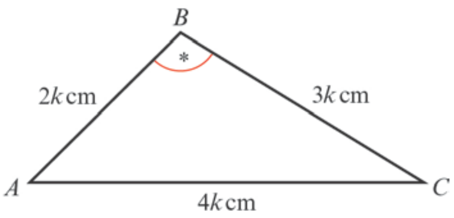
The largest angle is B as it is opposite AC.

$$\text{Using } \cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\cos B = \frac{6.2^2 + 9.3^2 - 12.7^2}{2 \times 6.2 \times 9.3} = -0.3152\ldots$$

$$B = 108.37\ldots = 108° \,(3 \text{ s.f.})$$

11



The largest angle will be opposite the side of length $4k$ cm, the longest side.

$$\text{Using } \cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

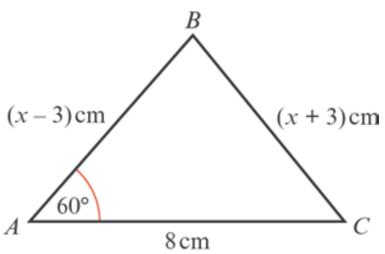
$$\cos B = \frac{9k^2 + 4k^2 - 16k^2}{2 \times 3k \times 2k}$$

$$= -0.25$$

$$B = 104.477\ldots$$

$$= 104° \,(3 \text{ s.f.})$$

12



Using $a^2 = b^2 + c^2 - 2bc\cos A$

$$(x+3)^2 = (x-3)^2 + 8^2 - 2 \times 8 \times (x-3)\cos 60°$$

$$x^2 + 6x + 9 = x^2 - 6x + 9 + 64 - 8(x-3)$$

$$x^2 + 6x + 9 = x^2 - 6x + 9 + 64 - 8x + 24$$

12

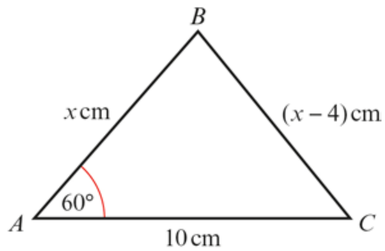
$$6x + 6x + 8x = 64 + 24$$

$$20x = 88$$

$$x = \frac{88}{20}$$

$$= 4.4 \text{ cm}$$

13



Using $a^2 = b^2 + c^2 - 2bc\cos A$

$$(x-4)^2 = 10^2 + x^2 - 2 \times 10 \times x\cos 60°$$

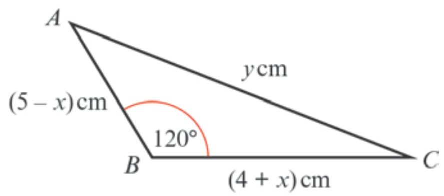
$$x^2 - 8x + 16 = 100 + x^2 - 10x$$

$$10x - 8x = 100 - 16$$

$$2x = 84$$

$$x = 42 \text{ cm}$$

14 a



Using $b^2 = a^2 + c^2 - 2ac\cos B$

$y^2 = (4 + x)^2 + (5 - x)^2 - 2(4 + x)(5 - x)\cos 120°$

$y^2 = 16 + 8x + x^2 + 25 - 10x + x^2 + (4 + x)(5 - x)$

(Note : $2\cos 120° = -1$)

$y^2 = 16 + 8x + x^2 + 25 - 10x + x^2 + 20 + x - x^2$

$= x^2 - x + 61$

b Completing the square:

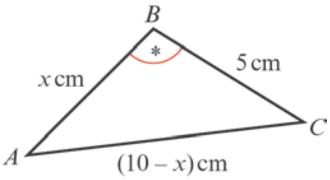
$$y^2 = \left(x - \tfrac{1}{2}\right)^2 + 61 - \tfrac{1}{4}$$

$$\Rightarrow y^2 = \left(x - \tfrac{1}{2}\right)^2 + 60\tfrac{3}{4}$$

The minimum value of y^2 occurs when $\left(x - \tfrac{1}{2}\right)^2 = 0$ i.e. when $x = \tfrac{1}{2}$.

So the minimum value of y^2 is 60.75.

15 a



$$\cos B = \frac{5^2 + x^2 - (10-x)^2}{2\times 5\times x}$$
$$= \frac{25 + x^2 - (100 - 20x + x^2)}{10x}$$
$$= \frac{25 + x^2 - 100 + 20x - x^2}{10x}$$
$$= \frac{20x - 75}{10x}$$
$$= \frac{5(4x-15)}{10x}$$
$$= \frac{4x-15}{2x}$$

b As $\cos B = -\frac{1}{7}$

$$\frac{4x-15}{2x} = -\frac{1}{7}$$
$$7(4x-15) = -2x$$
$$28x - 105 = -2x$$
$$30x = 105$$
$$x = \frac{105}{30}$$
$$= 3\tfrac{1}{2}$$

16 First find the length of the diagonal *BD*.

$a^2 = b^2 + c^2 - 2bc\cos A$

$a^2 = 120^2 + 75^2 - 2\times 120\times 75\cos 74°$

$a^2 = 14\,400 + 5625 - 4961.4724$

$a^2 = 15\,063.5276$

So a = 122.733 56...

So the length of the diagonal *BD* is 122.733 56... m.

Note that in this question you do not have to find the value of a since you only need a^2 in the next part of the calculation.

16 To find the angle between fences *BC* and *CD*:

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$
$$\cos C = \frac{135^2 + 60^2 - 122.733\,56^2}{2(135)(60)}$$
$$\cos C = \frac{18\,225 + 3600 - 15\,063.5276}{16\,200}$$
$$\cos C = 0.417\,37...$$
$$C = \cos^{-1} 0.417\,37...$$
$$= 65.33...°$$

So the angle between fences *BC* and *CD* is 65.3° (3 s.f.).

17 a $a^2 = b^2 + c^2 - 2bc\cos A$

$a^2 = 70^2 + 50^2 - 2\times 70\times 50\cos 20°$

$a^2 = 4900 + 2500 - 6577.848...$

$a^2 = 822.151\,65...$

So a = 28.673...

So the distance between ships *B* and *C* is 28.7 km (3 s.f.).

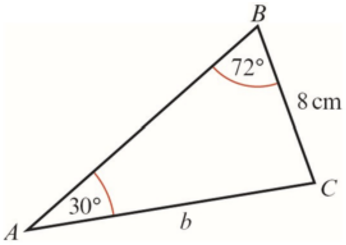
b

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$
$$\cos B = \frac{28.673^2 + 50^2 - 70^2}{2(28.673)(50)}$$
$$\cos B = \frac{822.151\,65 + 2500 - 4900}{2867.3187}$$
$$\cos B = -0.550\,28...$$
$$B = \cos^{-1} 0.550\,28...$$
$$= 123.3867...°$$

The bearing is 180° − 123.3867° = 56.6°.
So the bearing of ship *C* from ship *B* is 056.6°.

Trigonometric ratios 9B

1 a



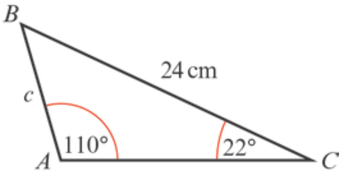
Using $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$

$\dfrac{b}{\sin 72°} = \dfrac{8}{\sin 30°}$

$\Rightarrow b = \dfrac{8\sin 72°}{\sin 30°} = 15.2 \text{ cm } (3 \text{ s.f.})$

$(\text{As } 72° > 30°,\ b > 8 \text{ cm})$

b



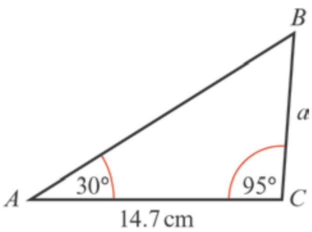
Using $\dfrac{a}{\sin A} = \dfrac{c}{\sin C}$

$\dfrac{c}{\sin 22°} = \dfrac{24}{\sin 110°}$

$\Rightarrow c = \dfrac{24\sin 22°}{\sin 110°} = 9.57 \text{ cm } (3 \text{ s.f.})$

$(\text{As } 110° > 22°,\ 24 \text{ cm} > c.)$

c



$\angle ABC = 180° - (30 + 95)°$

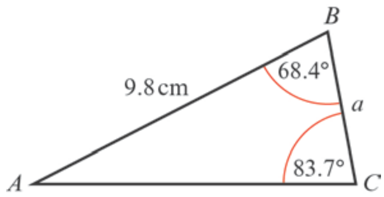
$= 55°$

Using $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$

$\dfrac{a}{\sin 30°} = \dfrac{14.7}{\sin 55°}$

$\Rightarrow a = \dfrac{14.7\sin 30°}{\sin 55°} = 8.97 \text{ cm } (3 \text{ s.f.})$

d



$\angle ABC = 180° - (68.4 + 83.7)°$

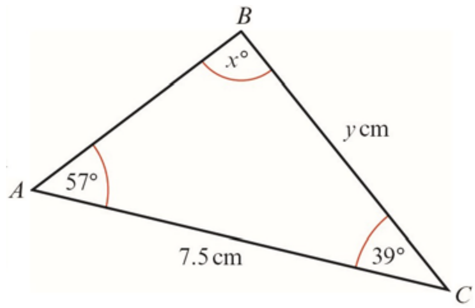
$= 27.9°$

Using $\dfrac{a}{\sin A} = \dfrac{c}{\sin C}$

$\dfrac{a}{\sin 27.9°} = \dfrac{9.8}{\sin 83.7°}$

$\Rightarrow a = \dfrac{9.8\sin 27.9°}{\sin 83.7°} = 4.61 \text{ cm } (3 \text{ s.f.})$

2 a



$x = 180° - (57 + 39)°$

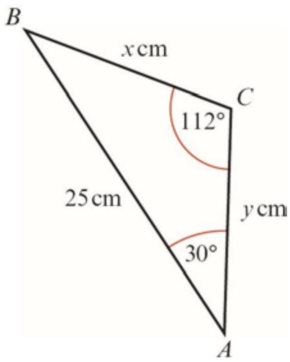
$= 84°$

Using $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$

$\dfrac{y}{\sin 57°} = \dfrac{7.5}{\sin 84°}$

$\Rightarrow y = \dfrac{7.5\sin 57°}{\sin 84°} = 6.32 \text{ cm } (3 \text{ s.f.})$

b



Using $\dfrac{a}{\sin A} = \dfrac{c}{\sin C}$

2 b $\dfrac{x}{\sin 30°}=\dfrac{25}{\sin 112°}$

$\Rightarrow x=\dfrac{25\sin 30°}{\sin 112°}=13.5 \text{ cm } (3 \text{ s.f.})$

$\angle B=180°-(112+30)°$

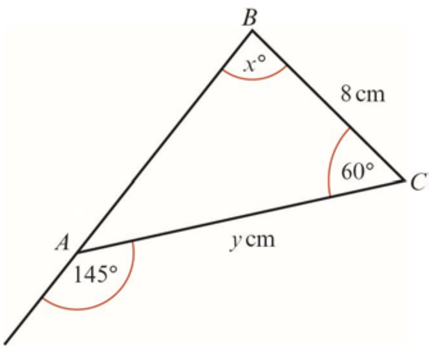
$=38°$

Using $\dfrac{b}{\sin B}=\dfrac{c}{\sin C}$

$\dfrac{y}{\sin 38°}=\dfrac{25}{\sin 112°}$

$\Rightarrow y=\dfrac{25\sin 38°}{\sin 112°}=16.6 \text{ cm } (3 \text{ s.f.})$

c



$x=180°-(60+35)°$

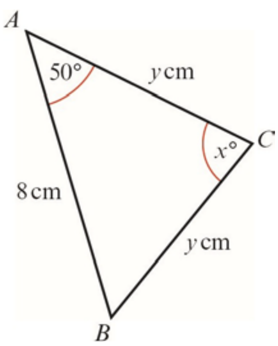
$=85°$

Using $\dfrac{b}{\sin B}=\dfrac{a}{\sin A}$

$\dfrac{y}{\sin 85°}=\dfrac{8}{\sin 35°}$

$\Rightarrow y=\dfrac{8\sin 85°}{\sin 35°}=13.9 \text{ cm } (3 \text{ s.f.})$

d



$x=180°-(50+50)°$

$=80°$

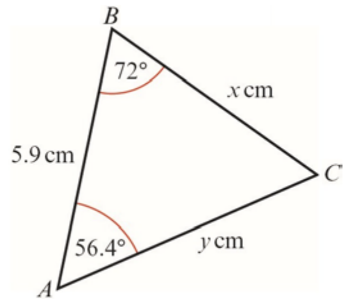
Using $\dfrac{a}{\sin A}=\dfrac{c}{\sin C}$

d $\dfrac{y}{\sin 50°}=\dfrac{8}{\sin 80°}$

$\Rightarrow y=\dfrac{8\sin 50°}{\sin 80°}=6.22 \text{ cm } (3 \text{ s.f.})$

(Note: You could use the line of symmetry to split the triangle into two right-angled triangles and use $\cos 50°=\dfrac{4}{y}$.)

e



$\angle C=180°-(56.4+72)°$

$=51.6°$

Using $\dfrac{a}{\sin A}=\dfrac{c}{\sin C}$

$\dfrac{x}{\sin 56.4°}=\dfrac{5.9}{\sin 51.6°}$

$\Rightarrow x=\dfrac{5.9\sin 56.4°}{\sin 51.6°}$

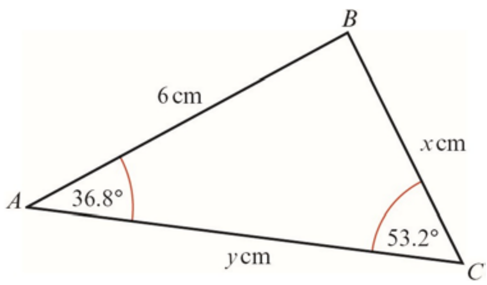
$=6.27 \text{ cm } (3 \text{ s.f.})$

Using $\dfrac{b}{\sin B}=\dfrac{c}{\sin C}$

$\dfrac{y}{\sin 72°}=\dfrac{5.9}{\sin 51.6°}$

$\Rightarrow y=\dfrac{5.9\sin 72°}{\sin 51.6°}=7.16 \text{ cm } (3 \text{ s.f.})$

f



Using $\dfrac{a}{\sin A}=\dfrac{c}{\sin C}$

$\dfrac{x}{\sin 36.8°}=\dfrac{6}{\sin 53.2°}$

2 f $\Rightarrow x = \dfrac{6\sin 36.8°}{\sin 53.2°} = 4.49\text{ cm}\ (3\text{ s.f.})$

$\angle B = 180° - (36.8 + 53.2)°$

$= 90°$

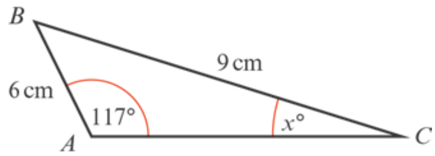
Using $\dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

$\dfrac{6}{\sin 53.2°} = \dfrac{y}{\sin 90°}$

$\Rightarrow y = \dfrac{6\sin 90°}{\sin 53.2°} = 7.49\text{ cm}\ (3\text{ s.f.})$

(Note: The third angle is 90° so you could solve the problem using sine or cosine; the sine rule is not necessary.)

3 a



Using $\dfrac{\sin C}{c} = \dfrac{\sin A}{a}$

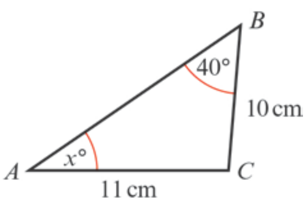
$\dfrac{\sin x°}{6} = \dfrac{\sin 117°}{9}$

$\Rightarrow \sin x° = \dfrac{6\sin 117°}{9}\ (= 0.5940\ldots)$

$\Rightarrow x° = \sin^{-1}\left(\dfrac{6\sin 117°}{9}\right) = 36.4°\ (3\text{ s.f.})$

$\Rightarrow x = 36.4$

b



Using $\dfrac{\sin A}{a} = \dfrac{\sin B}{b}$

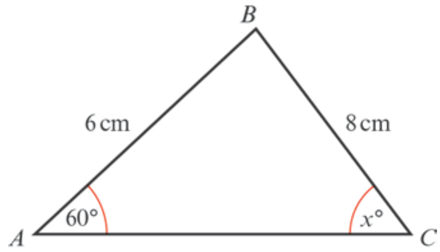
$\dfrac{\sin x°}{10} = \dfrac{\sin 40°}{11}$

$\Rightarrow \sin x° = \dfrac{10\sin 40°}{11}\ (= 0.5843\ldots)$

$\Rightarrow x° = \sin^{-1}\left(\dfrac{10\sin 40°}{11}\right) = 35.8°\ (3\text{ s.f.})$

$\Rightarrow x = 35.8$

3 c



Using $\dfrac{\sin C}{c} = \dfrac{\sin A}{a}$

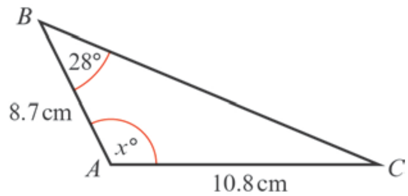
$\dfrac{\sin x°}{6} = \dfrac{\sin 60°}{8}$

$\Rightarrow \sin x° = \dfrac{6\sin 60°}{8}\ (= 0.6495\ldots)$

$\Rightarrow x° = \sin^{-1}\left(\dfrac{6\sin 60°}{8}\right) = 40.5°\ (3\text{ s.f.})$

$\Rightarrow x = 40.5$

d



Using $\dfrac{\sin C}{c} = \dfrac{\sin B}{b}$

$\dfrac{\sin C}{8.7} = \dfrac{\sin 28°}{10.8}$

$\Rightarrow \sin C = \dfrac{8.7\sin 28°}{10.8}\ (= 0.3781\ldots)$

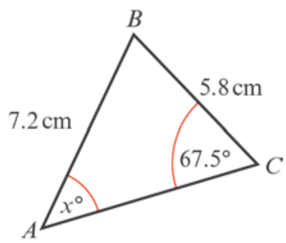
$\Rightarrow C = \sin^{-1}\left(\dfrac{8.7\sin 28°}{10.8}\right)$

$\Rightarrow C = 22.2°\ (3\text{ s.f.})$

$\Rightarrow x° = 180° - (28 + 22.2)° = 130°\ (3\text{ s.f.})$

$\Rightarrow x = 130$

4 a



Using $\dfrac{\sin A}{a} = \dfrac{\sin C}{c}$

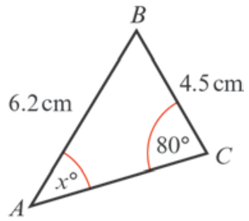
$\dfrac{\sin x°}{5.8} = \dfrac{\sin 67.5°}{7.2}$

4 a $\Rightarrow \sin x° = \dfrac{5.8\sin 67.5°}{7.2} (= 0.7442\ldots)$

$\Rightarrow x° = \sin^{-1}\left(\dfrac{5.8\sin 67.5°}{7.2}\right) = 48.09°$

$\Rightarrow x = 48.1\,(3\text{ s.f.})$

b



Using $\dfrac{\sin A}{a} = \dfrac{\sin C}{c}$

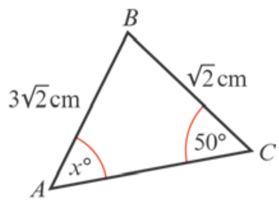
$\dfrac{\sin x°}{4.5} = \dfrac{\sin 80°}{6.2}$

$\Rightarrow \sin x° = \dfrac{4.5\sin 80°}{6.2} (= 0.7147\ldots)$

$\Rightarrow x° = \sin^{-1}\left(\dfrac{4.5\sin 80°}{6.2}\right) = 45.63°$

$\Rightarrow x = 45.6\,(3\text{ s.f.})$

c



Using $\dfrac{\sin A}{a} = \dfrac{\sin C}{c}$

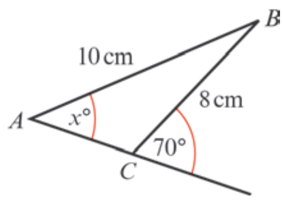
$\dfrac{\sin x°}{\sqrt{2}} = \dfrac{\sin 50°}{3\sqrt{2}}$

$\Rightarrow \sin x° = \dfrac{\sqrt{2}\sin 50°}{3\sqrt{2}} (= 0.2553\ldots)$

$\Rightarrow x° = \sin^{-1}\left(\dfrac{\sin 50°}{3}\right) = 14.79°$

$\Rightarrow x = 14.8\,(3\text{ s.f.})$

d



d Angle $ACB = 180° - 70° = 110°$

Using $\dfrac{\sin A}{a} = \dfrac{\sin C}{c}$

$\dfrac{\sin x°}{8} = \dfrac{\sin 110°}{10}$

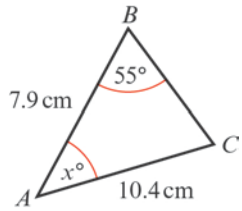
$\Rightarrow \sin x° = \dfrac{8\sin 110°}{10} (= 0.7517\ldots)$

$\Rightarrow x° = \sin^{-1}\left(\dfrac{8\sin 110°}{10}\right)$

$= 48.74°$

$\Rightarrow x = 48.7\,(3\text{ s.f.})$

e



Using $\dfrac{\sin C}{c} = \dfrac{\sin B}{b}$

$\dfrac{\sin C}{7.9} = \dfrac{\sin 55°}{10.4}$

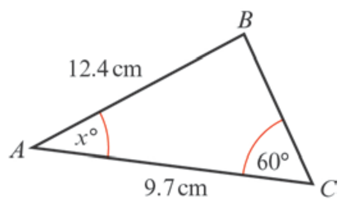
$\Rightarrow \sin C = \dfrac{7.9\sin 55°}{10.4} (= 0.6222\ldots)$

$\Rightarrow C = \sin^{-1}\left(\dfrac{7.9\sin 55°}{10.4}\right) = 38.48°$

$x° = 180° - (55° + C)$

$\Rightarrow x = 86.52 = 86.5\,(3\text{ s.f.})$

f



Using $\dfrac{\sin B}{b} = \dfrac{\sin C}{c}$

$\dfrac{\sin B}{9.7} = \dfrac{\sin 60°}{12.4}$

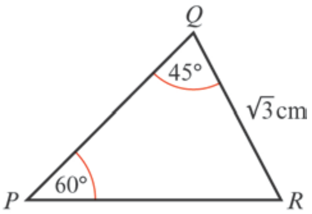
$\Rightarrow \sin B = \dfrac{9.7\sin 60°}{12.4} (= 0.6774\ldots)$

$\Rightarrow B = 42.65°$

$x° = 180° - (60 + B)° = 77.35°$

$\Rightarrow x = 77.4\ (3\text{ s.f.})$

5



a Using $\dfrac{q}{\sin Q} = \dfrac{p}{\sin P}$

$$\frac{PR}{\sin 45°} = \frac{\sqrt{3}}{\sin 60°}$$

$$\Rightarrow PR = \frac{\sqrt{3}\sin 45°}{\sin 60°} = 1.41 \text{ cm } (3 \text{ s.f.})$$

(The exact answer is $\sqrt{2}$ cm.)

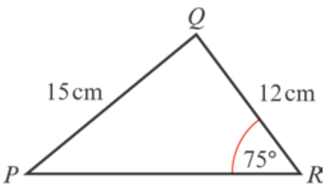
b Using $\dfrac{r}{\sin R} = \dfrac{p}{\sin P}$

$$(R = 180° - (60 + 45)° = 75°)$$

$$\frac{PQ}{\sin 75°} = \frac{\sqrt{3}}{\sin 60°}$$

$$\Rightarrow PQ = \frac{\sqrt{3}\sin 75°}{\sin 60°} = 1.93 \text{ cm } (3 \text{ s.f.})$$

6



Using $\dfrac{\sin P}{p} = \dfrac{\sin R}{r}$

$$\frac{\sin P}{12} = \frac{\sin 75°}{15}$$

$$\Rightarrow \sin P = \frac{12\sin 75°}{15} (= 0.7727\ldots)$$

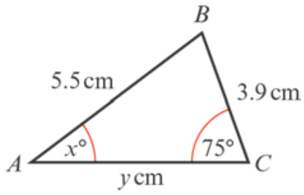
$$\Rightarrow P = \sin^{-1}\left(\frac{12\sin 75°}{15}\right) = 50.60°$$

Angle $QPR = 50.6°$ (3 s.f.)

Angle $PQR = 180° - (75 + 50.6)°$

$= 54.4°$ (3 s.f.)

7 a



Using $\dfrac{\sin A}{a} = \dfrac{\sin C}{c}$

$$\frac{\sin x°}{3.9} = \frac{\sin 75°}{5.5}$$

$$\Rightarrow \sin x° = \frac{3.9\sin 75°}{5.5}$$

$$\Rightarrow x° = \sin^{-1}\left(\frac{3.9\sin 75°}{5.5}\right)$$

$= 43.23°$

$\Rightarrow x = 43.2$ (3 s.f.)

So $\angle ABC = 180° - (75 + 43.2)° = 61.8°$

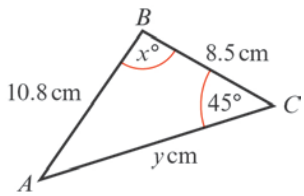
Using $\dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

$$\frac{y}{\sin 61.8°} = \frac{5.5}{\sin 75°}$$

$$\Rightarrow y = \frac{5.5\sin 61.8°}{\sin 75°} = 5.018$$

$\Rightarrow y = 5.02$ (3 s.f.)

b



Using $\dfrac{\sin A}{a} = \dfrac{\sin C}{c}$

$$\frac{\sin A}{8.5} = \frac{\sin 45°}{10.8}$$

$$\Rightarrow \sin A = \frac{8.5\sin 45°}{10.8}$$

$$\Rightarrow A = \sin^{-1}\left(\frac{8.5\sin 45°}{10.8}\right) = 33.815°$$

$x° = 180° - (45° + A) = 101.2°$

$\Rightarrow x = 101$ (3 s.f.)

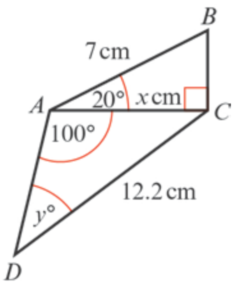
7 b Using $\dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

$$\frac{y}{\sin x^\circ} = \frac{10.8}{\sin 45^\circ}$$

$$\Rightarrow y = \frac{10.8 \sin x^\circ}{\sin 45^\circ} = 14.98$$

$$\Rightarrow y = 15.0\ (3\text{ s.f.})$$

c



In ΔABC, $\dfrac{x}{7} = \cos 20^\circ$

$$\Rightarrow x = 7\cos 20^\circ$$
$$= 6.58\ (3\text{ s.f.})$$

Using $\dfrac{\sin D}{d} = \dfrac{\sin A}{a}$ in ΔADC

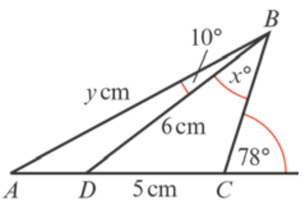
$$\frac{\sin y^\circ}{x} = \frac{\sin 100^\circ}{12.2}$$

$$\Rightarrow \sin y^\circ = \frac{x \sin 100^\circ}{12.2}$$

$$\Rightarrow y^\circ = \sin^{-1}\left(\frac{x\sin 100^\circ}{12.2}\right) = 32.07^\circ$$

$$\Rightarrow y = 32.1\ (3\text{ s.f.})$$

d



In triangle BDC:

$\angle C = 180^\circ - 78^\circ = 102^\circ$

Using $\dfrac{\sin B}{b} = \dfrac{\sin C}{c}$

$$\frac{\sin x^\circ}{5} = \frac{\sin 102^\circ}{6}$$

$$\Rightarrow \sin x^\circ = \frac{5\sin 102^\circ}{6}$$

d $\Rightarrow x^\circ = \sin^{-1}\left(\dfrac{5\sin 102^\circ}{6}\right) = 54.599^\circ$

$\Rightarrow x = 54.6\ (3\text{ s.f.})$

In triangle ABC:

$\angle BAC = 180^\circ - 102^\circ - (10 + x)^\circ = 13.4^\circ$

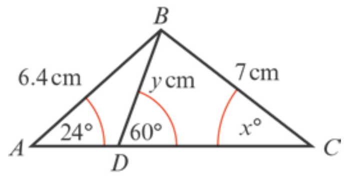
So $\angle ADB = 180^\circ - 10^\circ - 13.4^\circ = 156.6^\circ$

Using $\dfrac{d}{\sin D} = \dfrac{a}{\sin A}$ in ΔABD

$$\frac{y}{\sin 156.6^\circ} = \frac{6}{\sin 13.4^\circ}$$

$$\Rightarrow y = \frac{6\sin 156.6^\circ}{\sin 13.4^\circ}$$
$$= 10.28$$
$$= 10.3\ (3\text{ s.f.})$$

e



Using $\dfrac{\sin C}{c} = \dfrac{\sin A}{a}$ in ΔABC

$$\frac{\sin x^\circ}{6.4} = \frac{\sin 24^\circ}{7}$$

$$\Rightarrow x = 21.8\ (3\text{ s.f.})$$

Using $\dfrac{a}{\sin A} = \dfrac{d}{\sin D}$ in ΔABD

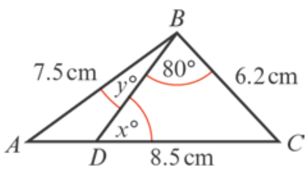
$$\frac{y}{\sin 24^\circ} = \frac{6.4}{\sin 120^\circ}$$

$$\Rightarrow y = \frac{6.4\sin 24^\circ}{\sin 120^\circ} = 3.0058$$

$$\Rightarrow y = 3.01\ (3\text{ s.f.})$$

(The above approach finds the two values independently. You could find y first and then use it to find x, but if your answer for y is wrong then x will be wrong as well.)

f



7 f Using $\frac{\sin D}{d}=\frac{\sin B}{b}$ in ΔBDC

$\frac{\sin x°}{6.2}=\frac{\sin 80°}{8.5}$

$\Rightarrow \sin x°=\frac{6.2\sin 80°}{8.5}$

$\Rightarrow x°=\sin^{-1}\left(\frac{6.2\sin 80°}{8.5}\right)=45.92°$

$\Rightarrow x=45.9\ (3\text{ s.f.})$

In triangle ABC:

$\angle ACB=180°-(80+x)°$

$=54.08°$

Using $\frac{\sin A}{a}=\frac{\sin C}{c}$

$\frac{\sin A}{6.2}=\frac{\sin 54.08°}{7.5}$

$\Rightarrow \sin A=\frac{6.2\sin 54.08°}{7.5}$

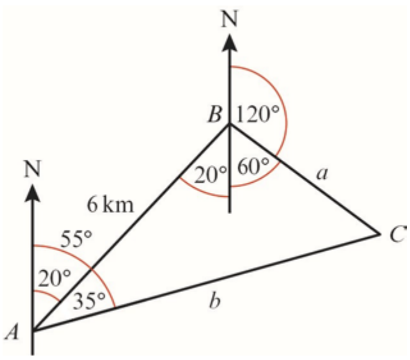
$\Rightarrow A=\sin^{-1}\left(\frac{6.2\sin 54.08°}{7.5}\right)$

$=42.03°$

So $y°=180°-(42.03+134.1)°$

$y=3.87\ (3\text{ s.f.})$

8



$\angle BAC=55°-20°=35°$

$\angle ABC=20°+60°=80°$

(Alternate angles and angles on a straight line.)

$\angle ACB=180°-(80+35)°=65°$

a Using $\frac{b}{\sin B}=\frac{c}{\sin C}$

$\Rightarrow \frac{AC}{\sin 80°}=\frac{6}{\sin 65°}$

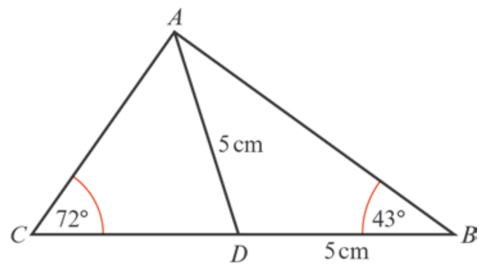
$\Rightarrow AC=\frac{6\sin 80°}{\sin 65°}=6.52\text{ km}\ (3\text{ s.f.})$

8 b Using $\frac{a}{\sin A}=\frac{c}{\sin C}$

$\Rightarrow \frac{BC}{\sin 35°}=\frac{6}{\sin 65°}$

$\Rightarrow BC=\frac{6\sin 35°}{\sin 65°}=3.80\text{ km}\ (3\text{ s.f.})$

9



a In triangle ABD:

$\angle DAB=43°$ (isosceles Δ)

So $\angle ADB=180°-(2\times 43°)=94°$

As the triangle is isosceles you could work with right-angled triangles, but using the sine rule

$\frac{d}{\sin D}=\frac{a}{\sin A}$

$\Rightarrow \frac{AB}{\sin 94°}=\frac{5}{\sin 43°}$

$\Rightarrow AB=\frac{5\sin 94°}{\sin 43°}=7.31\text{cm}\ (3\text{ s.f.})$

b In triangle ADC:

$\angle ADC=180°-94°=86°$

So $\angle CAD=180°-(72+86)°=22°$

Using $\frac{a}{\sin A}=\frac{c}{\sin C}$

$\Rightarrow \frac{CD}{\sin 22°}=\frac{5}{\sin 72°}$

$\Rightarrow CD=\frac{5\sin 22°}{\sin 72°}=1.97\text{ cm}\ (3\text{ s.f.})$

10 a In triangle ABD:

$\frac{\sin B}{b}=\frac{\sin A}{a}$

$\frac{\sin B}{76}=\frac{\sin 66°}{136}$

So $\sin B=\frac{76\sin 66°}{136}$

$B=30.6978...°$

10 a So the angle between AB and BD is 30.7°.
Using triangle BCD:

$$\frac{\sin B}{b} = \frac{\sin C}{c}$$

$$\frac{\sin B}{80} = \frac{\sin 98°}{136}$$

So $\sin B = \frac{80\sin 98°}{136}$

$B = 35.6273...°$

So the angle between BC and BD is 35.6°.
The angle between the fences AB and BC is 30.7° + 35.6° = 66.3°.

b In triangle ABD:
Angle ADB = 180° − 66° − 30.7° = 83.3°
Using the cosine rule:

$$d^2 = a^2 + b^2 - 2ab\cos D$$

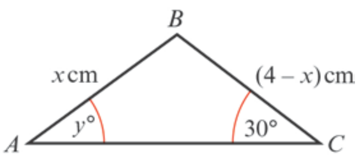
$$d^2 = 136^2 + 76^2 - 2\times136\times76\cos 83.3°$$

$$d^2 = 18\,496 + 5776 - 2411.817\,477$$

$$d^2 = 21\,860.182\,52$$

So d = 147.851...
So the length of the fence AB is 148 m (3 s.f.).

11



Using $\frac{a}{\sin A} = \frac{c}{\sin C}$

$$\Rightarrow \frac{4-x}{\sin y°} = \frac{x}{\sin 30°}$$

$$\Rightarrow (4-x)\sin 30° = x\sin y°$$

$$\Rightarrow (4-x)\times\frac{1}{2} = x\times\frac{1}{\sqrt{2}}$$

Multiply throughout by 2:

$$4 - x = x\sqrt{2}$$

$$x + \sqrt{2}x = 4$$

$$x(1+\sqrt{2}) = 4$$

$$x = \frac{4}{1+\sqrt{2}}$$

11 Multiply 'top and bottom' by $\sqrt{2}-1$:

$$x = \frac{4(\sqrt{2}-1)}{(\sqrt{2}-1)(\sqrt{2}+1)}$$

$$= \frac{4(\sqrt{2}-1)}{2-1}$$

$$= 4(\sqrt{2}-1)$$

12 a Using the left-hand triangle, the angles are 40°, 128° and 12°. (128° = 180° − 52° and 180° − (128° + 40°) = 12°)

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{a}{\sin 128°} = \frac{15}{\sin 12°}$$

$$a = \frac{15\sin 128°}{\sin 12°}$$

a = 56.8518...
Using the larger right-angled triangle:

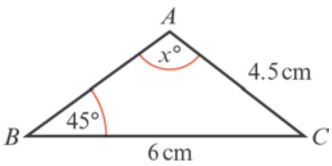
$$\sin 40° = \frac{\text{height}}{56.8518}$$

Height = 56.8518 sin 40° = 36.54...
The height of the building is 36.5 m (3 s.f.).

b Assume that the angles of elevation have been measured from ground level.

Trigonometric ratios 9C

1 a



$x > 45°$

So there are two possible results.

$$\text{Using } \frac{\sin A}{a} = \frac{\sin B}{b}$$

$$\frac{\sin x°}{6} = \frac{\sin 45°}{4.5}$$

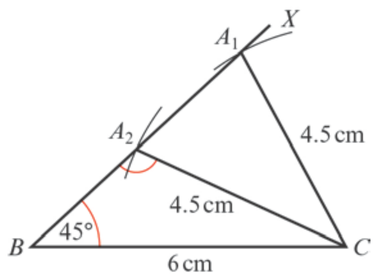
$$\sin x° = \frac{6\sin 45°}{4.5}$$

$$x° = \sin^{-1}\left(\frac{6\sin 45°}{4.5}\right) \text{ or}$$

$$x° = 180° - \sin^{-1}\left(\frac{6\sin 45°}{4.5}\right)$$

$$x° = 70.5° \, (3 \text{ s.f.}) \text{ or } x° = 109.5°$$

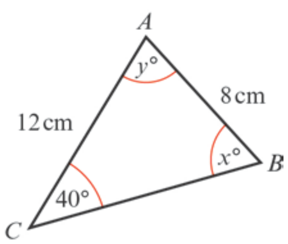
b



Draw *BC* = 6 cm.

Construct or draw an angle of 45° at *B* and extend the line as (*BX*).
Set the compasses to a radius of 4.5 cm.
Put the point on *C* and draw an arc.
The points where the arc meets *BX* are the two possible positions of *A*.

2 a



$$\text{Using } \frac{\sin B}{b} = \frac{\sin C}{c}$$

$$\frac{\sin x°}{12} = \frac{\sin 40°}{8}$$

$$\sin x° = \frac{12\sin 40°}{8}$$

$$x° = \sin^{-1}\left(\frac{12\sin 40°}{8}\right) \text{ or}$$

$$x° = 180° - \sin^{-1}\left(\frac{12\sin 40°}{8}\right)$$

$$x° = 74.6° \text{ or } x° = 105.4°$$

$$x = 74.6 \text{ or } 105 \, (3 \text{ s.f.})$$

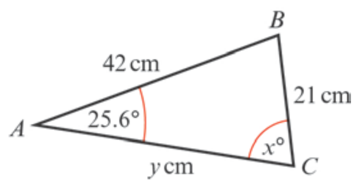
When $x = 74.6$:

$$y = 180 - (74.6 + 40)$$
$$= 180 - 114.6$$
$$= 65.4 \, (3 \text{ s.f.})$$

When $x = 105.4$:

$$y = 180 - (105.4 + 40)$$
$$= 180 - 145.4$$
$$= 34.6 \, (3 \text{ s.f.})$$

b



$$\text{Using } \frac{\sin C}{c} = \frac{\sin A}{a}$$

$$\frac{\sin x°}{42} = \frac{\sin 25.6°}{21}$$

$$\sin x° = \frac{42\sin 25.6°}{21}$$

$$x° = \sin^{-1}(2\sin 25.6°) \text{ or}$$

$$x° = 180° - \sin^{-1}(2\sin 25.6°)$$

$$x = 59.8 \text{ or } x = 120 \, (3 \text{ s.f.})$$

When $x = 59.8$:

angle $B = 180° - (59.8° + 25.6°) = 94.6°$

When $x = 120°$:

angle $B = 180° - (120.2° + 25.6°) = 34.2°$

2 b Using $\dfrac{b}{\sin B} = \dfrac{a}{\sin A}$

$$\frac{y}{\sin 94.6°} = \frac{21}{\sin 25.6°}$$

$$\Rightarrow y = \frac{21 \sin 94.6°}{\sin 25.6°}$$

$$= 48.4 \text{ (3 s.f.)}$$

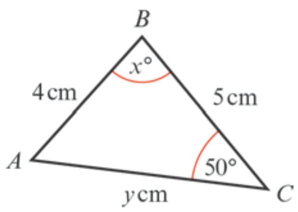
Using $\dfrac{b}{\sin B} = \dfrac{a}{\sin A}$

$$\frac{y}{\sin 34.2°} = \frac{21}{\sin 25.6°}$$

$$\text{So } y = \frac{21 \sin 34.2°}{\sin 25.6°}$$

$$= 27.3 \text{ (3 s.f.)}$$

c



Using $\dfrac{\sin A}{a} = \dfrac{\sin C}{c}$

$$\frac{\sin A}{5} = \frac{\sin 50°}{4}$$

$$\sin A = \frac{5 \sin 50°}{4}$$

$$A = \sin^{-1}\left(\frac{5 \sin 50°}{4}\right) \text{ or}$$

$$A = 180° - \sin^{-1}\left(\frac{5 \sin 50°}{4}\right)$$

$$A = 73.25 \text{ or } A = 106.75$$

When $A = 73.247$:

$$x = 180 - (50 + 73.247)$$

$$= 56.8 \text{ (3 s.f.)}$$

Using $\dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

$$\frac{y}{\sin x°} = \frac{4}{\sin 50°}$$

$$\text{So } y = \frac{4 \sin x°}{\sin 50°}$$

$$= 4.37 \text{ (3 s.f.)}$$

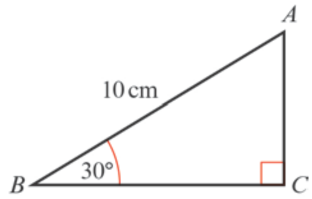
2 c When $A = 106.75$:

$$x = 180 - (50 + 106.75) = 23.2 \text{ (3 s.f.)}$$

As above:

$$y = \frac{4 \sin x°}{\sin 50°} = 2.06 \text{ (3 s.f.)}$$

3 a



The length of AC is least when it is at right angles to BC.

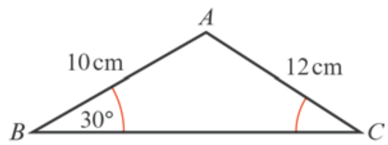
Using $\sin B = \dfrac{AC}{AB}$

$$\sin 30° = \frac{AC}{10}$$

$$AC = 10 \sin 30° = 5$$

$$AC = 5 \text{ cm}$$

b



Using $\dfrac{\sin C}{c} = \dfrac{\sin B}{b}$

$$\frac{\sin C}{10} = \frac{\sin 30°}{12}$$

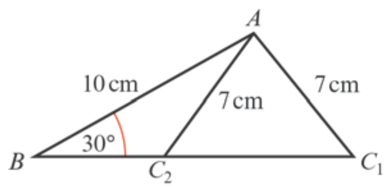
$$\sin C = \frac{10 \sin 30°}{12}$$

$$C = \sin^{-1}\left(\frac{10 \sin 30°}{12}\right)$$

$$= 24.62°$$

$$\angle ABC = 24.6° \text{ (3 s.f.)}$$

c



As 7 cm < 10 cm, $\angle ACB > 30°$.

3 c There are two possible results.
Using 7 cm instead of 12 cm in (b):

$$\sin C = \frac{10\sin 30°}{7}$$

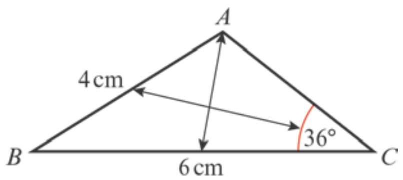
$$C = \sin^{-1}\left(\frac{10\sin 30°}{7}\right) \text{ or}$$

$$C = 180° - \sin^{-1}\left(\frac{10\sin 30°}{7}\right)$$

$$C = 45.58° \text{ or } 134.4°$$

$$\angle ABC = 45.6°\,(3\text{ s.f.}) \text{ or } 134°\,(3\text{ s.f.})$$

4



As 4 < 6, 36° < $\angle BAC$, so there are two possible values for angle A.

$$\text{Using } \frac{\sin A}{a} = \frac{\sin C}{c}$$

$$\frac{\sin A}{6} = \frac{\sin 36°}{4}$$

$$\sin A = \frac{6\sin 36°}{4}$$

$$A = \sin^{-1}\left(\frac{6\sin 36°}{4}\right) \text{ or}$$

$$A = 180° - \sin^{-1}\left(\frac{6\sin 36°}{4}\right)$$

$$A = 61.845\ldots° \text{ or } A = 118.154\ldots°$$

When $A = 118.154\ldots°$:

$$\angle ABC = 180° - (36° + 118.154\ldots°)$$

$$= 25.8°\ (3\text{ s.f.})$$

Using this value for $\angle ABC$ with

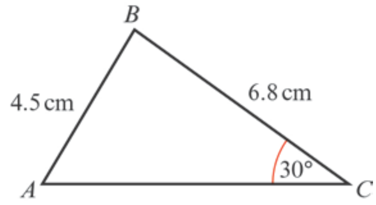
$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{AC}{\sin 25.8°} = \frac{4}{\sin 36°}$$

$$\text{So } AC = \frac{4\sin 25.8°}{\sin 36°}$$

$$= 2.96 \text{ cm } (3\text{ s.f.})$$

5



As 6.8 > 4.5, angle A > 30° and so there are two possible values for A.

$$\text{Using } \frac{\sin A}{a} = \frac{\sin C}{c}$$

$$\frac{\sin A}{6.8} = \frac{\sin 30°}{4.5}$$

$$A = \sin^{-1}\left(\frac{6.8\sin 30°}{4.5}\right) \text{ or}$$

$$A = 180° - \sin^{-1}\left(\frac{6.8\sin 30°}{4.5}\right)$$

$$A = 49.07\ldots° \text{ or } 130.926\ldots°$$

When A = 49.07…°, B is the largest angle.

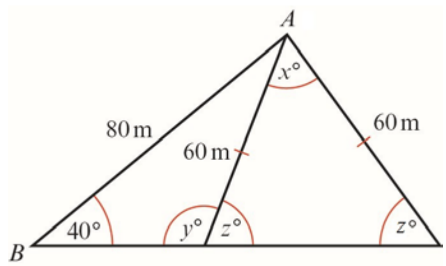
$$\angle ABC = 180° - (30° + 49.07\ldots°)$$

$$= 101°\ (3\text{ s.f.})$$

When A = 130.926…°, this the largest angle.

$$\angle BAC = 131°\,(3\text{ s.f.})$$

6 a



Using the sine rule:

$$\frac{\sin y}{80} = \frac{\sin 40°}{60}$$

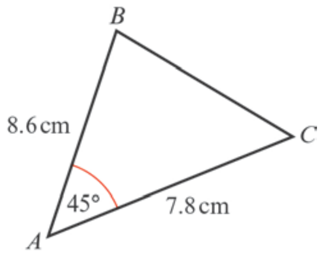
$$\sin y = \frac{80\sin 40°}{60}$$

y = 59° or 121°
y is obtuse, therefore, y = 121°
z = 59°
x = 180° − 2 × 59° = 62°
x = 62°

b The assumption is that the ball swings symmetrically.

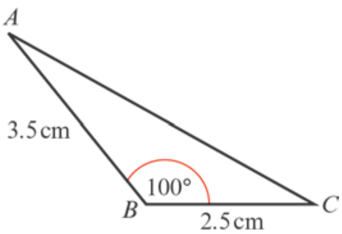
Trigonometric ratios 9D

1 a



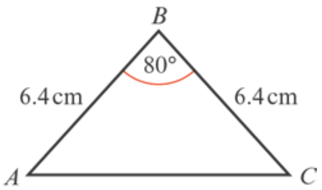
$$\text{Area} = \tfrac{1}{2}\times 7.8\times 8.6\times \sin 45°$$
$$= 23.71\ldots$$
$$= 23.7\text{ cm}^2\ (3\text{ s.f.})$$

b



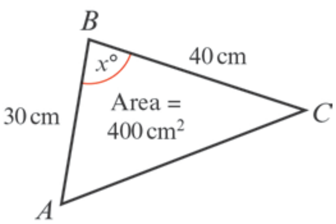
$$\text{Area} = \tfrac{1}{2}\times 2.5\times 3.5\times \sin 100°$$
$$= 4.308\ldots$$
$$= 4.31\text{ cm}^2\ (3\text{ s.f.})$$

c



$$\text{Area} = \tfrac{1}{2}\times 6.4\times 6.4\times \sin 80°$$
$$= 20.16\ldots$$
$$= 20.2\text{ cm}^2\ (3\text{ s.f.})$$

2 a



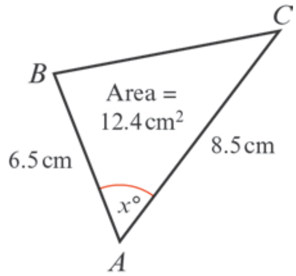
Using area $= \tfrac{1}{2}ac\sin B$

$$400 = \tfrac{1}{2}\times 40\times 30\times \sin x°$$

2 a So $\sin x° = \dfrac{400}{600} = \dfrac{2}{3}$

$$x° = \sin^{-1}\left(\tfrac{2}{3}\right) \text{ or } x° = 180° - \sin^{-1}\left(\tfrac{2}{3}\right)$$
$$x = 41.8\ (3\text{ s.f.}) \text{ or } x = 138\ (3\text{ s.f.})$$

b



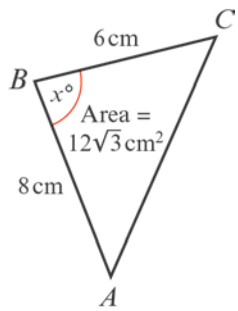
Using area $= \tfrac{1}{2}bc\sin A$

$$12.4 = \tfrac{1}{2}\times 8.5\times 6.5\times \sin x°$$

So $\sin x° = \dfrac{12.4}{27.625} = 0.04488\ldots$

$$x = 26.7\ (3\text{ s.f})\text{ or } x = 153\ (3\text{ s.f.})$$

c



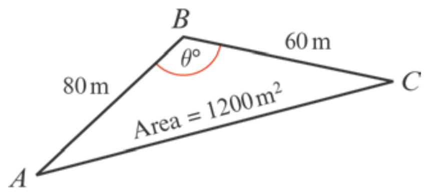
Using area $= \tfrac{1}{2}ac\sin B$

$$12\sqrt{3} = \tfrac{1}{2}\times 6\times 8\sin x°$$

So $\sin x° = \dfrac{12\sqrt{3}}{24} = \dfrac{\sqrt{3}}{2}$

$$x = 60 \text{ or } x = 120$$

3



Using area $= \tfrac{1}{2}ac\sin B$

$$1200 = \tfrac{1}{2}\times 60\times 80\times \sin\theta°$$

3 So $\sin\theta° = \frac{1200}{2400} = \frac{1}{2}$

$$\theta = 30 \text{ or } \theta = 150$$

But as AC is the largest side, θ must be the largest angle.
So θ = 150
Using the cosine rule to find AC:

$b^2 = a^2 + c^2 - 2ac\cos B$

$AC^2 = 60^2 + 80^2 - 2\times 60\times 80\times \cos 150°$

$= 18\,313.84\ldots$

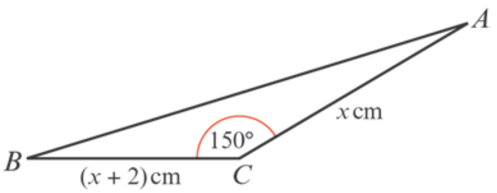
$AC = 135.3\ldots$

$AC = 135 \text{ m } (3 \text{ s.f.})$

So the perimeter $= 60 + 80 + 135$

$= 275\text{m } (3 \text{ s.f.})$

4



Area of $\Delta ABC = \frac{1}{2}x(x+2)\sin 150° \text{cm}^2$

So $5 = \frac{1}{2}x(x+2)\times\frac{1}{2}$

So $20 = x(x+2)$

$\Rightarrow x^2 + 2x - 20 = 0$

Using the quadratic formula:

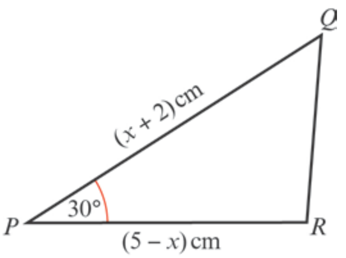
$$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$x = \frac{-2\pm\sqrt{84}}{2}$$

$x = 3.582\ldots$ or $x = -5.582\ldots$

As $x > 0$, $x = 3.58\ (3 \text{ s.f.})$

5



a Using area of $\Delta PQR = \frac{1}{2}qr\sin P$:

$A \text{ cm}^2 = \frac{1}{2}(5-x)(x+2)\sin 30° \text{ cm}^2$

5 a $\Rightarrow A = \frac{1}{2}(5x^2 - 2x + 10 - x^2)\times\frac{1}{2}$

$\Rightarrow A = \frac{1}{4}(10 + 3x - x^2)$

b Completing the square:

$10 + 3x - x^2 = -\left(\left(x - 1\frac{1}{2}\right)^2 - 2\frac{1}{4} - 10\right)$

$= -\left(\left(x - 1\frac{1}{2}\right)^2 - 12\frac{1}{4}\right)$

$= 12\frac{1}{4} - \left(x - 1\frac{1}{2}\right)^2$

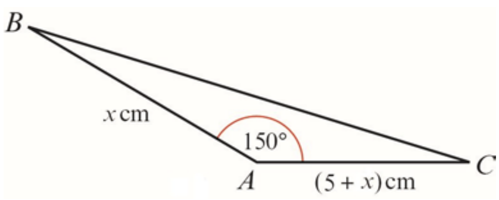
When $x = 1\frac{1}{2}$:

The maximum values of $10 + 3x - x^2 = 12\frac{1}{4}$ and the maximum value of A is

$\frac{1}{4}\left(12\frac{1}{4}\right) = 3\frac{1}{16}$

(You could use differentiation to find the maximum.)

6



a Using area of $\Delta BAC = \frac{1}{2}bc\sin A$

$3\frac{3}{4}\text{cm}^2 = \frac{1}{2}x(5+x)\sin 150° \text{ cm}^2$

$3\frac{3}{4} = \frac{1}{2}(5x + x^2)\times\frac{1}{2}$

$\Rightarrow 15 = 5x + x^2$

$\Rightarrow x^2 + 5x - 15 = 0$

b Using the quadratic equation formula:

$$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

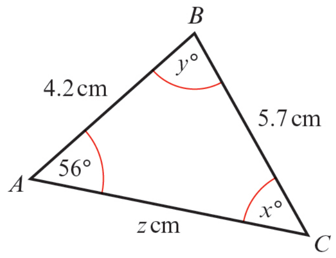
$$x = \frac{-5\pm\sqrt{85}}{2}$$

$x = 2.109\ldots$ or $x = -7.109\ldots$

As $x > 0$, $x = 2.11\ (3 \text{ s.f.})$

Trigonometric ratios 9E

1 a



Using $\frac{\sin C}{c} = \frac{\sin A}{a}$

$$\frac{\sin x°}{4.2} = \frac{\sin 56°}{5.7}$$

$$\sin x° = \frac{4.2 \sin 56°}{5.7}$$

$$x° = \sin^{-1}\left(\frac{4.2 \sin 56°}{5.7}\right)$$

$$= 37.65\ldots°$$

$$x = 37.7\,(3\text{ s.f.})$$

$$\text{So } y° = 180° - (56° + 37.7°)$$

$$= 86.3°$$

$$y = 86.3\,(3\text{ s.f.})$$

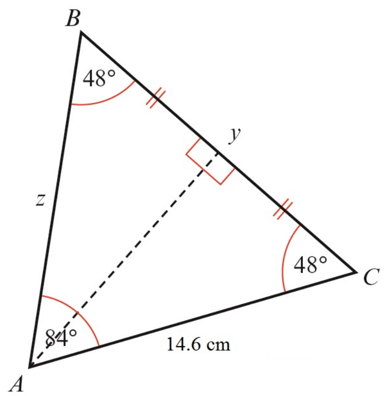
Using $\frac{b}{\sin B} = \frac{a}{\sin A}$

$$\frac{z}{\sin y°} = \frac{5.7}{\sin 56°}$$

$$\text{So } z = \frac{5.7 \sin y°}{\sin 56°}$$

$$= 6.86\,(3\text{ s.f.})$$

b $x° = 180° – (48 + 84)°$
$x° = 48°$



As angle B = angle C, z = 14.6 cm.

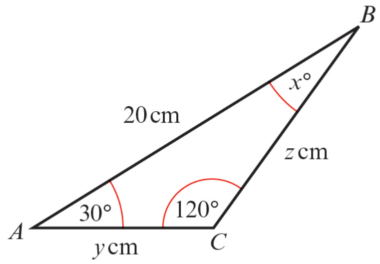
b Using the line of symmetry through A

$$\cos 48° = \frac{\frac{y}{2}}{14.6}$$

$$\text{So } y = 29.2 \cos 48°$$

$$= 19.5\,(3\text{ s.f.})$$

c



$$x° = 180° - (120° + 30°)$$

$$= 30°$$

Using the line of symmetry through C

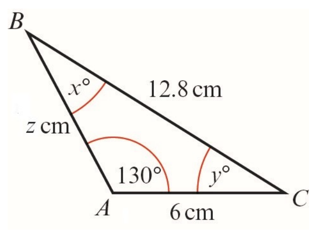
$$\cos 30° = \frac{10}{y}$$

$$\text{So } y = \frac{10}{\cos 30°}$$

$$= 11.5\,(3\text{ s.f.})$$

Since ΔABC is isosceles with $AC = CB$
$z = 11.5\,(3\text{ s.f.})$.

d



Using $\frac{\sin A}{a} = \frac{\sin B}{b}$

$$\frac{\sin 130°}{12.8} = \frac{\sin x°}{6}$$

$$\text{So } \sin x° = \frac{6 \sin 130°}{12.8}$$

$$= 0.359\,08\ldots$$

$$\Rightarrow x = 21.0\,(3\text{ s.f.})$$

$$\text{So } y° = 180° - (130° + x°)$$

$$= 28.956\ldots°$$

1 d $\Rightarrow y = 29.0\ (3\text{ s.f.})$

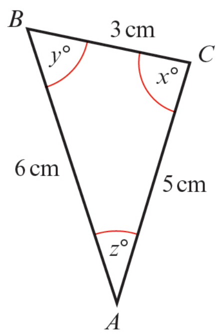
Using $\dfrac{c}{\sin C} = \dfrac{a}{\sin A}$

$$\frac{z}{\sin y°} = \frac{12.8}{\sin 130°}$$

So $z = \dfrac{12.8 \sin y°}{\sin 130°}$

$= 8.09\ (3\text{ s.f.})$

e



Using $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

$$\cos x° = \frac{3^2 + 5^2 - 6^2}{2 \times 3 \times 5}$$

$$= -0.0\dot{6}$$

$$x = 93.8\ (3\text{ s.f.})$$

Using $\dfrac{\sin B}{b} = \dfrac{\sin C}{c}$

$$\frac{\sin y°}{5} = \frac{\sin x°}{6}$$

$$\sin y° = \frac{5 \sin x°}{6}$$

$$y° = \sin^{-1}\left(\frac{5 \sin x°}{6}\right)$$

$$= 56.25\ldots°$$

$$y = 56.3\ (3\text{ s.f.})$$

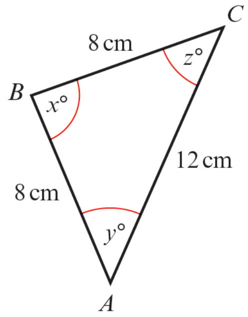
Using the angle sum for a triangle

$z° = 180° - (x + y)°$

$= 29.926\ldots°$

$z = 29.9\ (3\text{ s.f.})$

f



Using the line of symmetry through B

$\cos y° = \frac{6}{8}$

$= \frac{3}{4}$

$y° = \cos^{-1}\left(\frac{3}{4}\right)$

$= 41.40\ldots$

$y = 41.4\ (3\text{ s.f.})$

As the triangle is isosceles

$z = y$

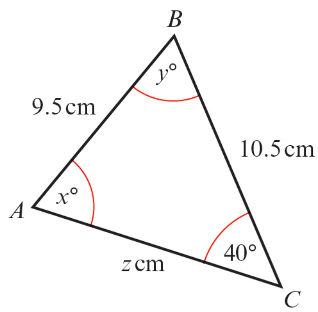
$= 41.4\ (3\text{ s.f.})$

So $x° = 180° - (y + z)°$

$= 97.2°$

$x = 97.2\ (3\text{ s.f.})$

g



Using $\dfrac{\sin A}{a} = \dfrac{\sin C}{c}$

$$\frac{\sin x°}{10.5} = \frac{\sin 40°}{9.5}$$

$$\sin x° = \frac{10.5 \sin 40°}{9.5}$$

$$x° = \sin^{-1}\left(\frac{10.5 \sin 40°}{9.5}\right) \text{ or}$$

$$x° = 180° - \sin^{-1}\left(\frac{10.5 \sin 40°}{9.5}\right)$$

$x° = 45.27°$ or $x° = 134.728\ldots°$

$x = 45.3\ (3\text{ s.f.})$ or $x = 135\ (3\text{ s.f.})$

1 g Using $\frac{b}{\sin B} = \frac{c}{\sin C}$

$$\frac{z}{\sin y°} = \frac{9.5}{\sin 40°}$$

$$z = \frac{9.5 \sin y°}{\sin 40°}$$

When $x = 45.3$

$$y° = 180° - (40 + 45.3)°$$
$$= 94.7°$$

So $y = 94.7$ (3 s.f.)

$$z = \frac{9.5 \sin y°}{\sin 40°}$$
$$= 14.7 \text{ (3 s.f.)}$$

When $x = 134.728\ldots$

$$y° = 180° - (40 + 134.72\ldots)°$$
$$= 5.27°$$

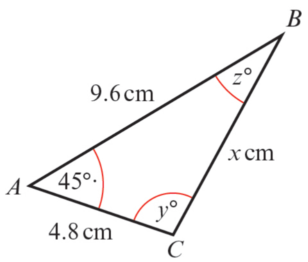
So $y = 5.27$ (3 s.f.)

$$z = \frac{9.5 \sin y°}{\sin 40°}$$
$$= 1.36 \text{ (3 s.f.)}$$

So $x = 45.3,\ y = 94.7,\ z = 14.7$

or $x = 135,\ y = 5.27,\ z = 1.36$

h



Using $a^2 = b^2 + c^2 - 2bc \cos A$

$$x^2 = 4.8^2 + 9.6^2 - 2 \times 4.8 \times 9.6 \times \cos 45°$$
$$= 50.03\ldots$$
$$x = 7.07 \text{ (3 s.f.)}$$

Using $\frac{\sin C}{c} = \frac{\sin A}{a}$

$$\frac{\sin y°}{9.6} = \frac{\sin 45°}{x}$$

$$\sin y° = \frac{9.6 \sin 45°}{x}$$

$$y° = \sin^{-1}\left(\frac{9.6 \sin 45°}{x}\right)$$

h $y° = 180 - 73.68\ldots°$ (y cannot be acute)

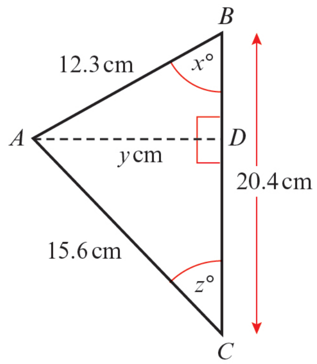
$$y° = 106.32\ldots°$$
$$y = 106 \text{ (3 s.f.)}$$

Then

$$z° = 180° - (45 + 106.32\ldots)°$$
$$= 28.68\ldots°$$
$$z = 28.7 \text{ (3 s.f.)}$$

So $x = 7.07,\ y = 107,\ z = 28.7$

i



Using $\cos B = \frac{a^2 + c^2 - b^2}{2ac}$

$$\cos x° = \frac{20.4^2 + 12.3^2 - 15.6^2}{2 \times 20.4 \times 12.3}$$
$$= 0.6458\ldots$$
$$x = 49.77\ldots°$$
$$x = 49.8 \text{ (3 s.f.)}$$

In right-angled triangle ABD

$$\sin x° = \frac{y}{12.3}$$

So $y = 12.3 \sin x°$

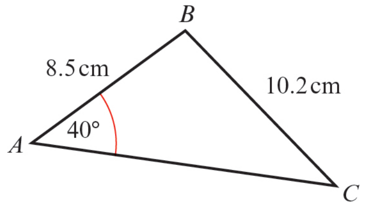
$$= 9.39 \text{ (3 s.f.)}$$

In right-angled triangle ACD

$$\sin z° = \frac{y}{15.6}$$
$$= 0.601\,99\ldots$$
$$z° = 37.01\ldots°$$
$$z = 37.0 \text{ (3 s.f.)}$$

So $x = 49.8,\ y = 9.39,\ z = 37.0$

2 a



Using $\dfrac{\sin C}{c} = \dfrac{\sin A}{a}$

$\dfrac{\sin C}{8.5} = \dfrac{\sin 40°}{10.2}$

$\sin C = \dfrac{8.5 \sin 40°}{10.2}$

$C = \sin^{-1}\left(\dfrac{8.5\sin 40°}{10.2}\right)$

$= 32.388\ldots°$

$= 32.4°\ (3 \text{ s.f.})$

$B = 180° - (40 + C)°$

$= 107.6\ldots°$

$B = 108°\ (3 \text{ s.f.})$

Using $\dfrac{b}{\sin B} = \dfrac{a}{\sin A}$

$b = \dfrac{10.2 \sin B}{\sin 40°}$

$= 15.1 \text{ cm}\ (3 \text{ s.f.})$

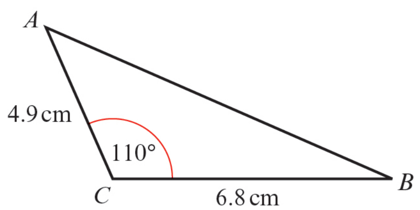
Area $= \frac{1}{2} ac \sin B$

$= \frac{1}{2} \times 10.2 \times 8.5 \times \sin 108°$

$= 41.228$

$= 41.2 \text{ cm}^2\ (3 \text{ s.f.})$

b



Using $c^2 = a^2 + b^2 - 2ab\cos C$

$AB^2 = 6.8^2 + 4.9^2 - 2 \times 6.8 \times 4.9 \times \cos 110°$

$= 93.04\ldots$

$AB = 9.6458\ldots$

$= 9.65 \text{ cm}\ (3 \text{ s.f.})$

Using $\dfrac{\sin A}{a} = \dfrac{\sin C}{c}$

2 b $\sin A = \dfrac{6.8 \sin 110°}{AB}$

$= 0.662\,45\ldots$

$A = 41.49°$

$= 41.5°\ (3 \text{ s.f.})$

So $B = 180° - (110 + A)°$

$= 28.5°\ (3 \text{ s.f.})$

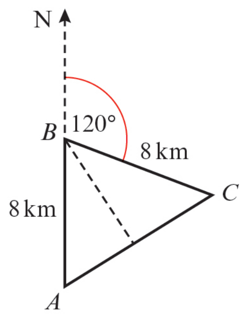
Area $= \frac{1}{2} ac \sin B$

$= \frac{1}{2} \times 6.8 \times 4.9 \times \sin 110°$

$= 15.655\ldots$

$= 15.7 \text{ cm}^2$ (3 s.f.)

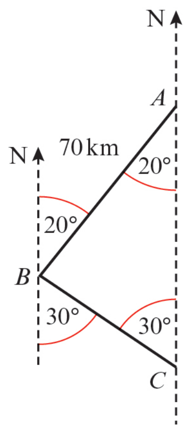
3



a Angle $ABC = 180° - 120°$
$= 60°$
As $\angle A = \angle C$, all angles are 60°.
It is an equilateral triangle.
So $AC = 8$ km.

b As $\angle BAC = 60°$,
the bearing of C from A is 060°.

4



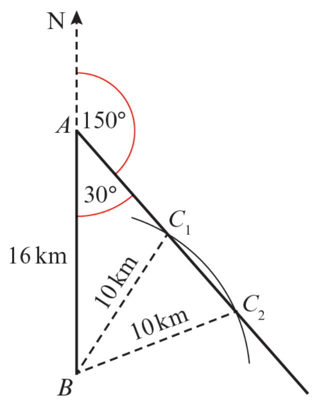
From the diagram
$\angle ABC = 180° - (20 + 30)°$
$= 130°$

4 Using $\dfrac{b}{\sin B}=\dfrac{c}{\sin C}$

$$\frac{AC}{\sin 130°}=\frac{70}{\sin 30°}$$
$$AC=\frac{70\sin 130°}{\sin 30°}$$
$$=107.246\ldots$$
$$AC=107\text{ km }(3\text{ s.f.})$$

From the diagram, the bearing of C from A is 180˚.

5



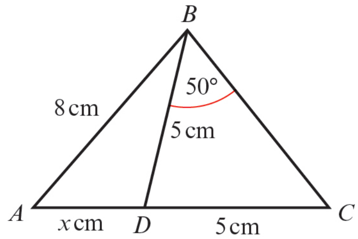
Using the sine rule

$$\frac{\sin C}{c}=\frac{\sin A}{a}$$
$$\frac{\sin C}{16}=\frac{\sin 30°}{10}$$
$$\sin C=\frac{16\sin 30°}{10}$$
$$=0.8$$
$$C=\sin^{-1}(0.8)\text{ or }C=180°-\sin^{-1}(0.8)$$
$$C=53.1°\text{ or }C=126.9°$$
$$\angle AC_2B=53.1°,\ \angle AC_1B=127°\ (3\text{ s.f.})$$

(Store the correct values; these are not required answers.)
Triangle BC_1C_2 is isosceles, so C_1C_2 can be found using this triangle, without finding AC_1 and AC_2.
Use the line of symmetry through B

$$\cos\angle C_1C_2B=\frac{\frac{1}{2}C_1C_2}{10}$$
$$\Rightarrow C_1C_2=20\cos\angle C_1C_2B$$
$$=20\cos\angle AC_2B$$
$$=20\cos 53.1°$$
$$\Rightarrow C_1C_2=12\text{ km}$$

6 a



In the isosceles ΔBDC

$$\angle BDC=180°-(50+50)°$$
$$=80°$$
$$\text{So }\angle BDA=180°-80°$$
$$=100°$$

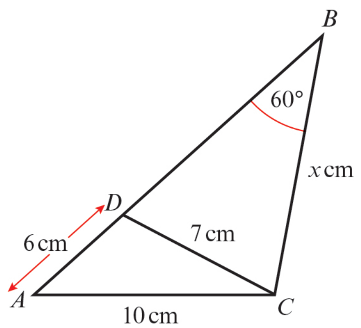
Using the sine rule in ΔABD

$$\frac{\sin A}{a}=\frac{\sin D}{d}$$
$$\Rightarrow\frac{\sin A}{5}=\frac{\sin 100°}{8}$$
$$\Rightarrow\sin A=\frac{5\sin 100°}{8}$$
$$\text{So }A=\sin^{-1}\left(\frac{5\sin 100°}{8}\right)$$
$$=37.9886\ldots$$
$$\angle ABD=180°-(100+A)°$$
$$=42.01\ldots°$$

Using $\dfrac{b}{\sin B}=\dfrac{d}{\sin D}$

$$\frac{x}{\sin B}=\frac{8}{\sin 100°}$$
$$x=\frac{8\sin B}{\sin 100°}$$
$$=5.436\ldots$$
$$x=5.44\ (3\text{ s.f.})$$

b



In ΔADC, using $\cos A=\dfrac{c^2+d^2-a^2}{2cd}$

6 b $\cos A = \dfrac{6^2 + 10^2 - 7^2}{2 \times 6 \times 10}$

$= 0.725$

So $A = 43.53\ldots°$

Using the sine rule in ΔABC

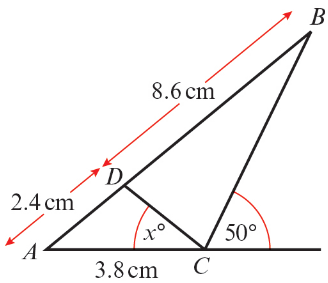
$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

So $\dfrac{x}{\sin A} = \dfrac{10}{\sin 60°}$

$$\Rightarrow x = \frac{10 \sin A}{\sin 60°}$$

So $x = 7.95$ (3 s.f.)

c



In ΔABC, $c = 11$ cm, $b = 3.8$ cm,
$\angle ACB = 130°$, $(180° - 50°)$

Using $\dfrac{\sin B}{b} = \dfrac{\sin C}{c}$

$\sin B = \dfrac{3.8 \sin 130°}{11}$

$= 0.2646\ldots$

$B = 15.345\ldots°$

So $A = 180° - (130 + B)°$

$= 34.654\ldots°$

In ΔADC, $c = 2.4$ cm, $d = 3.8$ cm,
$A = 34.654\ldots°$

Using the cosine rule:

$a^2 = c^2 + d^2 - 2cd \cos A$

So $DC^2 = 2.4^2 + 3.8^2 - 2 \times 2.4 \times 3.8 \times \cos A$

$= 5.1959\ldots$

$\Rightarrow DC = 2.279\ldots$cm

Using the sine rule:

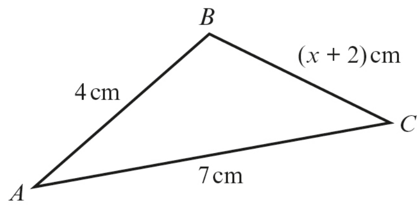
$$\frac{\sin C}{c} = \frac{\sin A}{a}$$

$\sin x° = \dfrac{2.4 \sin A}{DC}$

$= 0.598\,69\ldots$

$x = 36.8$ (3 s.f.)

7



a As $AB + BC > AC$

$4 + (x + 2) > 7$

$\Rightarrow x + 2 > 3$

$\Rightarrow x > 1$

As $AB + AC > BC$

$4 + 7 > x + 2$

$\Rightarrow 9 > x$

So $1 < x < 9$

b Using $b^2 = a^2 + c^2 - 2ac \cos B$

i $7^2 = (x+2)^2 + 4^2 - 2(x+2) \times 4 \times \cos 60°$

$49 = x^2 + 4x + 4 + 16 - 4(x + 2)$

$49 = x^2 + 4x + 4 + 16 - 4x - 8$

So $x^2 = 37$

$\Rightarrow x = 6.08$ (3 s.f.)

Area $= \frac{1}{2} ac \sin B$

$= \frac{1}{2} \times 8.08 \times 4 \times \sin 60°$

$= 13.9949\ldots$

$= 14.0$ cm^2 (3 s.f.)

ii $7^2 = (x + 2)^2 + 4^2$
$- 2 \times (x + 2) \times 4 \times \cos 45°$

$49 = x^2 + 4x + 4 + 16$
$- (8 \cos 45°)x - 16 \cos 45°$

So:

$x^2 + (4 - 8 \cos 45°)x$
$- (29 + 16 \cos 45°) = 0$

or $x^2 + 4\left(1 - \sqrt{2}\right)x$
$- \left(29 + 8\sqrt{2}\right) = 0$

Use the quadratic formula

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ with $a = 1$

$b = 4 - 8 \cos 45°$

$= 4\left(1 - \sqrt{2}\right)$

$= -1.6568\ldots$

7 b ii $c = -(29 + 16\cos 45°)$

$= -(29 + 8\sqrt{2})$

$= -40.313\ldots$

$x = 7.23$ (3 s.f.)

(The other value of x is less than –2.)

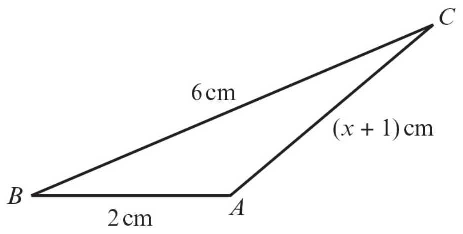
Area $= \frac{1}{2}ac\sin B$

$= \frac{1}{2} \times 4 \times 9.23 \times \sin 45°$

$= 13.05\ldots$

$= 13.1\text{ cm}^2$ (3 s.f.)

8 a



Using $b^2 = a^2 + c^2 - 2ac\cos B$ where $\cos B = \frac{5}{8}$

$(x+1)^2 = 6^2 + 2^2 - 2 \times 6 \times 2 \times \frac{5}{8}$

$x^2 + 2x + 1 = 36 + 4 - 15$

$x^2 + 2x - 24 = 0$

$(x+6)(x-4) = 0$

So $x = 4 (x > -1)$

b Use identity, $\cos^2 x + \sin^2 x = 1$.

$\cos B = \frac{5}{8}$

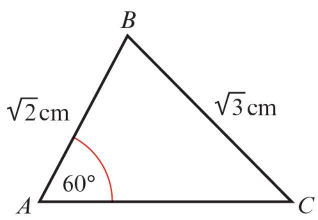
so $\sin \angle ABC = \frac{\sqrt{39}}{8}$

Area $= \frac{1}{2}ac\sin B$

$= \frac{1}{2} \times 6 \times 2 \times \frac{\sqrt{39}}{8}$

$= 4.68\text{ cm}^2$ (3 s.f.)

9



Using $\frac{\sin C}{c} = \frac{\sin A}{a}$

$\sin C = \frac{\sqrt{2}\sin 60°}{\sqrt{3}}$

$= 0.7071\ldots$

$C = \sin^{-1}\left(\frac{\sqrt{2}\sin 60°}{\sqrt{3}}\right)$

$= 45°$

$B = 180° - (60 + 45)°$

$= 75°$

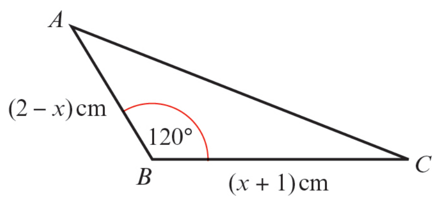
Using $\frac{b}{\sin B} = \frac{a}{\sin A}$

$\frac{AC}{\sin 75°} = \frac{\sqrt{3}}{\sin 60°}$

So $AC = \frac{\sqrt{3}\sin 75°}{\sin 60°}$

$= 1.93\text{ cm}$ (3 s.f.)

10



a Using the cosine rule:

$b^2 = a^2 + c^2 - 2ac\cos B$

$AC^2 = (x+1)^2 + (2-x)^2$
$\quad - 2(x+1)(2-x)\cos 120°$

$AC^2 = (x^2 + 2x + 1) + (4 - 4x + x^2)$
$\quad + (x+1)(2-x)$

$AC^2 = x^2 + 2x + 1 + 4 - 4x + x^2$
$\quad - x^2 + 2x - x + 2$

$AC^2 = x^2 - x + 7$

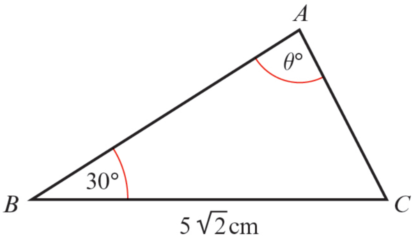
b Completing the square:

$x^2 - x + 7 \equiv \left(x - \frac{1}{2}\right)^2 + 7 - \frac{1}{4}$

$\equiv \left(x - \frac{1}{2}\right)^2 + 6\frac{3}{4}$

This is a minimum when $x - \frac{1}{2} = 0 \Rightarrow x = \frac{1}{2}$.

11



Using $\frac{b}{\sin B} = \frac{a}{\sin A}$

$$\frac{AC}{\sin 30°} = \frac{5\sqrt{2}}{\sin \theta°}$$

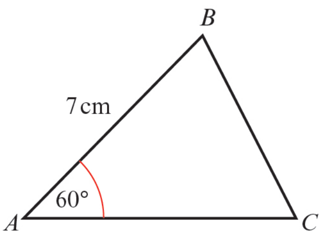
$$AC = \frac{5\sqrt{2}\sin 30°}{\left(\frac{\sqrt{5}}{8}\right)}$$

$$AC = \frac{5\sqrt{2}\sin 30° \times 8}{\sqrt{5}}$$

$$= \left(\sqrt{5}\sqrt{2}\right)\left(8\sin 30°\right)$$

$$= 4\sqrt{10} \text{ cm}$$

12



Using the cosine rule

$a^2 = b^2 + c^2 - 2bc\cos A$

with $a = x$, $b = (8 - x)$, $c = 7$ and $A = 60°$

$$x^2 = (8-x)^2 + 49 - 2(8-x) \times 7 \times \cos 60°$$

$$x^2 = 64 - 16x + x^2 + 49 - 7(8-x)$$

$$x^2 = 64 - 16x + x^2 + 49 - 56 + 7x$$

$$\Rightarrow 9x = 57$$

$$\Rightarrow x = \tfrac{57}{9} = \tfrac{19}{3} = 6\tfrac{1}{3}$$

So $BC = 6\tfrac{1}{3}$ cm and

$$AC = \left(8 - 6\tfrac{1}{3}\right)\text{cm}$$

$$= 1\tfrac{2}{3}\text{cm}$$

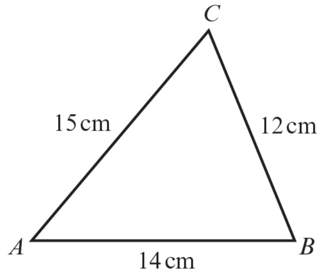
$$\text{Area } = \tfrac{1}{2}bc\sin A$$

$$= \tfrac{1}{2} \times 7 \times \tfrac{5}{3} \times \sin 60°$$

$$= 5.0518\ldots$$

$$= 5.05 \text{ cm}^2 \text{ (3 s.f.)}$$

13 a



$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\cos C = \frac{12^2 + 15^2 - 14^2}{2(12)(15)}$$

$$\cos C = \frac{144 + 225 - 196}{360}$$

$C = 61.278...$

$C = 61.3°$ (3 s.f.)

b Use the formula.

$$\text{Area } = \tfrac{1}{2}ab\sin C$$

$$= \tfrac{1}{2} \times 12 \times 15 \times \sin 61.3°$$

$$= 78.943\ldots$$

$$= 78.9 \text{ cm}^2 \text{ (3 s.f.)}$$

14 a

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos A = \frac{2.1^2 + 4.2^2 - 5.9^2}{2(2.1)(4.2)}$$

$$\cos A = \frac{4.41 + 17.64 - 34.81}{17.64}$$

A = 136.33...

∴ Angle $DAB = 136.3°$ (1 d.p.)

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\cos C = \frac{3.5^2 + 7.5^2 - 5.9^2}{2(3.5)(7.5)}$$

$$\cos C = \frac{12.25 + 56.25 - 34.81}{52.5}$$

$C = 50.080...$

∴ Angle $BCD = 50.1°$

b Area ABD $= \tfrac{1}{2}bc\sin A$

$$= \tfrac{1}{2} \times 2.1 \times 4.2 \times \sin 136.3°$$

$$= 3.046\ 79...$$

14 b Area $BCD = \frac{1}{2}ab\sin C$

$= \frac{1}{2} \times 3.5 \times 7.5 \times \sin 50.1°$

$= 10.069\ 04...$

Total area = 3.046 79 + 10.069 04

= 13.11583

$\therefore$ The area of the flower bed is 13.1 m^2.

c First find angle ADB:

$$\cos D = \frac{a^2 + b^2 - d^2}{2ab}$$

$$\cos D = \frac{5.9^2 + 2.1^2 - 4.2^2}{2(5.9)(2.1)}$$

$$\cos D = \frac{34.81 + 4.41 - 17.64}{24.78}$$

So $D = 29.440\ 849...$

Now find angle BDC:

$$\cos D = \frac{b^2 + c^2 - d^2}{2bc}$$

$$\cos D = \frac{3.5^2 + 5.9^2 - 7.5^2}{2(3.5)(5.9)}$$

$$\cos D = \frac{12.25 + 34.81 - 56.25}{41.3}$$

So D = 102.856 97...

Angle ADC = 29.440849 + 102.85697

= 132.298°

Now find the length AC:

$$d^2 = a^2 + c^2 - 2ac\cos D$$

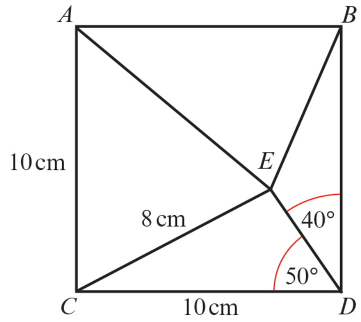
$$d^2 = 3.5^2 + 2.1^2 - 2 \times 3.5 \times 2.1 \times \cos 132.298°$$

$$d^2 = 12.25 + 4.41 + 9.8929$$

So $d = 5.15$

So the length of AC is 5.15 m.

15



Use the sine rule to work out angle CED.

$$\frac{\sin E}{e} = \frac{\sin D}{d}$$

$$\frac{\sin E}{10} = \frac{\sin 50°}{8}$$

$$\sin E = \frac{10\sin 50°}{8}$$

E = 73.246 86° or 106.753 14°

The angle is obtuse so

Angle CED = 106.753 14°

Angle ECD = 180° − 50° − 106.753 14°

= 23.25°

Use trigonometry to work out the height of triangle CDE.

$$\sin 23.25° = \frac{\text{height}}{8}$$

Height = 3.1575 cm

The height of triangle ABE = 10 − 3.1575

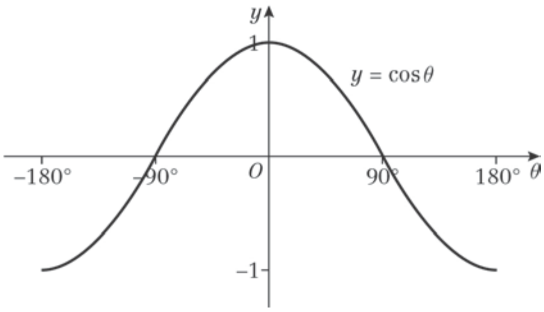
= 6.84 cm

Area of triangle = $\frac{1}{2} \times 10 \times 6.84 = 34.2$

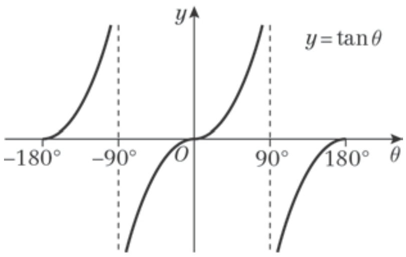
$\therefore$ Area of the shaded triangle is 34.2 cm^2.

Trigonometric ratios 9F

1



2



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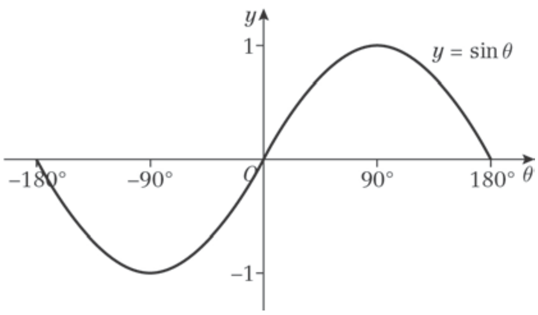
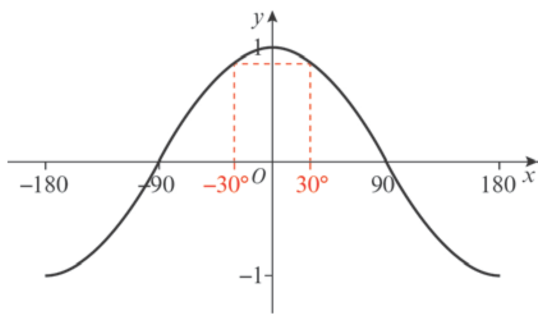
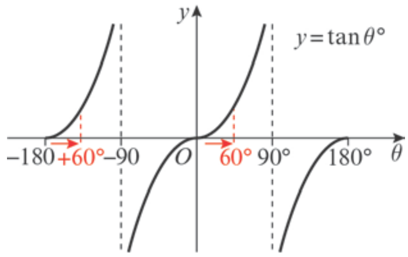


4 a



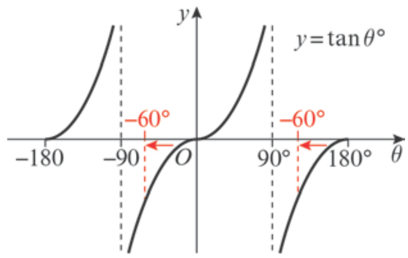
$\theta = -30°$

b i

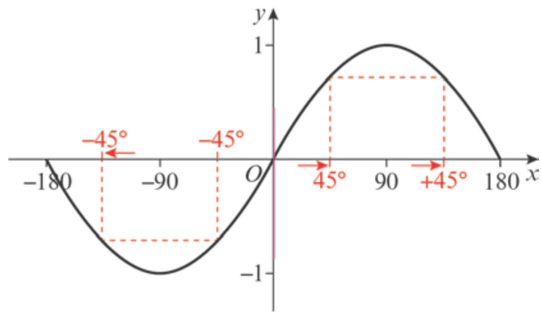


$\theta = -120°$, $(-180° + 60°)$

ii $\theta = -60°$ or $\theta = 120°$



c i



$\theta = 135°$

ii $\theta = -45°$ or $\theta = 225°$

Trigonometric ratios 9G

1 a i The maximum value of cos $x°$ is 1. This occurs when $x = 0$.

ii Minimum value is –1, which occurs when $x = 180$.

b i Maximum value of sin $x°$ is 1, so maximum value of 4 sin $x°$ is 4. This occurs when $x = 90$.

ii Minimum value of 4 sin $x°$ is = –4. This occurs when $x = 270$.

c The graph, cos $(–x)°$ is a reflection of the graph of cos $x°$ in the y-axis. This is the same curve; cos $(–x)° =$ cos $x°$.

i Maximum value of cos $(–x)°$ is 1. This occurs when $x = 0$.

ii Minimum value of cos $(–x)°$ is –1. This occurs when $x = 180$.

d The graph of 3 + sin $x°$ is the graph of sin $x°$ translated by +3 vertically.

i Maximum is 4 when $x = 90$.
ii Minimum is 2 when $x = 270$.

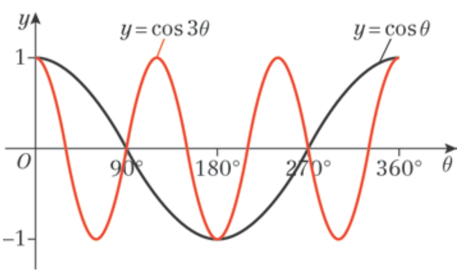
e The graph of –sin $x°$ is the reflection of the graph of sin $x°$ in the x-axis.

i Maximum is 1 when $x = 270$.
ii Minimum is –1 when $x = 90$.

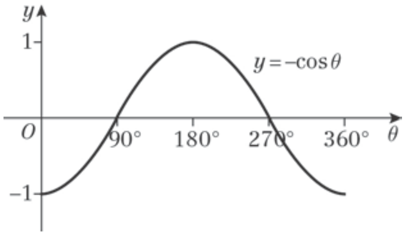
f The graph of sin $3x°$ is the graph of sin $x°$ stretched by $\frac{1}{3}$ in the x direction.

i Maximum is 1 when $x = 30$.
ii Minimum is –1 when $x = 90$.

2

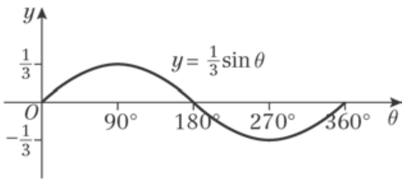


3 a The graph of $y = –\cos\theta$ is the graph of $y = \cos\theta$ reflected in the θ-axis.



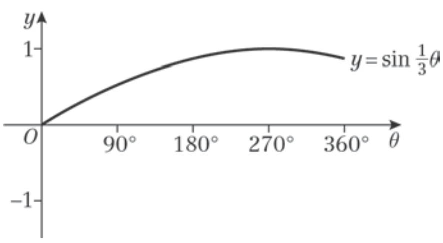
The graph:
meets the θ-axis at (90°, 0), (270°, 0)
meets the y-axis at (0°, –1)
has a maximum at (180°, 1)
has minima at (0°, –1) and (360°, –1).

b The graph of $y = \frac{1}{3}\sin\theta$ is the graph of $y = \sin\theta$ stretched by scale factor $\frac{1}{3}$ in the y direction.



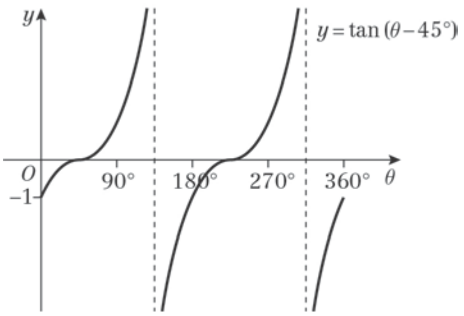
The graph:
meets θ-axis at (0°, 0), (180°, 0), (360°, 0)
meets y-axis at (0°, 0)
has a maximum at $\left(90°, \frac{1}{3}\right)$
has a minimum at $\left(270°, -\frac{1}{3}\right)$.

c The graph of $y = \sin\frac{1}{3}\theta$ is the graph of $y = \sin\theta$ stretched by scale factor 3 in θ direction.



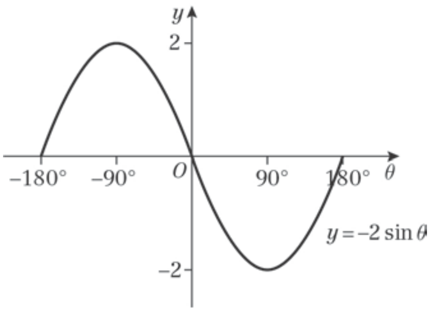
The graph:
only meets the axes at the origin,
has a maximum at (270°, 1).

3 d The graph of $y = \tan(\theta - 45°)$ is the graph of $\tan\theta$ translated by 45° to the right.



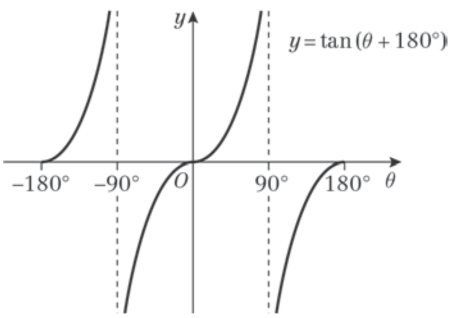
The graph:
meets the θ-axis at (45°, 0), (225°, 0),
meets the y-axis at (0°, –1),
has asymptotes at θ = 135° and θ = 315°.

4 a This is the graph of $y = \sin\theta°$ stretched by scale factor –2 in the y-direction (i.e. reflected in the θ-axis and scaled by 2 in the y-direction).



The graph:
meets the θ-axis at (–180°, 0), (0°, 0), (180°, 0),
has a maximum at (–90°, 2),
has a minimum at (90°, –2).

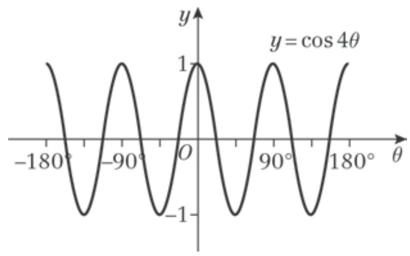
b This is the graph of $y = \tan\theta°$ translated by 180° to the left.



As $\tan\theta°$ has a period of 180°,
$\tan(\theta + 180)° = \tan\theta$

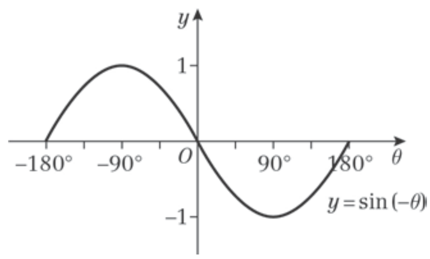
4 b The graph meets the θ-axis at (–180°, 0), (0°, 0), (180°, 0)

c This is the graph of $y = \cos\theta°$ stretched by scale factor $\frac{1}{4}$ horizontally.



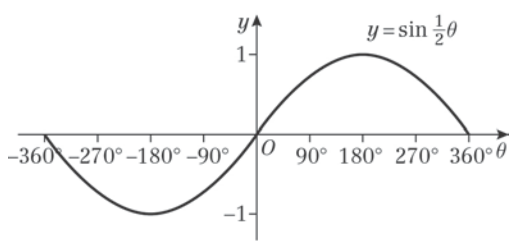
The graph:
meets the θ-axis at $\left(-157\frac{1}{2}°, 0\right)$, $\left(-112\frac{1}{2}°, 0\right)$, $\left(-67\frac{1}{2}°, 0\right)$, $\left(-22\frac{1}{2}°, 0\right)$, $\left(22\frac{1}{2}°, 0\right)$, $\left(67\frac{1}{2}°, 0\right)$, $\left(112\frac{1}{2}°, 0\right)$, $\left(157\frac{1}{2}°, 0\right)$
meets the y-axis at (0°, 1)
has maxima at (–180°, 1), (–90°, 1), (0°, 1), (90°, 1), (180°, 1)
has minima at (–135°, –1), (–45°, –1), (45°, –1), (135°, –1).

d This is the graph of $y = \sin\theta°$ reflected in the y-axis.
(This is the same as $y = -\sin\theta°$.)

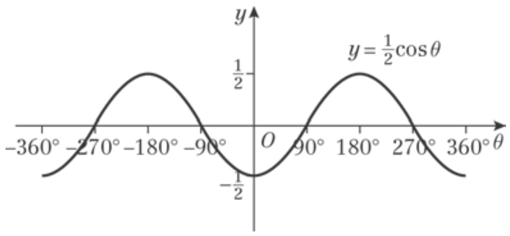


The graph:
meets the θ-axis at (–180°, 0), (0°, 0), (180°, 0)
has a maximum at (–90°, 1)
has a minimum at (90°, –1).

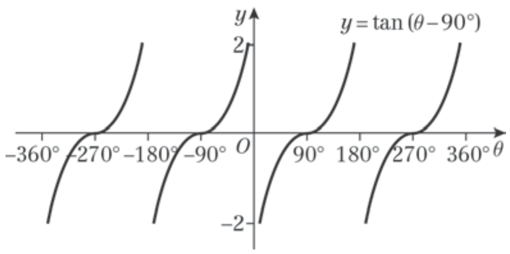
5 a Period = 720°



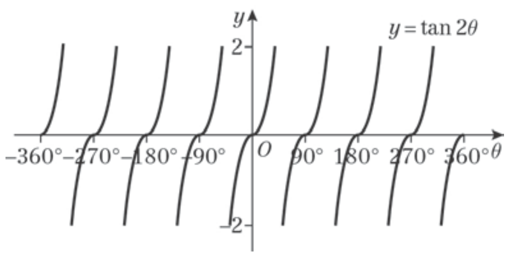
5 b Period = 360°



c Period = 180°

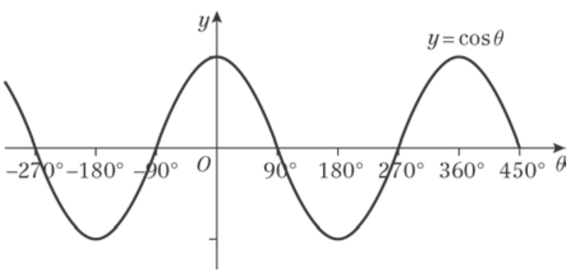


d Period = 90°

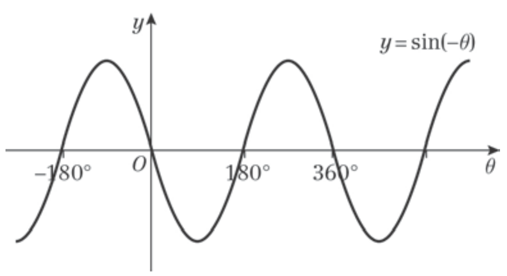


6 a i $y = \cos(-\theta)$ is a reflection of

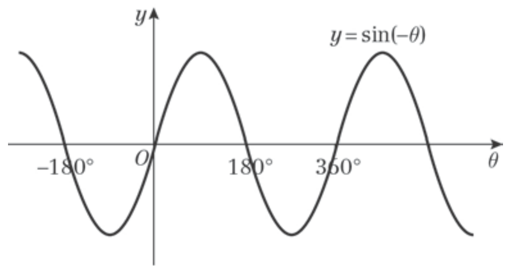
$y = \cos\theta$ in the y-axis, which is the same curve, so $\cos\theta = \cos(-\theta)$.



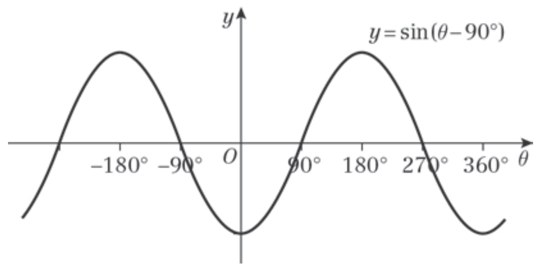
ii $y = \sin(-\theta)$ is a reflection of $y = \sin\theta$ in the y-axis.



6 a ii $y = -\sin(-\theta)$ is a reflection of $y = \sin(-\theta)$ in the θ-axis, which is the graph of $y = \sin\theta$, so $-\sin(-\theta) = \sin\theta$.



iii $y = \sin(\theta - 90°)$ is the graph of $y = \sin\theta$ translated by 90° to the right, which is the graph of $y = -\cos\theta$.
So $\sin(\theta - 90°) = -\cos\theta$.



b Using **a ii**

$$\sin(90° - \theta) = -\sin(-(90° - \theta))$$
$$= -\sin(\theta - 90°)$$

Using **a iii**

$$-\sin(\theta - 90°) = -(-\cos\theta)$$
$$= \cos\theta$$

So $\sin(90° - \theta) = \cos\theta$.

c Using **a i**

$$\cos(90° - \theta) = \cos(\theta - 90°)$$
$$= \sin\theta$$

So $\cos(90° - \theta) = \sin\theta$.

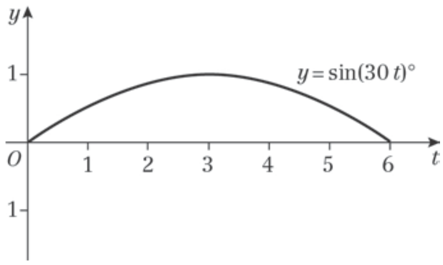
7 a The curve crosses the x-axis at −270° − 30°, −90° − 30°, 90° − 30° and 270° − 30°; θ = −300°, −120°, 60° and 240°.
Coordinates are (−300°, 0), (−120°, 0), (60°, 0) and (240°, 0)

b $\cos 30° = \dfrac{\sqrt{3}}{2}$; $\left(0, \dfrac{\sqrt{3}}{2}\right)$

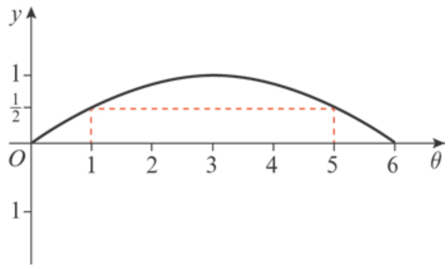
8 a The graph is a translation left 60° of the sine graph.
Therefore, $y = \sin(x + 60°)$
$k = 60°$

b Yes, the graph could be a translation right 300°, so $y = \sin(x - 300°)$

9 a



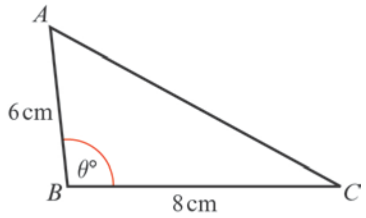
9 b



Between 1 p.m. and 5 p.m.

Trigonometric ratios, Mixed Exercise 9

1



a Using area of $\Delta ABC = \frac{1}{2}ac\sin B$

$10\,\text{cm}^2 = \frac{1}{2}\times 6\times 8\times \sin\theta°\text{cm}^2$

So $10 = 24\sin\theta°$

So $\sin\theta° = \frac{10}{24} = \frac{5}{12}$

$\Rightarrow \theta = 24.6 \text{ or } 155\ (3\text{ s.f.})$

As θ is obtuse, $\angle ABC = 155°\ (3\text{ s.f.})$

b Using the cosine rule

$b^2 = a^2 + c^2 - 2ac\cos B$

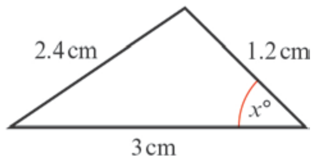
$AC^2 = 8^2 + 6^2 - 2\times 8\times 6\times \cos B$

$= 187.26\ldots$

$AC = 13.68\ldots$

The third side has length 13.7 m (3 s.f.).

2 a



Using the cosine rule

$$\cos x° = \frac{3^2 + 1.2^2 - 2.4^2}{2\times 3\times 1.2}$$

$= 0.65$

$x = \cos^{-1}(0.65)$

$= 49.458\ldots$

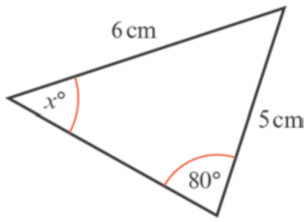
$x = 49.5\ (3\text{ s.f.})$

Using the area of a triangle formula

area $= \frac{1}{2}\times 1.2\times 3\times \sin x°\text{cm}^2$

$= 1.37\text{ cm}^2\ (3\text{ s.f.})$

2 b



Using the sine rule

$$\frac{\sin x°}{5} = \frac{\sin 80°}{6}$$

$$\sin x° = \frac{5\sin 80°}{6}$$

$= 0.8206\ldots$

$x = 55.2\ (3\text{ s.f.})$

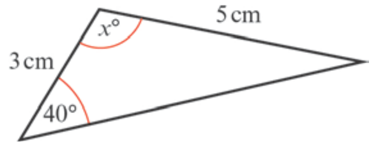
The angle between the 5 cm and 6 cm sides is $180° - (80 + x)° = (100 - x)°$.

Using the area of a triangle formula:

area $= \frac{1}{2}\times 5\times 6\times \sin(100 - x)°\text{ cm}^2$

$= 10.6\text{ cm}^2\ (3\text{ s.f.})$

c



Use the sine rule to find the angle opposite the 3 cm side. Call this $y°$.

$$\frac{\sin y°}{3} = \frac{\sin 40°}{5}$$

$$\sin y° = \frac{3\sin 40°}{5}$$

$\Rightarrow y = 22.68\ldots$

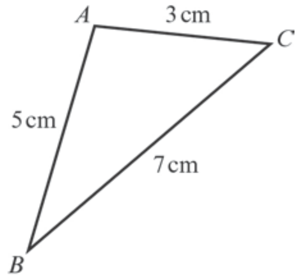
So $x = 180 - (40 + y)$

$= 117\ (3\text{ s.f.})$

Area of triangle $= \frac{1}{2}\times 3\times 5\times \sin x°$

$= 66.6\text{ cm}^2\ (3\text{ s.f.})$

3

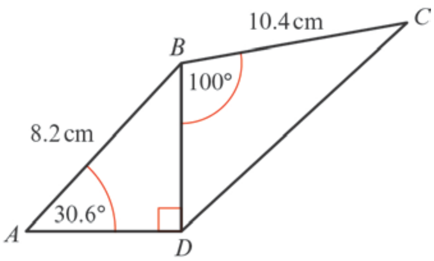


Use the cosine rule to find angle A.

$$\cos A = \frac{3^2+5^2-7^2}{2\times3\times5}$$
$$= -0.5$$
$$A = \cos^{-1}(-0.5)$$
$$= 120°$$

$$\text{Area of triangle} = \tfrac{1}{2}\times3\times5\times\sin A\,\text{cm}^2$$
$$= 6.495\ldots\,\text{cm}^2$$
$$= 6.50\ \text{cm}^2\ (3\text{ s.f.})$$

4 a



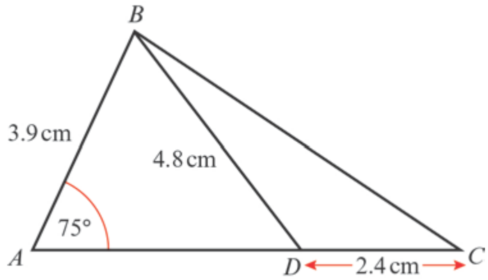
In ΔBDA, $\dfrac{BD}{8.2} = \sin 30.6°$

$$\text{So } BD = 8.2\sin 30.6°$$
$$= 4.174\ldots$$
$$\frac{AD}{8.2} = \cos 30.6$$
$$AD = 8.2\cos 30.6 = 7.0580...$$
$$\angle ABD = 90° - 30.6°$$
$$= 59.4°$$

We can use AD and BD to calculate the area of ΔABD or use:

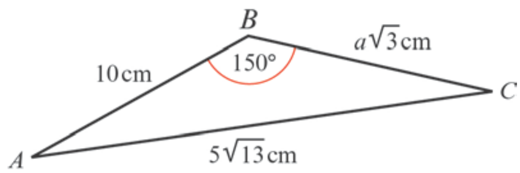
$$\text{Area of } \Delta ABD = \tfrac{1}{2}\times8.2\times BD\times\sin 59.4°$$
$$= 14.7307\ldots\,\text{cm}^2$$
$$\text{Area of } \Delta BDC = \tfrac{1}{2}\times10.4\times BD\times\sin 100°$$
$$= 21.375\ldots\,\text{cm}^2$$
$$\text{Total area} = \text{area } \Delta ABD + \text{area } \Delta BDC$$
$$= 36.1\,\text{cm}^2\ (3\text{ s.f.})$$

4 b



$$\frac{\sin\angle ADB}{3.9} = \frac{\sin 75°}{4.8}$$
$$\sin\angle ADB = \frac{3.9\sin 75°}{4.8}$$
$$\angle ADB = \sin^{-1}\left(\frac{3.9\sin 75°}{4.8}\right)$$
$$= 51.7035\ldots°$$
$$\text{So } \angle ABD = 180° - (75 + \angle ADB)°$$
$$= 53.296\ldots°$$
$$\text{Area of } \Delta ABD = \tfrac{1}{2}\times3.9\times4.8\times\sin\angle ABD$$
$$= 7.504\ldots\,\text{cm}^2$$
$$\text{In } \Delta BDC, \angle BDC = 180° - \angle ADB$$
$$= 128.29\ldots°$$
$$\text{Area of } \Delta BDC = \tfrac{1}{2}\times2.4\times4.8\times\sin\angle BDC$$
$$= 4.520\ldots\,\text{cm}^2$$
$$\text{Total area} = \text{area } \Delta ABD + \text{area } \Delta BDC$$
$$= 12.0\,\text{cm}^2\ (3\text{ s.f.})$$

5



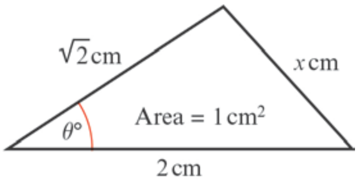
a Using the cosine rule:

$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$\left(5\sqrt{13}\right)^2 = \left(a\sqrt{3}\right)^2 + 10^2$$
$$-2\times a\sqrt{3}\times10\times\cos 150°$$
$$325 = 3a^2 + 100 + 30a$$
$$3a^2 + 30a - 225 = 0$$
$$a^2 + 10a - 75 = 0$$
$$(a+15)(a-5) = 0$$
$$\Rightarrow a = 5 \text{ as } a > 0$$

5 b Area $\Delta ABC = \frac{1}{2} \times 10 \times 5\sqrt{3} \times \sin 150°$

$= 12.5\sqrt{3}\text{ cm}^2$

6



Using the area formula:

$1 = \frac{1}{2} \times 2 \times \sqrt{2} \times \sin\theta°$

$\Rightarrow \sin\theta° = \dfrac{1}{\sqrt{2}}$

$\Rightarrow \theta = 45$ or 135

But as θ is not the largest angle, θ must be 45.

Use the cosine rule to find x.

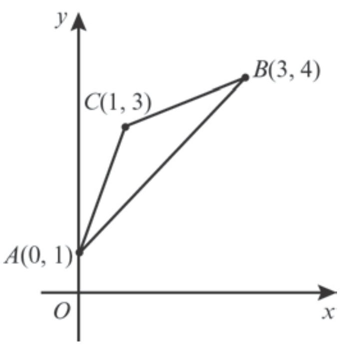
$x^2 = 2^2 + \left(\sqrt{2}\right)^2 - 2 \times 2 \times \sqrt{2} \times \cos 45°$

$x^2 = 4 + 2 - 4 = 2$

So $x = \sqrt{2}$

So the triangle is isosceles with two angles of 45°. It is a right-angled isosceles triangle.

7



a Use Pythagoras' theorem.

$AC = \sqrt{(1-0)^2 + (3-1)^2}$

$= \sqrt{5}$

$= b$

$BC = \sqrt{(3-1)^2 + (4-3)^2}$

$= \sqrt{5}$

$= a$

7 a $AB = \sqrt{(3-0)^2 + (4-1)^2}$

$= \sqrt{18}$

$= c$

Using the cosine rule

$\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

$\cos C = \dfrac{5 + 5 - 18}{2 \times \sqrt{5} \times \sqrt{5}}$

$= \dfrac{-8}{10}$

$= \dfrac{-4}{5}$

Find sin C by using the identity, $\cos^2 x + \sin^2 x = 1$ or by drawing a 3,4,5 triangle and looking at the ratio of the sides.

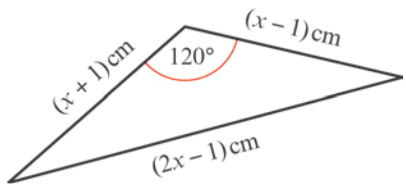
b Using the area formula:

area of $\Delta ABC = \frac{1}{2}ab\sin C$

$= \frac{1}{2} \times \sqrt{5} \times \sqrt{5} \times \sin C$

$= 1.5\text{ cm}^2$

8



a Using the cosine rule

$(2x-1)^2 = (x+1)^2 + (x-1)^2$
$- 2(x+1)(x-1)\cos 120°$

$4x^2 - 4x + 1 = (x^2 + 2x + 1)$
$+ (x^2 - 2x + 1) + (x^2 - 1)$

$4x^2 - 4x + 1 = 3x^2 + 1$

$x^2 - 4x = 0$

$x(x-4) = 0$

$\Rightarrow x = 4 \quad x > 1$

b Area of $\Delta = \frac{1}{2} \times (x+1) \times (x-1) \times \sin 120°$

$= \frac{1}{2} \times 5 \times 3 \times \sin 120°$

$= \dfrac{1}{2} \times 5 \times 3 \times \dfrac{\sqrt{3}}{2}$

8 b Area of $\Delta = \dfrac{15\sqrt{3}}{4}$

$= 6.50\text{ cm}^2$ (3 s.f.)

9 a $b^2 = a^2 + c^2 - 2ac\cos B$

$= 1.4^2 + 1.2^2 - 2\times1.4\times1.2\times\cos 70°$

$= 1.96 + 1.44 - 1.149\,187\,68$

So $b = 1.500\,027\ldots$

So point C is 1.50 km from the park keeper's hut.

b $\dfrac{\sin A}{a} = \dfrac{\sin B}{b}$

$\dfrac{\sin A}{1.4} = \dfrac{\sin 70°}{1.5}$

$\sin A = \dfrac{1.4\sin 70°}{1.5}$

So $A = 61.288\,10°$

Bearing = 360° − (180° − 61.288 10°)

= 241.29°

The bearing of the hut from point C is 241°.

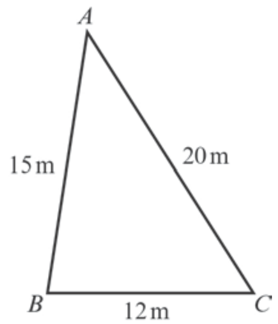
c Area of $\Delta = \frac{1}{2}ac\sin B$

$= \frac{1}{2}\times 1.4\times 1.2\times \sin 70°$

$= 0.789\,34\ldots$

$= 0.789\text{ km}^2$ (3 s.f.)

10



$\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

$\cos A = \dfrac{20^2 + 15^2 - 12^2}{2(20)(15)}$

$\cos A = \dfrac{400 + 225 - 144}{600}$

So $A = 36.7°$

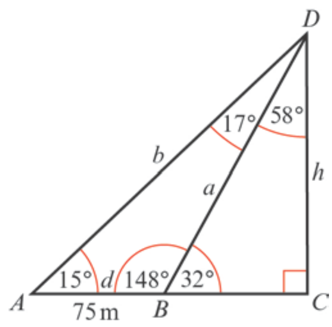
Area of one sail = $\frac{1}{2}bc\sin A$

$= \frac{1}{2}\times 20\times 15\times \sin 36.7°$

$= 89.665\ldots$

Area of all four sails = 359 m^2 (3 s.f.)

11



Using triangle ABD, the angles are 15°, 148° and 17°.

$\dfrac{b}{\sin B} = \dfrac{d}{\sin D}$

$\dfrac{b}{\sin 148°} = \dfrac{75}{\sin 17°}$

$b = \dfrac{75\sin 148°}{\sin 17°}$

$b = 135.936\ldots$

Using the larger right-angled triangle:

$\sin 15° = \dfrac{\text{height}}{135.936}$

height = 135.936 sin15°

= 35.1829...

The height of the church tower is 35.2 m (3 s.f.).

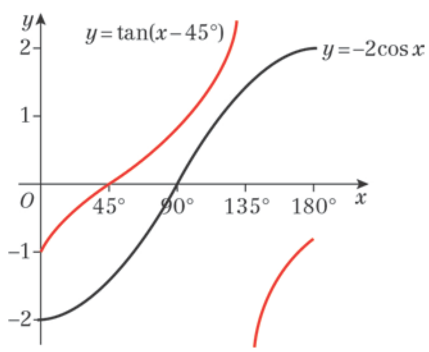
12 a A stretch of scale factor 2 in the x direction.

b A translation of +3 in the y direction.

c A reflection in the x-axis.

d A translation of 20 in the negative x direction (i.e. 20 to the left).

13 a



b $\tan(x - 45°) + 2\cos x = 0$

$\tan(x - 45°) = -2\cos x$

The graphs do not intersect so there are no solutions.

14 a As it is the graph of $y = \sin x°$ translated, the gap between A and B is 180, so $p = 300$.

b The difference in the x-coordinates of D and A is 90, so the x-coordinate of D is 30. The maximum value of y is 1, so D is the point (30, 1).

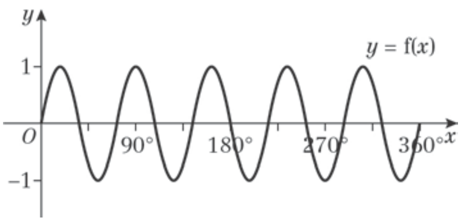
c For the graph of $y = \sin x°$, the first positive intersection with the x-axis would occur at 180. The point A is at 120 and so the curve has been translated by 60 to the left.

$k = 60$

d The equation of the curve is $y = \sin(x + 60)°$.

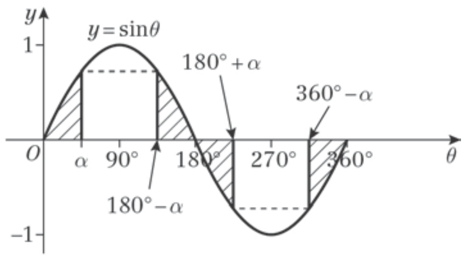
When $x = 0,\ y = \sin 60° = \frac{\sqrt{3}}{2}$, so $q = \frac{\sqrt{3}}{2}$.

15 a The graph of $y = \sin x$ crosses the x-axis at (180°, 0).
f(x) = sin px is a stretch horizontally with scale factor $\frac{36}{180} = \frac{1}{5}$.
f(x) = sin 5x
$p = 5$



b The period of f(x) is 360 ÷ 5 = 72°.

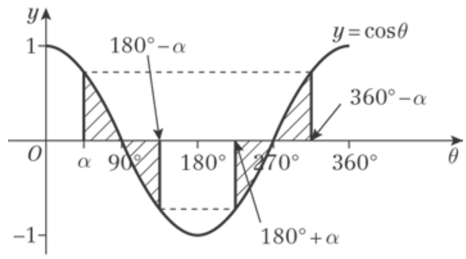
16 a

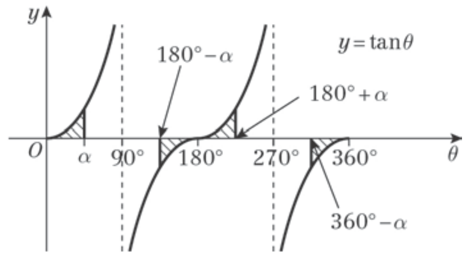


b The four shaded regions are congruent therefore the magnitude of the y value is the same for sin α.
Sin α° and sin (108 – α)° have the same y value (call it k).

16 b So sin α° = sin (180 – α)°, sin(180 \+ α)° and sin(360 – α)° have the same y value, which will be –k.
So sinα° = sin(180 – α)°
= –sin(180 + α)°
= –sin(360 – α)°

17 a





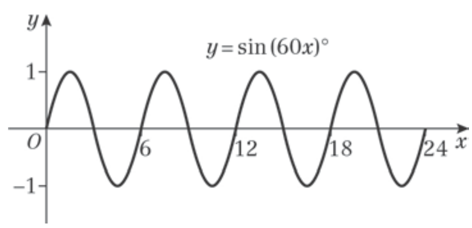
b i From the graph of $y = \cos \theta°$, which shows four congruent shaded regions, if the y value at $\alpha°$ is k, then y at (180 – α)° is –k, y at (180 + α)° is –k and y at (360 – α)° it is +k.

So cosα° = –cos (180° – α)
= –cos (180° + α)
= cos (360° – α)

ii From the graph of $y = \tan \theta°$, if the y value at $\alpha°$ is k, then at (180 – α)° it is is –k, at (180° + α) it is +k and at (360° – α) it is –k.

So tan α° = –tan (180° – α)
= +tan (180° + α)
= −tan (360° – α)

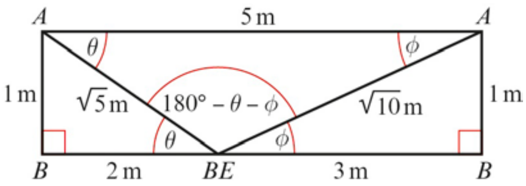
18 a



b There are 4 complete waves in the interval $0 \le x \le 24°$ so there are 4 sand dunes in this model.

18 c The sand dunes may not all be the same height.

Challenge



$\angle ACB = \tan^{-1} 1 = 45°$

Show that $\theta + \phi = 45°$

$\sin\theta = \dfrac{1}{\sqrt{5}}$

Using the sine rule:

$$\frac{\sin(180° - \theta - \phi)}{5} = \frac{\sin\theta}{\sqrt{10}}$$

$$\sin(180° - \theta - \phi) = \frac{5\sin\theta}{\sqrt{10}}$$

Substituting $\sin\theta = \dfrac{1}{\sqrt{5}}$:

$$\sin(180° - \theta - \phi) = \frac{5\left(\frac{1}{\sqrt{5}}\right)}{\sqrt{10}}$$

$$= \frac{5}{\sqrt{50}}$$

$$= \frac{5}{5\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}}$$

$\sin^{-1}\left(\dfrac{1}{\sqrt{2}}\right) = 45°$, but angle $180° - \theta - \phi$ is obtuse.

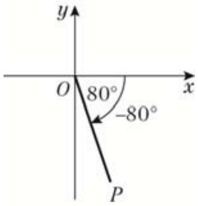
So, $180° - \theta - \phi = 180° - 45° = 135°$

Therefore, $\theta + \phi = 45°$

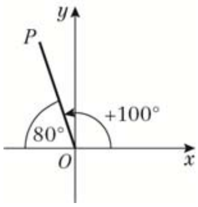
So, $\angle AEB + \angle ADB = \angle ACB$

Trigonometric identities and equations 10A

1 a



b



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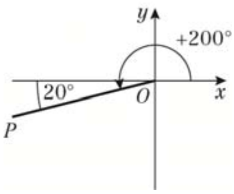
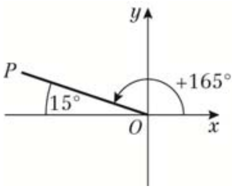
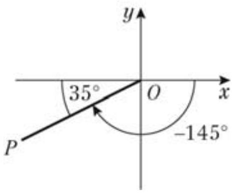


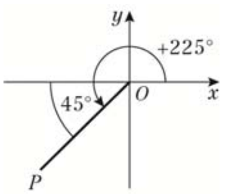
d



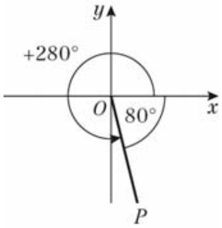
e



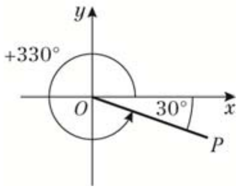
f



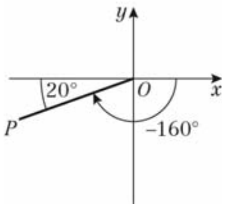
g



h



i



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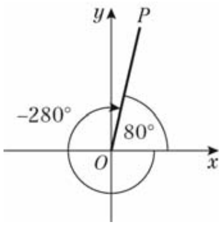
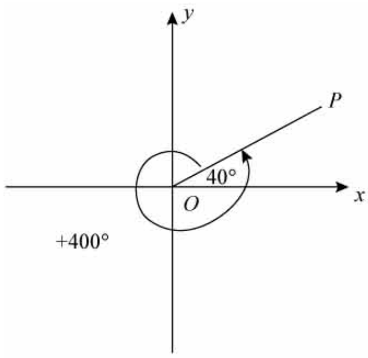
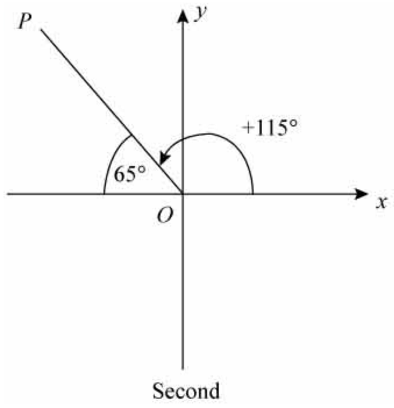


2 a



First

2 b



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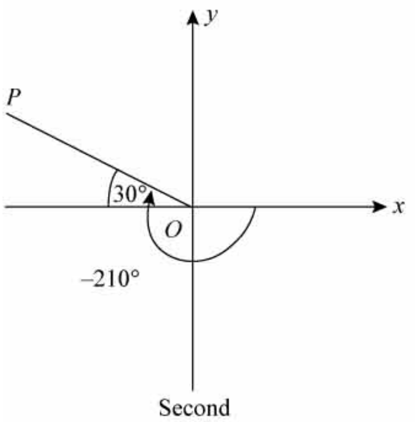

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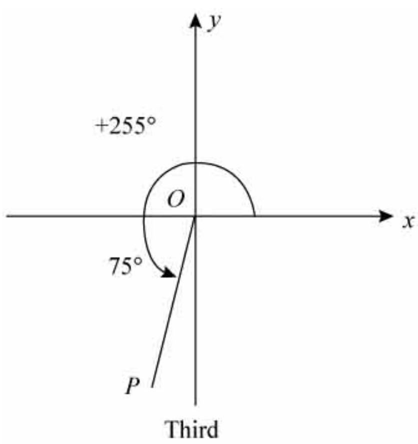
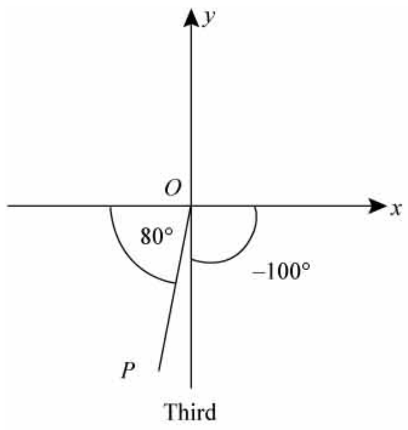
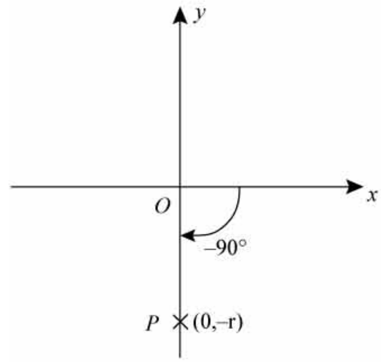


2 e

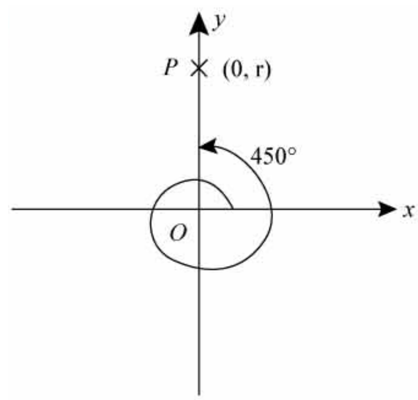


3 a



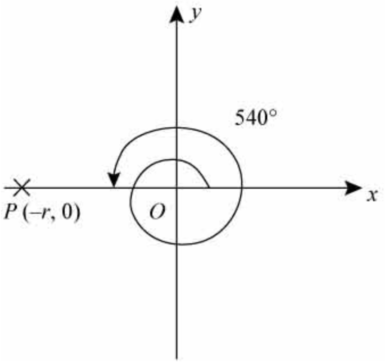
$$\sin(-90)° = \frac{-r}{r} = -1$$

b



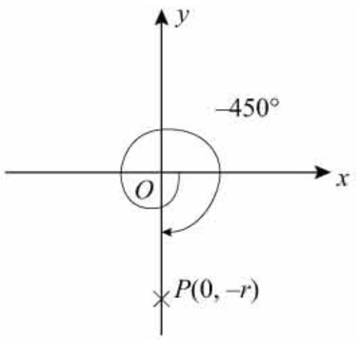
$$\sin 450° = \frac{r}{r} = 1$$

3 c



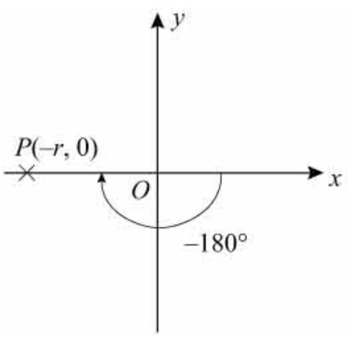
$$\sin 540° = \frac{0}{r} = 0$$

d



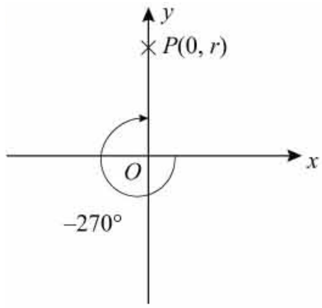
$$\sin(-450)° = \frac{-r}{r} = -1$$

e



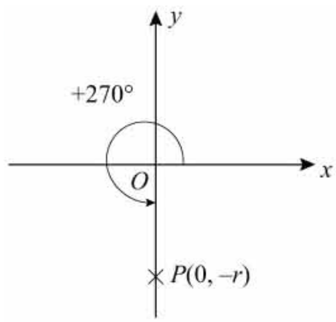
$$\cos(-180)° = \frac{-r}{r} = -1$$

f



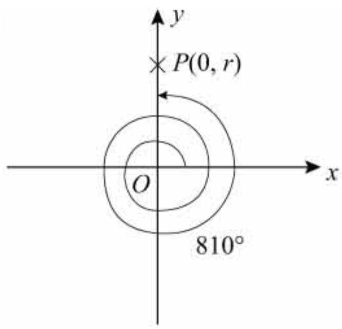
$$\cos(-270)° = \frac{0}{r} = 0$$

g



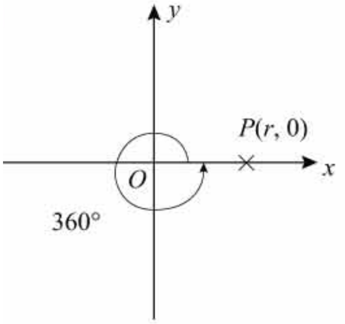
$$\cos 270° = \frac{0}{r} = 0$$

h



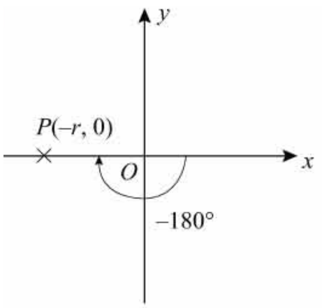
$$\cos 810° = \frac{0}{r} = 0$$

3 i



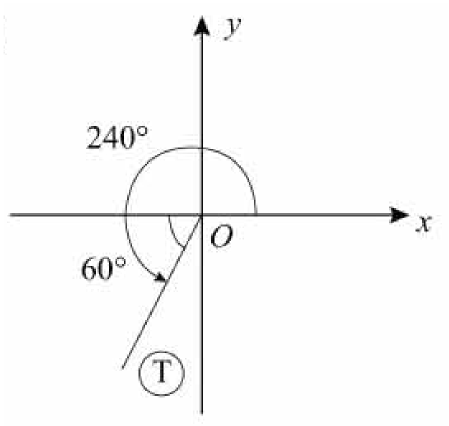
$$\tan 360° = \frac{0}{r} = 0$$

j



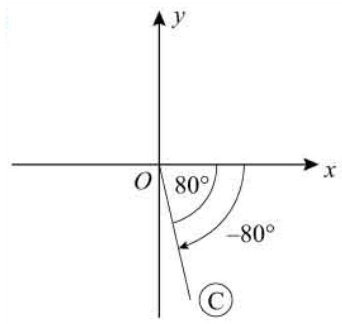
$$\tan(-180)° = \frac{0}{-r} = 0$$

4 a



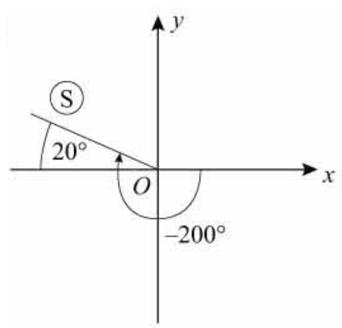
60° is the acute angle.
In the third quadrant sin is − ve.
So $\sin 240° = -\sin 60°$

b



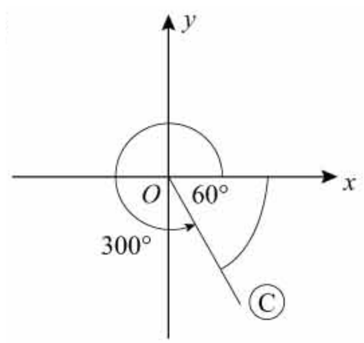
80° is the acute angle.
In the fourth quadrant sin is − ve.
So $\sin(-80)° = -\sin 80°$

c



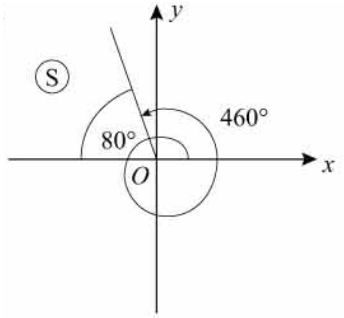
20° is the acute angle.
In the second quadrant sin is + ve.
So $\sin(-200)° = +\sin 20°$

d



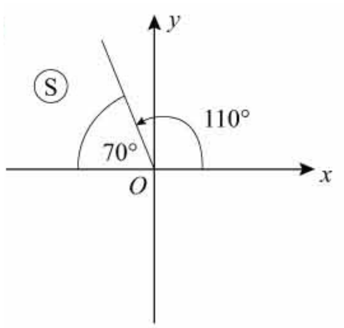
60° is the acute angle.
In the fourth quadrant sin is − ve.
So $\sin 300° = -\sin 60°$

4 e



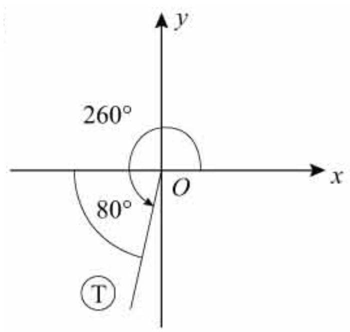
80° is the acute angle.
In the second quadrant sin is + ve.
So $\sin 460° = +\sin 80°$

f



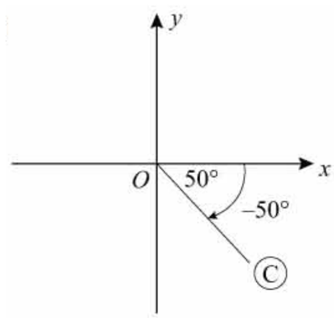
70° is the acute angle.
In the second quadrant cos is − ve.
So $\cos 110° = -\cos 70°$

g



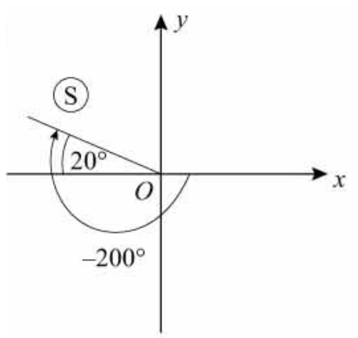
80° is the acute angle.
In the third quadrant cos is − ve.
So $\cos 260° = -\cos 80°$

h



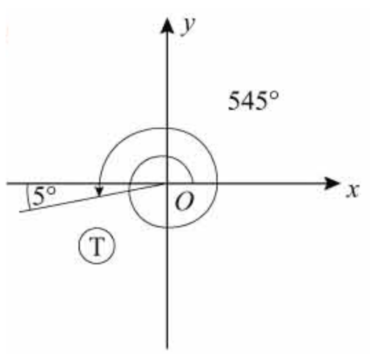
50° is the acute angle.
In the fourth quadrant cos is + ve.
So $\cos(-50)° = +\cos 50°$

i



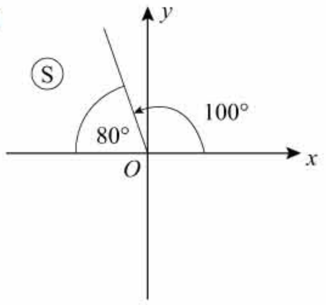
20° is the acute angle.
In the second quadrant cos is − ve.
So $\cos(-200)° = -\cos 20°$

j



5° is the acute angle.
In the third quadrant cos is − ve.
So $\cos 545° = -\cos 5°$

4 k

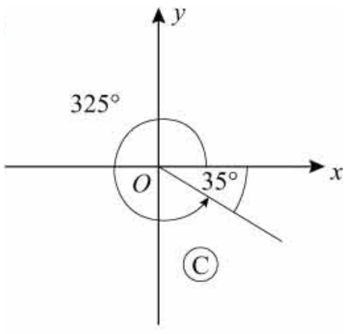


80° is the acute angle.

In the second quadrant tan is $-$ve.

So $\tan 100° = -\tan 80°$

l

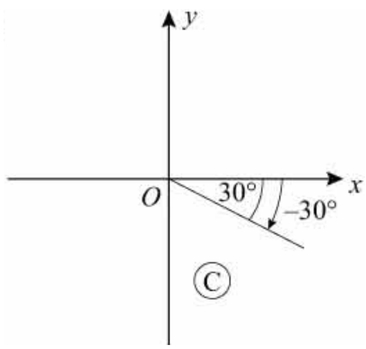


35° is the acute angle.

In the fourth quadrant tan is $-$ve.

So $\tan 325° = -\tan 35°$

m

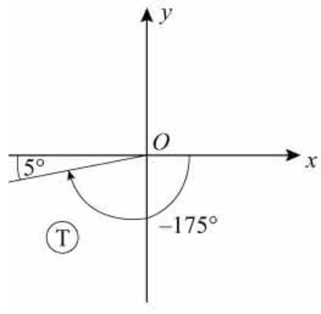


30° is the acute angle.

In the fourth quadrant tan is $-$ve.

So $\tan(-30)° = -\tan 30°$

n

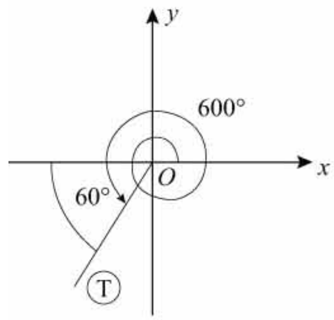


5° is the acute angle.

In the third quadrant tan is $+$ve.

So $\tan(-175)° = +\tan 5°$

o

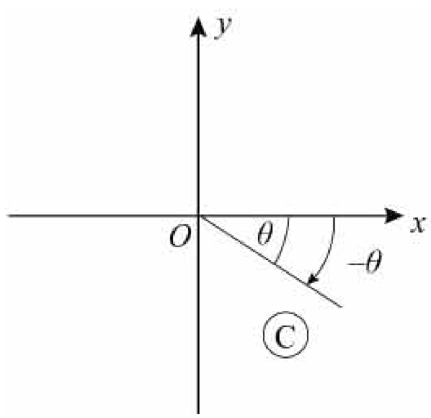


60° is the acute angle.

In the third quadrant tan is $+$ve.

So $\tan 600° = +\tan 60°$

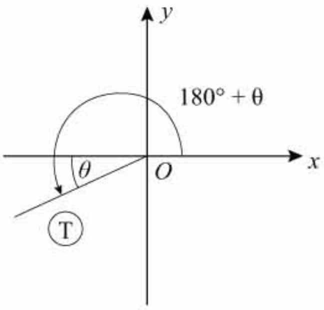
5 a



sin is $-$ve in this quadrant.

So $\sin(-\theta) = -\sin\theta$

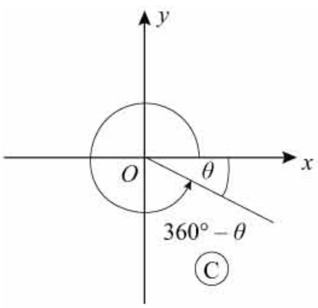
5 b



sin is $-$ve in this quadrant.

So $\sin(180° + \theta) = -\sin\theta$

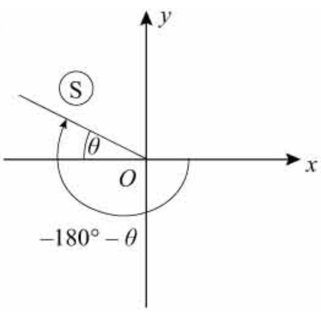
c



sin is $-$ve in this quadrant.

So $\sin(360° - \theta) = -\sin\theta$

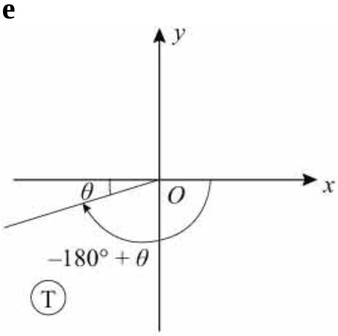
d



sin is $+$ve in this quadrant.

So $\sin-(180° + \theta) = +\sin\theta$

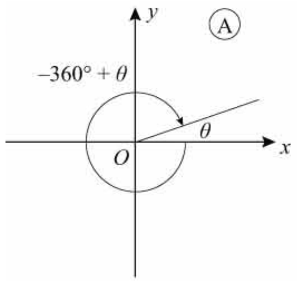
e



sin is $-$ve in this quadrant.

So $\sin(-180° + \theta) = -\sin\theta$

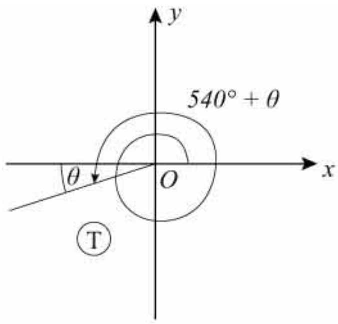
f



sin is $+$ve in this quadrant.

So $\sin(-360° + \theta) = +\sin\theta$

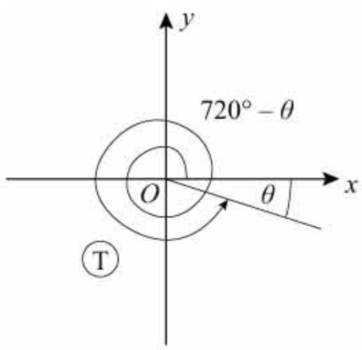
g



sin is $-$ve in this quadrant.

So $\sin(540° + \theta) = -\sin\theta$

h



sin is $+$ve in this quadrant.

So $\sin(720° - \theta) = -\sin\theta$

i $\theta + 720°$ is in the first quadrant with θ to the horizontal.

So $\sin(\theta + 720°) = +\sin\theta$

6 a $180° - \theta$ is in the second quadrant where cos is –ve, and the angle to the horizontal is θ.

So $\cos(180° - \theta) = -\cos\theta$

6 b $180° + \theta$ is in the third quadrant, at θ to the horizontal.
So $\cos(180° + \theta) = -\cos\theta$

c $-\theta$ is in the fourth quadrant, at θ to the horizontal.
So $\cos(-\theta) = +\cos\theta$

d $-(180° - \theta)$ is in the third quadrant, at θ to the horizontal.
So $\cos -(180° - \theta) = -\cos\theta$

e $\theta - 360°$ is in the first quadrant, at θ to the horizontal.
So $\cos(\theta - 360°) = \cos\theta$

f $\theta - 540°$ is in the third quadrant, at θ to the horizontal.
So $\cos(\theta - 540°) = -\cos\theta$

g $-\theta$ is in the fourth quadrant.
So $\tan(-\theta) = -\tan\theta$

h $(180° - \theta)$ is in the second quadrant.
So $\tan(180° - \theta) = -\tan\theta$

i $(180° + \theta)$ is in the third quadrant.
So $\tan(180° + \theta) = +\tan\theta$

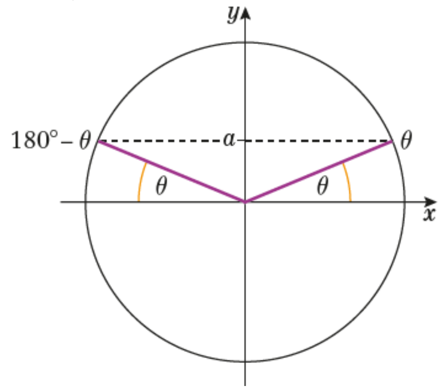
j $(-180° + \theta)$ is in the third quadrant.
So $\tan(-180° + \theta) = +\tan\theta$

k $(540° - \theta)$ is in the second quadrant.
So $\tan(540° - \theta) = -\tan\theta$

l $(\theta - 360°)$ is in the first quadrant.
So $\tan(\theta - 360°) = +\tan\theta$

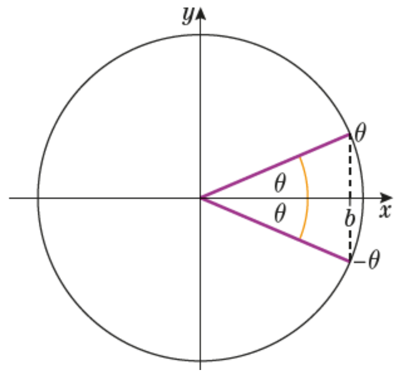
Challenge

a Diagram showing the positions of θ and $(180° - \theta)$ on the unit circle:



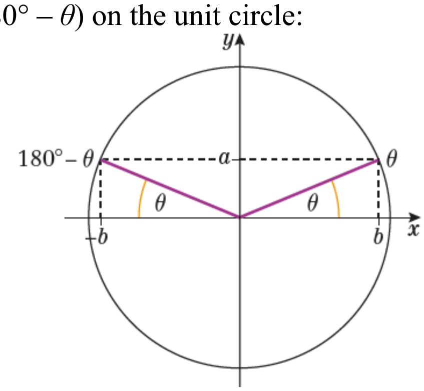
From the diagram, $\sin\theta = \alpha$
and $\sin(180° - \theta) = \alpha$
so $\sin\theta = \sin(180° - \theta)$

b Diagram showing the positions of $-\theta$ and θ on the unit circle:



From the diagram, $\cos\theta = b$
and $\cos(-\theta) = b$
so $\cos\theta = \cos(-\theta)$

c Diagram showing the positions of θ and $(180° - \theta)$ on the unit circle:



From the diagram, $\tan\theta = \dfrac{a}{b}$

and $\tan(180° - \theta) = -\dfrac{a}{b}$

so $\tan(180° - \theta) = -\tan(\theta)$

Trigonometric identities and equations 10B

1 **a** $\sin 135° = +\sin 45°$
(135° is in the second quadrant at 45° to the horizontal.)
So $\sin 135° = \frac{\sqrt{2}}{2}$

b $\sin(-60)° = -\sin 60°$
(–60° is in the fourth quadrant at 60° to the horizontal.)
So $\sin(-60°) = -\frac{\sqrt{3}}{2}$

c $\sin 330° = -\sin 30°$
(330° is in the fourth quadrant at 30° to the horizontal.)
So $\sin 330° = -\frac{1}{2}$

d $\sin 420° = +\sin 60°$
(on second revolution)
So $\sin 420° = \frac{\sqrt{3}}{2}$

e $\sin(-300°) = +\sin 60°$
(–300° is in the first quadrant at 60° to the horizontal.)
So $\sin(-300°) = \frac{\sqrt{3}}{2}$

f $\cos 120° = -\cos 60°$
(120° is in the second quadrant at 60° to the horizontal.)
So $\cos 120° = -\frac{1}{2}$

g $\cos 300° = +\cos 60°$
(300° is in the fourth quadrant at 60° to the horizontal.)
So $\cos 300° = \frac{1}{2}$

h $\cos 225° = -\cos 45°$
(225° is in the third quadrant at 45° to the horizontal.)
So $\cos 225° = -\frac{\sqrt{2}}{2}$

i $\cos(-210°) = -\cos 30°$
(–210° is in the second quadrant at 30° to the horizontal.)
So $\cos(-210°) = -\frac{\sqrt{3}}{2}$

j $\cos 495° = -\cos 45°$
(495° is in the second quadrant at 45° to the horizontal.)
So $\cos 495° = -\frac{\sqrt{2}}{2}$

k $\tan 135° = -\tan 45°$
(135° is in the second quadrant at 45° to the horizontal.)
So $\tan 135° = -1$

l $\tan(-225°) = -\tan 45°$
(–225° is in the second quadrant at 45° to the horizontal.)
So $\tan(-225°) = -1$

m $\tan 210° = +\tan 30°$
(210° is in the third quadrant at 30° to the horizontal.)
So $\tan 210° = \frac{\sqrt{3}}{3}$

n $\tan 300° = -\tan 60°$
(300° is in the fourth quadrant at 60° to the horizontal.)
So $\tan 300° = -\sqrt{3}$

o $\tan(-120°) = +\tan 60°$
(–120° is in the third quadrant at 60° to the horizontal.)
So $\tan(-120°) = \sqrt{3}$

Challenge

a i $\tan 30° = \dfrac{1}{CE}$

$CE = \dfrac{1}{\tan 30°}$

$= \dfrac{1}{\frac{\sqrt{3}}{3}}$

$= \dfrac{3}{\sqrt{3}}$

$= \dfrac{3\sqrt{3}}{3}$

$= \sqrt{3}$

ii Using Pythagoras' theorem

$CD^2 = 1^2 + \sqrt{3}^2$

$CD = \sqrt{1+3}$

$CD = 2$

iii Using Pythagoras' theorem on the isosceles triangle ABC

$AB^2 + BC^2 = (1+\sqrt{3})^2$

$AB = BC$ so $BC^2 + BC^2 = (1+\sqrt{3})^2$

$2BC^2 = 4 + 2\sqrt{3}$

$BC^2 = 2 + \sqrt{3}$

$BC = \sqrt{2+\sqrt{3}}$

iv $DB = AB - AD$

Using Pythagoras' theorem

$AD = \sqrt{1^2 + 1^2}$

$= \sqrt{2}$

$DB = \sqrt{2+\sqrt{3}} - \sqrt{2}$

b Angle $BCD = 45° - 30° = 15°$

c i $\sin 15° = \dfrac{DB}{CD}$

$= \dfrac{\sqrt{2+\sqrt{3}} - \sqrt{2}}{2}$

ii $\cos 15° = \dfrac{BC}{CD}$

$= \dfrac{\sqrt{2+\sqrt{3}}}{2}$

Trigonometric identities and equations 10C

1 a As $\sin^2 \frac{1}{2}\theta + \cos^2 \frac{1}{2}\theta \equiv 1$

So $1 - \cos^2 \frac{1}{2}\theta = \sin^2 \frac{1}{2}\theta$

b As $\sin^2 3\theta + \cos^2 3\theta \equiv 1$

So:

$$5\sin^2 3\theta + 5\cos^2 3\theta = 5\left(\sin^2 3\theta + \cos^2 3\theta\right)$$
$$= 5$$

c As $\sin^2 A + \cos^2 A \equiv 1$

So $\sin^2 A - 1 \equiv -\cos^2 A$

d $$\frac{\sin\theta}{\tan\theta} = \frac{\sin\theta}{\sin\theta / \cos\theta}$$
$$= \sin\theta \times \frac{\cos\theta}{\sin\theta}$$
$$= \cos\theta$$

e $$\frac{\sqrt{1-\cos^2 x}}{\cos x} = \frac{\sqrt{\sin^2 x}}{\cos x}$$
$$= \frac{\sin x}{\cos x}$$
$$= \tan x$$

f $$\frac{\sqrt{1-\cos^2 3A}}{\sqrt{1-\sin^2 3A}} = \frac{\sqrt{\sin^2 3A}}{\sqrt{\cos^2 3A}}$$
$$= \frac{\sin 3A}{\cos 3A}$$
$$= \tan 3A$$

g $$\left(1+\sin x\right)^2 + \left(1-\sin x\right)^2 + 2\cos^2 x$$
$$= 1 + 2\sin x + \sin^2 x + 1 - 2\sin x$$
$$+ \sin^2 x + 2\cos^2 x$$
$$= 2 + 2\sin^2 x + 2\cos^2 x$$
$$= 2 + 2\left(\sin^2 x + \cos^2 x\right)$$
$$= 2 + 2$$
$$= 4$$

h $$\sin^4\theta + \sin^2\theta\cos^2\theta$$
$$= \sin^2\theta\left(\sin^2\theta + \cos^2\theta\right)$$
$$= \sin^2\theta$$

1 i $$\sin^4\theta + 2\sin^2\theta\cos^2 + \cos^4\theta$$
$$= \left(\sin^2\theta + \cos^2\theta\right)^2$$
$$= 1^2$$
$$= 1$$

2 Given $2\sin\theta = 3\cos\theta$

So $\dfrac{\sin\theta}{\cos\theta} = \dfrac{3}{2}$

(divide both side by $2\cos\theta$)

So $\tan\theta = \dfrac{3}{2}$

3 As $\sin x \cos y = 3\cos x \sin y$

So $\dfrac{\sin x\cos y}{\cos x\cos y} = 3\dfrac{\cos x\sin y}{\cos x\cos y}$

So $\tan x = 3\tan y$

4 a As $\sin^2\theta + \cos^2\theta \equiv 1$

So $\cos^2\theta \equiv 1 - \sin^2\theta$

b $$\tan^2\theta \equiv \frac{\sin^2\theta}{\cos^2\theta} \equiv \frac{\sin^2\theta}{1-\sin^2\theta}$$

c $$\cos\theta\tan\theta = \cos\theta \times \frac{\sin\theta}{\cos\theta}$$
$$= \sin\theta$$

d $$\frac{\cos\theta}{\tan\theta} = \frac{\cos\theta}{\sin\theta / \cos\theta}$$
$$= \cos\theta \times \frac{\cos\theta}{\sin\theta}$$
$$= \frac{\cos^2\theta}{\sin\theta}$$

So $\dfrac{\cos\theta}{\tan\theta} = \dfrac{1-\sin^2\theta}{\sin\theta}$ or $\dfrac{1}{\sin\theta} - \sin\theta$

e $$\left(\cos\theta - \sin\theta\right)\left(\cos\theta + \sin\theta\right)$$
$$= \cos^2\theta - \sin^2\theta$$
$$= \left(1 - \sin^2\theta\right) - \sin^2\theta$$
$$= 1 - 2\sin^2\theta$$

5 a $$\begin{aligned}\text{LHS} &= (\sin\theta+\cos\theta)^2\\ &= \sin^2\theta+2\sin\theta\cos\theta+\cos^2\theta\\ &= (\sin^2\theta+\cos^2\theta)+2\sin\theta\cos\theta\\ &= 1+2\sin\theta\cos\theta\\ &= \text{RHS}\end{aligned}$$

b $$\begin{aligned}\text{LHS} &= \frac{1}{\cos\theta}-\cos\theta\\ &= \frac{1-\cos^2\theta}{\cos\theta}\\ &= \frac{\sin^2\theta}{\cos\theta}\\ &= \sin\theta\times\frac{\sin\theta}{\cos\theta}\\ &= \sin\theta\tan\theta\\ &= \text{RHS}\end{aligned}$$

c $$\begin{aligned}\text{LHS} &= \tan x+\frac{1}{\tan x}\\ &= \frac{\sin x}{\cos x}+\frac{\cos x}{\sin x}\\ &= \frac{\sin^2 x+\cos^2 x}{\sin x\cos x}\\ &= \frac{1}{\sin x\cos x}\\ &= \text{RHS}\end{aligned}$$

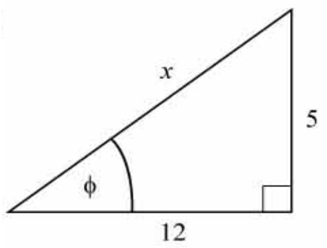
d $$\begin{aligned}\text{LHS} &= \cos^2 A-\sin^2 A\\ &\equiv \cos^2 A-(1-\cos^2 A)\\ &\equiv \cos^2 A-1+\cos^2 A\\ &\equiv 2\cos^2 A-1 \checkmark\\ &\equiv 2(1-\sin^2 A)-1\\ &\equiv 2-2\sin^2 A-1\\ &\equiv 1-2\sin^2 A \checkmark\end{aligned}$$

e $$\begin{aligned}\text{LHS} = {} & (2\sin\theta-\cos\theta)^2+(\sin\theta+2\cos\theta)^2\\ &\equiv 4\sin^2\theta-4\sin\theta\cos\theta+\cos^2\theta\\ &\quad +\sin^2\theta+4\sin\theta\cos\theta+4\cos^2\theta\\ &\equiv 5\sin^2\theta+5\cos^2\theta\\ &\equiv 5(\sin^2\theta+\cos^2\theta)\\ &\equiv 5\\ &\equiv \text{RHS}\end{aligned}$$

5 f $$\begin{aligned}\text{LHS} &= 2-(\sin\theta-\cos\theta)^2\\ &= 2-(\sin^2\theta-2\sin\theta\cos\theta+\cos^2\theta)\\ &= 2-(1-2\sin\theta\cos\theta)\\ &= 1+2\sin\theta\cos\theta\\ &= \sin^2\theta+\cos^2\theta+2\sin\theta\cos\theta\\ &= (\sin\theta+\cos\theta)^2\\ &= \text{RHS}\end{aligned}$$

g $$\begin{aligned}\text{LHS} &= \sin^2 x\cos^2 y-\cos^2 x\sin^2 y\\ &= \sin^2 x(1-\sin^2 y)\\ &\quad -(1-\sin^2 x)\sin^2 y\\ &= \sin^2 x-\sin^2 x\sin^2 y\\ &\quad -\sin^2 y+\sin^2 x\sin^2 y\\ &= \sin^2 x-\sin^2 y\\ &= \text{RHS}\end{aligned}$$

6 a



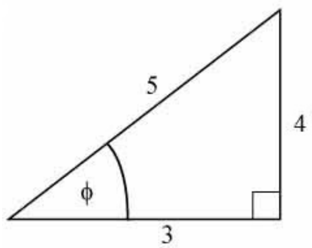
Using Pythagoras' theorem:

$x^2 = 12^2+5^2 = 169$

$x = 13$

So $\sin\theta = \dfrac{5}{13}$ and $\cos\theta = \dfrac{12}{13}$

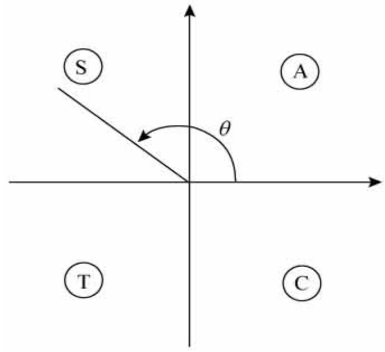
b



Using Pythagoras' theorem, $x = 4$

So $\sin\phi = \frac{4}{5}$ and $\tan\phi = -\frac{4}{3}$

6 b



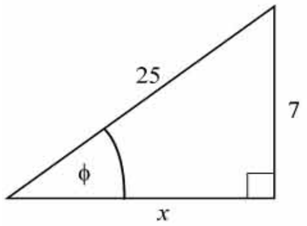
As θ is obtuse:

$\sin\theta = \sin\phi = \frac{4}{5}$

and

$\tan\theta = -\tan\phi = -\frac{4}{3}$

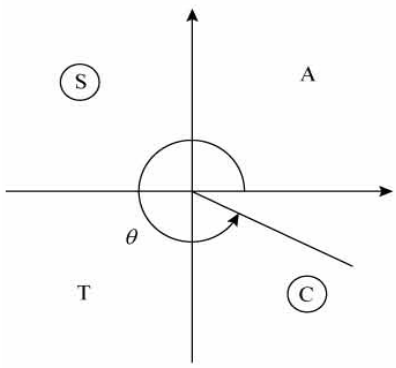
c



Using Pythagoras' theorem

$$x^2 + 7^2 = 25^2$$
$$x^2 = 25^2 - 7^2$$
$$= 576$$
$$x = 24$$

So $\cos\phi = \frac{24}{25}$ and $\tan\phi = \frac{7}{24}$



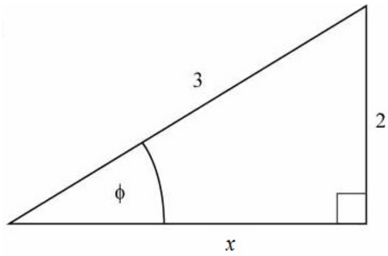
As θ is in the fourth quadrant

$$\cos\theta = +\cos\phi$$
$$= \frac{24}{25}$$

and $\tan\theta = -\tan\phi$

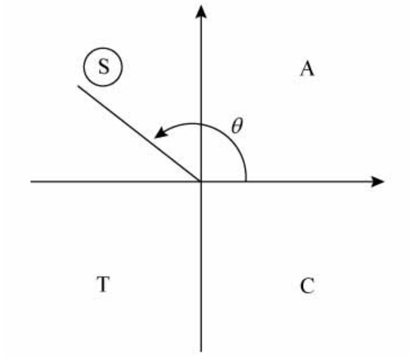
$$= -\frac{7}{24}$$

7 Consider the angle ϕ where $\sin\phi = \frac{2}{3}$.



Using Pythagoras' theorem, $x = \sqrt{5}$

a So $\cos\phi = \dfrac{\sqrt{5}}{3}$



As θ is obtuse, $\cos\theta = -\cos\phi = -\dfrac{\sqrt{5}}{3}$

b From the triangle

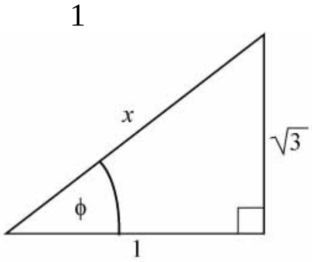
$$\tan\phi = \frac{2}{\sqrt{5}}$$
$$= \frac{2\sqrt{5}}{5}$$

Using the quadrant diagram

$$\tan\theta = -\tan\phi$$
$$= -\frac{2\sqrt{5}}{5}$$

8 Draw a right-angled triangle with

$$\tan\phi = +\sqrt{3}$$
$$= \frac{\sqrt{3}}{1}$$

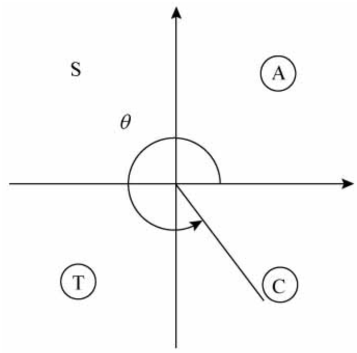


Using Pythagoras' theorem

$$x^2 = \left(\sqrt{3}\right)^2 + 1^2 = 4$$

So $x = 2$

8 a $\sin\phi = \frac{\sqrt{3}}{2}$



As θ is reflex and $\tan\phi$ is –ve, ϕ is in the fourth quadrant.

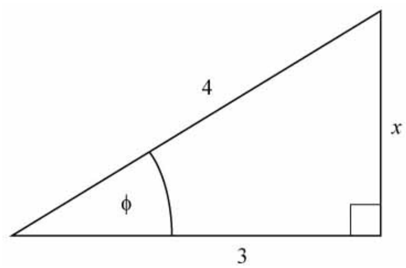
So $\sin\theta = -\sin\phi$

$= \frac{-\sqrt{3}}{2}$

b $\cos\phi = \frac{1}{2}$

As $\cos\theta = \cos\phi$, $\cos\theta = \frac{1}{2}$

9 Draw a right-angled triangle with $\cos\phi = \frac{3}{4}$.



Using Pythagoras' theorem

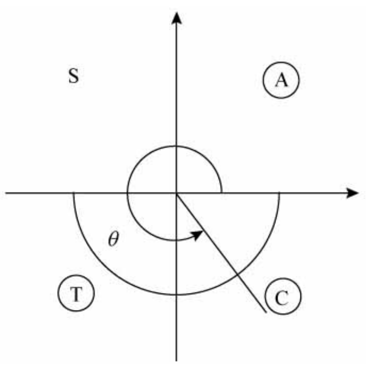
$x^2 + 3^2 = 4^2$

$x^2 = 4^2 - 3^2$

$= 7$

$x = \sqrt{7}$

So $\sin\phi = \frac{\sqrt{7}}{4}$ and $\tan\phi = \frac{\sqrt{7}}{3}$



As θ is reflex and $\cos\theta$ is +ve, θ is in the fourth quadrant.

9 a $\sin\theta = -\sin\phi$

$= -\frac{\sqrt{7}}{4}$

b $\tan\theta = -\tan\phi$

$= -\frac{\sqrt{7}}{3}$

10 a As $\sin^2\theta + \cos^2\theta \equiv 1$

$x^2 + y^2 = 1$

b $\sin\theta = x$ and $\cos\theta = \frac{y}{2}$

So, using $\sin^2\theta + \cos^2\theta \equiv 1$

$x^2 + \left(\frac{y}{2}\right)^2 = 1$

or $x^2 + \frac{y^2}{4} = 1$

or $4x^2 + y^2 = 4$

c As $\sin\theta = x$

$\sin^2\theta = x^2$

Using $\sin^2\theta + \cos^2\theta \equiv 1$

$x^2 + y = 1$

d As $\tan\theta = \frac{\sin\theta}{\cos\theta}$

$\cos\theta = \frac{\sin\theta}{\tan\theta}$

So $\cos\theta = \frac{x}{y}$

Using $\sin^2\theta + \cos^2\theta \equiv 1$

$x^2 + \frac{x^2}{y^2} = 1$ or $x^2y^2 + x^2 = y^2$

e $\sin\theta + \cos\theta = x$

$-\sin\theta + \cos\theta = y$

Adding the two equations:

$2\cos\theta = x + y$

So $\cos\theta = \frac{x+y}{2}$

Subtracting the two equations:

$2\sin\theta = x - y$

So $\sin\theta = \frac{x-y}{2}$

10 e Using $\sin^2\theta + \cos^2\theta \equiv 1$

$$\left(\frac{x-y}{2}\right)^2 + \left(\frac{x+y}{2}\right)^2 = 1$$
$$x^2 - 2xy + y^2 + x^2 + 2xy + y^2 = 4$$
$$2x^2 + 2y^2 = 4$$
$$x^2 + y^2 = 2$$

11 a Using the cosine rule

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$
$$\cos B = \frac{8^2 + 12^2 - 10^2}{2 \times 8 \times 12}$$
$$\cos B = \frac{64 + 144 - 100}{192}$$
$$\cos B = \tfrac{108}{192}$$
$$\cos B = \tfrac{9}{16}$$

b Since $\sin^2\theta + \cos^2\theta = 1$

$$\sin^2 B + \left(\tfrac{9}{16}\right)^2 = 1$$
$$\sin^2 B = 1 - \tfrac{81}{256}$$
$$= \tfrac{175}{256}$$

So $\sin B = \sqrt{\dfrac{175}{256}}$

$$= \frac{5\sqrt{7}}{16}$$

12 a Using the sine rule

$$\frac{\sin Q}{q} = \frac{\sin P}{p}$$
$$\frac{\sin Q}{8} = \frac{\sin 30°}{6}$$
$$\sin Q = \frac{8 \sin 30°}{6}$$
$$= \frac{8 \times \frac{1}{2}}{6}$$
$$= \frac{2}{3}$$

b Since $\sin^2\theta + \cos^2\theta = 1$

$$\left(\tfrac{2}{3}\right)^2 + \cos^2 Q = 1$$
$$\cos^2 Q = 1 - \tfrac{4}{9}$$
$$= \tfrac{5}{9}$$

Since Q is obtuse Q is in the second quadrant where cosine is negative.

So $\cos Q = -\dfrac{\sqrt{5}}{3}$

Trigonometric identities and equations 10D

1 a Consider $\tan x = -2$
$x = \tan^{-1}(-2)$
$= 63.4°$ (3 s.f.) in the first quadrant
The principal solution marked by A in the diagram is $180° - 63.4° = 116.6°$

b The solutions between 0° and 360°:
$-63.4° + 180° = 116.6°$
$-63.4° + 360° = 296.6°$

2 a $\cos x = 0.4$
$x = \cos^{-1}(0.4)$
$= 66.4$ (3 s.f.)

b $360° - 66.4° = 293.6°$
$180° - 66.4° = 113.6°$
$180° + 66.4° = 246.4°$
$x = 66.4°, 113.6°, 246.4°$ and $293.6°$

3 a Using the graph of $y = \sin\theta$
$\sin\theta = -1$ when $\theta = 270°$

b $\tan\theta = \sqrt{3}$
The calculator solution is $60°$ $\left(\tan^{-1}\sqrt{3}\right)$
and, as $\tan\theta$ is +ve, θ lies in the first and third quadrants.
$\theta = 60°$ and $(180° + 60°) = 60°, 240°$

c $\cos\theta = \frac{1}{2}$
The calculator solution is 60° and as $\cos\theta$ is +ve, θ lies in the first and fourth quadrants.
$\theta = 60°$ and $(360° - 60°) = 60°, 300°$

d $\sin\theta = \sin 15°$
The acute angle satisfying the equation is $\theta = 15°$.
As $\sin\theta$ is +ve, θ lies in the 1st and 2nd quadrants, so
$\theta = 15°$ and $(180° - 15°) = 15°, 165°$

e A first solution is $\cos^{-1}(-\cos 40°) = 140°$
A second solution of $\cos\theta = k$ is $360° -$ 1st solution.

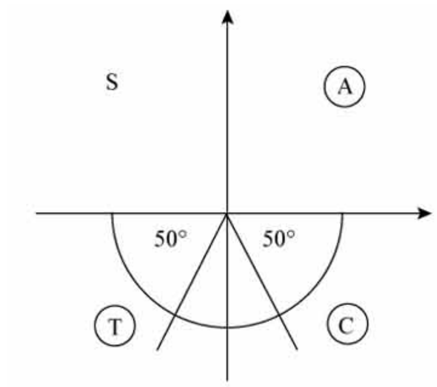
3 e So second solution is 220°.
(Use the quadrant diagram as a check.)

f A first solution is $\tan^{-1}(-1) = -45°$
Use the quadrant diagram, noting that as tan is −ve, solutions are in the 2nd and 4th quadrants.
(−45° is not in the given interval.)
So solutions are 135° and 315°.

g From the graph of $y = \cos\theta$
$\cos\theta = 0$ when $\theta = 90°, 270°$

h $\sin\theta = -0.766$
$\sin^{-1}(-0.766) = -50°$

$360° - 50° = 310°$

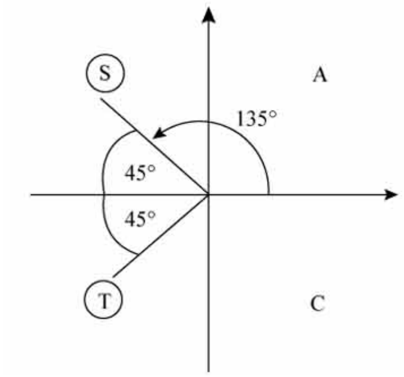


From the diagram, the second solution is $180° + 50° = 230°$.
$\theta = 230°, 310°$

4 a $\sin\theta = \frac{5}{7}$
The first solution is
$\sin^{-1}\left(\frac{5}{7}\right) = 45.6°$
The second solution is
$180° - 45.6° = 134.4°$

b $\cos\theta = -\frac{\sqrt{2}}{2}$
Calculator solution is 135°.
As $\cos\theta$ is −ve, θ is in the second and third quadrants.

4 b



Solutions are 135° and 225°
$(135° \text{ and } 360° - 135°)$.

c Calculator solution is
$\cos^{-1}\left(-\frac{2}{3}\right) = 131.8° \ (1 \text{ d.p.})$
Second solution is $360° - 131.8° = 228.2°$

d $\sin\theta = -\frac{3}{4}$
$\theta = -48.6°$
$\theta = 360° - 48.6°, \text{or } 180° + 48.6°$
$= 311.4°, 228.6°$

e $\tan\theta = \frac{1}{7}$
$\theta = 8.13° \text{ or } 188°$

f $\tan\theta = \frac{15}{8}$
$\theta = 61.9° \text{ or } 242°$

g $\tan\theta = -\frac{11}{3}$
$\theta = -74.7°$
$\theta = 105.3° \text{ or } 285°$

h $\cos\theta = \frac{\sqrt{5}}{3}$
$\theta = 41.8°, 318°$

5 a $\sqrt{3}\sin\theta = \cos\theta$
So dividing both sides by $\sqrt{3}\cos\theta$
$\tan\theta = \frac{1}{\sqrt{3}}$

5 a Calculator solution is 30°.
As $\tan\theta$ is + ve, θ is in the first and third quadrants.
Solutions are 30°, 210°
$(30° \text{ and } 180° + 30°)$.

b $\sin\theta + \cos\theta = 0$
So $\sin\theta = -\cos\theta \Rightarrow \tan\theta = -1$
Calculator solution $(-45°)$ is not in the given interval.
As $\tan\theta$ is − ve, θ is in the second and fourth quadrants.
Solutions are 135° and 315°
$(180° + \tan^{-1}(-1), 360° + \tan^{-1}(-1))$.

c $3\sin\theta = 4\cos\theta$
$\tan\theta = \frac{4}{3}$
$\theta = 53.1° \text{ or } 233°$

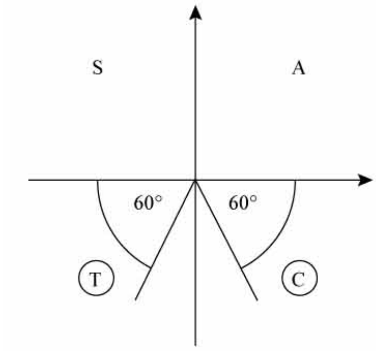
d $2\sin\theta - 3\cos\theta = 0$
$\tan\theta = \frac{3}{2}$
$\theta = 56.3° \text{ or } 236°$

e $\sqrt{2}\sin\theta = 2\cos\theta$
$\tan\theta = \frac{2}{\sqrt{2}} = \sqrt{2}$
$\theta = 54.7° \text{ or } 235°$

f $\sqrt{5}\sin\theta + \sqrt{2}\cos\theta = 0$
$\sqrt{5}\tan\theta + \sqrt{2} = 0$
$\tan\theta = -\frac{\sqrt{2}}{\sqrt{5}}$
$\theta = -32.3° \ \theta > 0$
$\theta = 148° \text{ or } 328°$

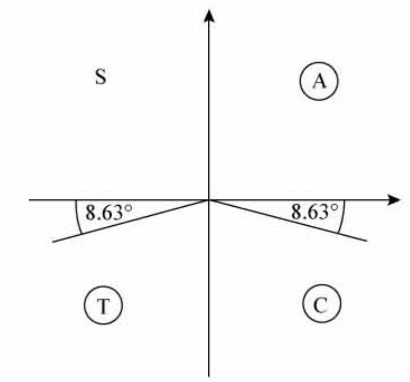
6 a Calculator solution of
$\sin x° = -\frac{\sqrt{3}}{2}$ is $x = -60°$
As $\sin x°$ is − ve, x is in the third and fourth quadrants.

6 a



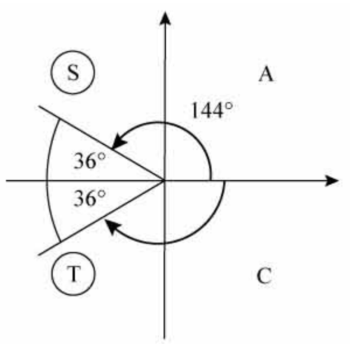
Read off all solutions in the interval
$-180° \leq x \leq 540°$.
$x = -120°, -60°, 240°, 300°$

b $2\sin x° = -0.3$
$\sin x° = -0.15$
First solution is $x = \sin^{-1}(-0.15)$
$= -8.63°$ (3 s.f.)
As $\sin x°$ is $-$ ve, x is in the third and fourth quadrants.



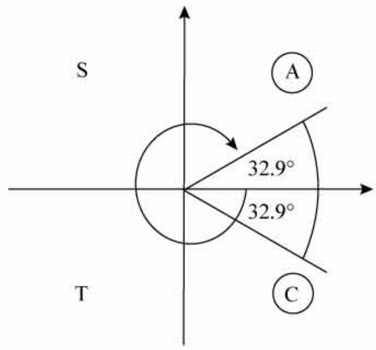
Read off all solutions in the interval
$-180° \leq x \leq 180°$.
$x = -171.37°, \ -8.63°$
$x = -171°, \ -8.63°$ (3 s.f.)

c $\cos x° = -0.809$
Calculator solution is 144° (3 s.f.)
As $\cos x°$ is $-$ ve, x is in the second and third quadrants.



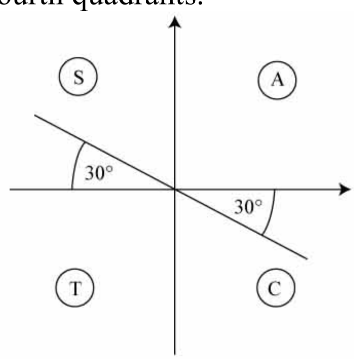
c Read off all the solutions in the interval
$-180° \leq x \leq 180°$.
$x = -144°, +144°$
Note: Here solutions are $\cos^{-1}(-0.809)$ and $(360° - \cos^{-1}(-0.809))$.

d $\cos x° = 0.84$
Calculator solution is 32.9° (3 s.f.)
(not in interval).
As $\cos x°$ is + ve, x is in the first and fourth quadrants.



Read off all the solutions in the interval
$-360° < x < 0°$.
$x = -327°, -32.9°$ (3 s.f.)
(*Note*: Here solutions are
$\cos^{-1}(0.84) - 360°$ and
$(360° - \cos^{-1}(0.84)) - 360°$)

e $\tan x° = -\dfrac{\sqrt{3}}{3}$
Calculator solution is
$\tan^{-1}\left(-\dfrac{\sqrt{3}}{3}\right) = -30°$ (not in interval)
As $\tan x°$ is $-$ve, x is in the second and fourth quadrants.

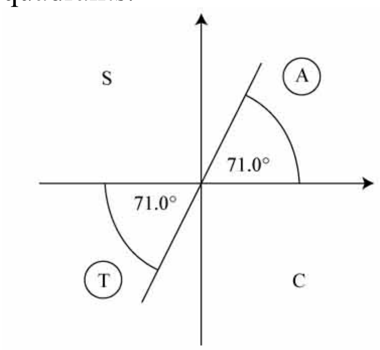


Read off all the solutions in the interval
$0° \leq x \leq 720°$.
$x = 150°, 330°, 510°, 690°$

6 e *Note*: Here solutions are

$$\tan^{-1}\left(-\frac{\sqrt{3}}{3}\right)+180°,\ \tan^{-1}\left(-\frac{\sqrt{3}}{3}\right)+360°,$$
$$\tan^{-1}\left(-\frac{\sqrt{3}}{3}\right)+540°,\ \cos^{-1}\left(-\frac{\sqrt{3}}{3}\right)+720°.$$

f $\tan x° = 2.90$
Calculator solution is
$\tan^{-1}(2.90) = 71.0°\,(3\text{ s.f.})$
(not in interval).
As $\tan x°$ is +ve, x is in the first and third quadrants.



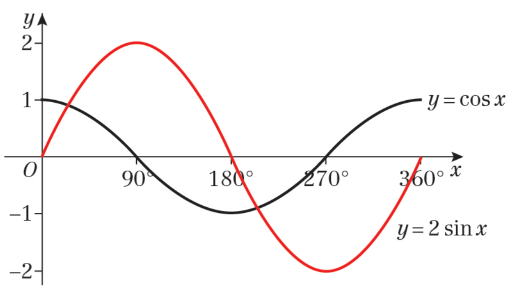
Read off all solutions in the interval
$80° \le x \le 440°$.
$x = 251°, 431°$
(*Note*: Here solutions are
$\tan^{-1}(2.90)+180°,\ \tan^{-1}(2.90)+360°$.)

7 a It should be $\tan x = \frac{2}{3}$, not $\frac{3}{2}$.

b Squaring both sides creates extra solutions.

c $\tan x = \frac{2}{3}$
$x = 33.7°$ or $x = -146.3°$

8 a



8 b The graphs intersect at 2 points in the given range so there are 2 solutions.

c $2\sin x = \cos x$
$\frac{\sin x}{\cos x} = \frac{1}{2}$
$\tan x = \frac{1}{2}$
$x = 26.6°$
$x = 26.6° + 180° = 206.6°$
$x = 26.6°$ or $206.6°$

9 $\tan\theta = \pm 3$
When $\tan\theta = 3$, $\theta = 71.6°$
or $\theta = 71.6° + 180° = 251.6°$
When $\tan\theta = -3$, $\theta = -71.6°$
or $\theta = -71.6° + 180° = 108.4°$ or
$\theta = 108.4° + 180° = 288.4°$
$\theta = 71.6°, 108.4°, 251.6°$ or $288.4°$

10 a $4\sin^2 x - 3\cos^2 x = 2$
$4\sin^2 x - 3(1 - \sin^2 x) = 2$
$4\sin^2 x - 3 + 3\sin^2 x = 2$
$7\sin^2 x = 5$

b $\sin^2 x = \frac{5}{7}$
$\sin x = \pm\sqrt{\frac{5}{7}}$
$x = 57.7°$ or $-57.7°$
$x = 180° - 57.7° = 122.3°$
$x = 180° + 57.7° = 237.7°$
$x = 360° - 57.7° = 302.3°$
$x = 57.7°, 122.3°, 237.7°$ or $302.3°$

11 a $2\sin^2 x + 5\cos^2 x = 1$
$2\sin^2 x + 5(1 - \sin^2 x) = 1$
$2\sin^2 x + 5 - 5\sin^2 x = 1$
$3\sin^2 x = 4$

b Using $3\sin^2 x = 4$
$\sin^2 x = \frac{4}{3}$
$\sin x > 1$, therefore there are no solutions.

Trigonometric identities and equations 10E

1 a $\sin 4\theta = 0 \quad 0° \le \theta \le 360°$

Let $X = 4\theta$ so $0° \le X \le 1440°$

Solve $\sin X = 0$ in the interval $0° \le X \le 1440°$

From the graph of $y = \sin X$, $\sin X = 0$ where

$X = 0°, 180°, 360°, 540°, 720°, 900°, 1080°, 1260°, 1440°$

$$\theta = \frac{X}{4}$$
$$= 0°, 45°, 90°, 135°, 180°, 225°, 270°, 315°, 360°$$

b $\cos 3\theta = -1 \quad 0° \le \theta \le 360°$

Let $X = 3\theta$ so $0° \le X \le 1080°$

Solve $\cos X = 0$ in the interval $0° \le X \le 1080°$

From the graph of $y = \cos X$, $\cos X = -1$ where

$X = 180°, 540°, 900°,$

$$\theta = \frac{X}{3}$$
$$= 60°, 180°, 300°$$

c $\tan 2\theta = 1 \quad 0° \le \theta \le 360°$

Let $X = 2\theta$

Solve $\tan X = 1$ in the interval $0° \le X \le 720°$.

A solution is $X = \tan^{-1}(1) \quad = 45°$

As $\tan X$ is +ve, X is in the first and third quadrants.

So $X = 45°, 225°, 405°, 585°$

$$\theta = \frac{X}{2}$$
$$= 22\tfrac{1}{2}°, 112\tfrac{1}{2}°, 202\tfrac{1}{2}°, 292\tfrac{1}{2}°$$

d $\cos 2\theta = \frac{1}{2} \quad 0 \le \theta \le 360°$

Let $X = 2\theta$

Solve $\cos X = \frac{1}{2}$ in the interval $0 \le X \le 720°$.

A solution is $X = \cos^{-1}\left(\frac{1}{2}\right) = 60°$

As $\cos X$ is +ve, X is in the first and fourth quadrants.

So $X = 60°, 300°, 420°, 660°$

d $\theta = \frac{X}{2}$

$= 30°, 150°, 210°, 330°$

e $\tan \frac{1}{2}\theta = -\frac{1}{\sqrt{3}} \quad 0 \le \theta \le 360°$

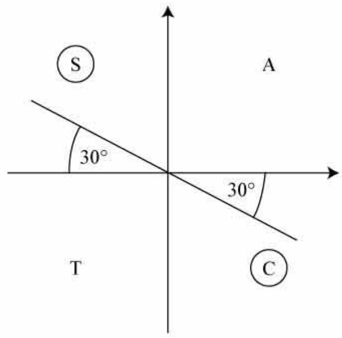
Let $X = \frac{1}{2}\theta$

Solve $\tan X = -\frac{1}{\sqrt{3}}$ in the interval $0° \le X \le 180°$.

A solution is $X = \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -30°$

(This is not in the interval.)

As $\tan X$ is −ve, X is in the second and fourth quadrants.



Read off solution in the interval $0° \le X \le 180°$.

$X = 150°$

So $\theta = 2X$

$= 300°$

f $\sin(-\theta) = \frac{1}{\sqrt{2}} \quad 0° \le \theta \le 360°$

Let $X = -\theta$

Solve $\sin X = \frac{1}{\sqrt{2}}$ in the interval $0° \le X \le 360°$.

A solution is $X = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = 45°$

As $\sin X$ is +ve, X is in the first and second quadrants.

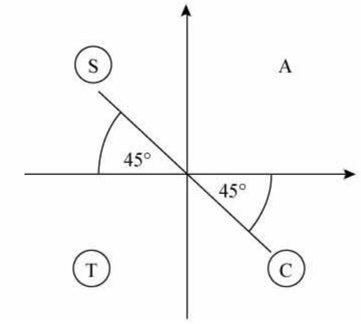
$X = -315°, -225°$

So $\theta = -X$

$= 225°, 315°$

2 a $\tan(45° - \theta) = -1 \quad 0° \le \theta \le 360°$

Let $X = 45° - \theta$ so $0° \ge -\theta \ge -360°$
Solve $\tan X = -1$ in the interval
$45° \ge X \ge -315°$
A solution is $X = \tan^{-1}(-1) = -45°$
As $\tan X$ is −ve, X is in the second and fourth quadrants.



$X = -225°, -45°$
So $\theta = 45° - X = 90°, 270°$

b $2\sin(\theta - 20°) = 1$

So $\sin(\theta - 20°) = \frac{1}{2} \quad 0° \le \theta \le 360°$

Let $X = \theta - 20°$

Solve $\sin X = \frac{1}{2}$ in the interval

$-20° \le X \le 340°$

A solution is $X = \sin^{-1}\left(\frac{1}{2}\right) = 30°$

As $\sin X$ is +ve, solutions are in the first and second quadrants.
$X = 30°, 150°$

So $\theta = X + 20°$

$= 50°, 170°$

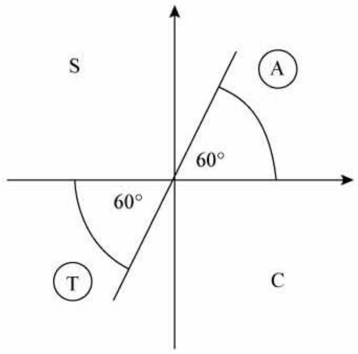
c Solve $\tan X = \sqrt{3}$ where $X = (\theta + 75°)$.

The interval for X is $75° \le X \le 435°$

One solution is $\tan^{-1}(\sqrt{3}) = 60°$

(This is not in the interval)
As $\tan X$ is +ve, solutions are in the first and third quadrants.

c



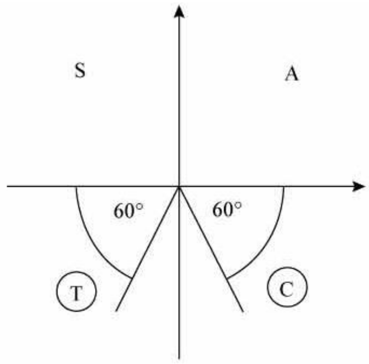
$X = 240°, 420°$
So $\theta = X - 75°$
$= 165°, 345°$

d Solve $\sin X = \frac{\sqrt{3}}{2}$ where $X = (\theta - 10°)$.

The interval for X is $-10° < X \le 350°$

First solution is $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -60°$

(This is not in the interval)
As $\sin X$ is –ve, X is in the third and fourth quadrants.



Read off solutions in the interval
$-10° < X \le 350°$
$X = 240°, 300°$
So $\theta = X + 10°$
$= 250°, 310°$

2 e Solve $\cos X° = -1$ where $X = (50° + 2\theta)$.
The interval for X is
$2(0) + 50° \le X \le 2(360°) + 50°$
i.e. $50° \le X \le 770°$
First solution is $\cos^{-1}(-1) = 180°$
Second solution in interval
$= 180° + 360° = 540°$

$\theta = \frac{X - 50°}{2}$

So $\theta = 65°, 245°$

2 f Solve $\tan X° = -0.51$
where $X = (3\theta + 25°)$.
The interval for X is
$3(-90°) + 25 \leq X \leq 3(180°) + 25°$
i.e. $-245° \leq X \leq 565°$
First solution is $\tan^{-1}(-0.51) = -27.0°$
Other solutions $= -27.0° \pm n180°$, where
n is an integer
Solution in range are therefore:
$X = -207°, -27.0°, 153°, 333°$ and $513°$
$\theta = \dfrac{X - 25°}{3}$
So $\theta = -77.3°, -17.3°, 42.7°, 103°, 163°$

3 a Let $X = 3\theta$
So $3 \sin X = 2 \cos X$
$\dfrac{\sin X}{\cos X} = \dfrac{2}{3}$
$\tan X = \dfrac{2}{3}$
As $X = 3\theta$, then as $0° \leq \theta \leq 180°$
So $3 \times 0° \leq X \leq 3 \times 180°$
So the interval for X is $0° \leq X \leq 540°$.
$X = 33.7°, 213.7°, 393.7°$
i.e. $3\theta = 33.7°, 213.7°, 393.7°$
So $\theta = 11.2°, 71.2°, 131.2°$

b Let $X = \theta + 45°$
So $4 \sin X = 5 \cos X$
$\dfrac{\sin X}{\cos X} = \dfrac{5}{4}$
$\tan X = \dfrac{5}{4}$
As $X = \theta + 45°$, then as $0° \leq \theta \leq 450°$
so $0 + 45 \leq X \leq 450° + 45$
So the interval for X is $45° \leq X \leq 495°$.
$X = 51.3°, 231.3°, 411.3°$
i.e. $\theta + 45° = 51.3°, 231.3°, 411.3°$
So $\theta = 6.3°, 186.3°, 366.3°$

c Let $X = 2x$
$2 \sin X - 7 \cos X = 0$
$2 \sin X = 7 \cos X$
$\dfrac{\sin X}{\cos X} = \dfrac{7}{2}$
$\tan X = \dfrac{7}{2}$
As $X = 2x$, then as $0° \leq x \leq 180°$
So $2 \times 0° \leq X \leq 2 \times 180°$

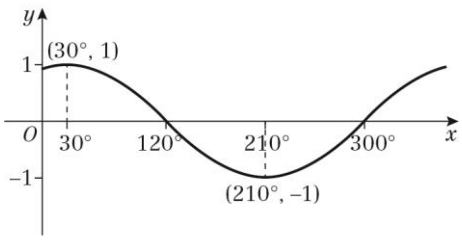
c So the interval for X is $0° \leq X \leq 360°$.
$X = 74.05°, 254.05°$
i.e. $2x = 74.05°, 254.05°$
So $x = 37.0°, 127.0°$

d Let $X = \theta - 60°$
So $\sqrt{3} \sin X + \cos X = 0$
$\sqrt{3} \sin X = -\cos X$
$\dfrac{\sin X}{\cos X} = \dfrac{-1}{\sqrt{3}}$
$\tan X = \dfrac{-1}{\sqrt{3}}$
As $X = \theta - 60°$, then as $-180° \leq \theta \leq 180°$
So $-180° - 60 \leq X \leq 180° - 60$
So the interval for X is $-240° \leq X \leq 120°$.
$X = -30°, -210°$
i.e. $\theta - 60° = -210°, -30°$
So $\theta = -150°, 30°$

4 a Let $X = x + 20°$
So $\sin X = \frac{1}{2}$
As $X = x + 20°$, then as $0 \leq x \leq 180°$
So $0 + 20 \leq x \leq 180° + 20$
So the interval for X is $20° \leq X \leq 200°$.
$X = 30°, 150°$
i.e. $x + 20° = 30°, 150°$
So $x = 10°, 130°$

b Let $X = 2x$
So $\cos X = -0.8$
As $X = 2x$, then as $0 \leq x \leq 180°$
So $2 \times 0 \leq X \leq 2 \times 180°$
So the interval for X is $0 \leq X \leq 360°$
$X = 143.13°, 216.87°$
i.e. $2x = 143.13°, 216.87°$
So $x = 71.6°, 108.4°$

5 a



b $\left(0°, \dfrac{\sqrt{3}}{2}\right)$, $(120°, 0)$, $(300°, 0)$

5 c Let $X = x + 60°$
So $\sin X = 0.55$
As $X = x + 60°$, then as $0° \le x \le 360°$
So $0° + 60° \le x \le 360° + 60°$
So the interval for X is $60° \le X \le 420°$.
$X = 33.4°, 146.6°, 393.4°$
i.e. $x + 60° = 33.4°, 146.6°, 393.4°$
So $x = 86.6°, 333.4°$

6 a $4\sin x = 3\cos x$
$$\frac{\sin x}{\cos x} = \frac{3}{4}$$
$$\tan x = \frac{3}{4}$$

b Let $X = 2\theta$
So $\tan X = \dfrac{3}{4}$
As $X = 2\theta$, then as $0° \le \theta \le 360°$
So $2 \times 0° \le X \le 2 \times 360°$
So the interval for X is $0° \le X \le 720°$.
$X = 36.87°, 216.87°, 396.87°, 576.87°$
i.e. $2\theta = 36.87°, 216.87°, 396.87°, 576.87°$
So $\theta = 18.4°, 108.4°, 198.4°, 288.4°$

7 a $\tan 60k° = -\dfrac{1}{\sqrt{3}}$
Let $X = 60k°$
So $\tan X = -\dfrac{1}{\sqrt{3}}$
$X = -30°, 150°, 330°,$
I.e. $60k° = -30°, 150°, 330°,$
$k = -0.5, 2.5,$
$k > 0$ so $k = 2.5$

b No because when $X = 330°$, $k = 5.5$.
As k increases, the period of the tan graph increases.

8 Let $X = 3x - 45°$
So $\sin X = \frac{1}{2}$
As $X = 3x - 45°$, then as $0° \le x \le 180°$
So $3 \times 0° - 45° \le x \le 3 \times 180° - 45°$
So the interval for X is $-45° \le X \le 495°$.
$X = 30°, 150°, 390°$
i.e. $3x - 45° = 30°, 150°, 390°$
So $x = 25°, 65°, 145°$

Trigonometric identities and equations 10F

1 a $4\cos^2\theta = 1 \Rightarrow \cos^2\theta = \frac{1}{4}$

So $\cos\theta = \pm\frac{1}{2}$

Solutions are $60°, 120°, 240°, 300°$.

b $2\sin^2\theta - 1 = 0 \Rightarrow \sin^2\theta = \frac{1}{2}$

So $\sin\theta = \pm\frac{1}{\sqrt{2}}$

Solutions are in all four quadrants, at 45° to the horizontal.

So $\theta = 45°, 135°, 225°, 315°$

c Factorising, $\sin\theta(3\sin\theta + 1) = 0$

So $\sin\theta = 0$ or $\sin\theta = -\frac{1}{3}$

Solutions of $\sin\theta = 0$ are

$\theta = 0°\ 180°, 360°$ (from graph)

Solutions of $\sin\theta = -\frac{1}{3}$ are

$\theta = 199°,\ 341°$ (3 s.f.)

These are in the third and fourth quadrants.

d $\tan^2\theta - 2\tan\theta - 10 = 0$

So $\tan\theta = \frac{2 \pm \sqrt{4+40}}{2}$

$= \frac{2 \pm \sqrt{44}}{2}$

$(= -2.3166\ldots \text{ or } 4.3166\ldots)$

Solutions of $\tan\theta = \frac{2-\sqrt{44}}{2}$ are in the second and fourth quadrants.

So $\theta = 113.35°, 293.3°$

Solutions of $\tan\theta = \frac{2+\sqrt{44}}{2}$ are in the first and third quadrants.

So $\theta = 76.95\ldots°, 256.95\ldots°$

Solution set: $77.0°, 113°, 257°, 293°$

e Factorising LHS of

$2\cos^2\theta - 5\cos\theta + 2 = 0$

$(2\cos\theta - 1)(\cos\theta - 2) = 0$

So $2\cos\theta - 1 = 0$ or $\cos\theta - 2 = 0$

As $\cos\theta \le 1, \cos\theta = 2$ has no solutions.

Solutions of $\cos\theta = \frac{1}{2}$ are $\theta = 60°,\ 300°$

f $\sin^2\theta - 2\sin\theta - 1 = 0$

So $\sin\theta = \frac{2 \pm \sqrt{8}}{2}$

Solve $\sin\theta = \frac{2-\sqrt{8}}{2}$ as $\frac{2+\sqrt{8}}{2} > 1$

$\theta = 204°, 336°$

The solutions are in the third and fourth quadrants as $\frac{2-\sqrt{8}}{2} < 0.$

g $\tan^2 2\theta = 3 \Rightarrow \tan 2\theta = \pm\sqrt{3}$

Solve $\tan X = +\sqrt{3}$ and $\tan X = -\sqrt{3}$, where $X = 2\theta$

The interval for X is $0 \le X \le 720°$.

For $\tan X = \sqrt{3}$,

$X = 60°, 240°, 420°, 600°$

So $\theta = \frac{X}{2} = 30°, 120°, 210°, 300°$

For $\tan X = -\sqrt{3}$,

$X = 120°, 300°, 480°, 660°$

So $\theta = 60°, 150°, 240°, 330°$

Solution set:

$\theta = 30°, 60°, 120°, 150°, 210°, 240°,$
$300°, 330°$

2 a Solve $\sin^2 X = 1$ where $X = 2\theta$

The interval for X is $-360° \le X \le 360°$.

$\sin X = +1$ gives $X = -270°, 90°$

$\sin X = -1$ gives $X = -90°, +270°$

$X = -270°, -90°, +90°, +270°$

So $\theta = \frac{X}{2}$

$= \pm 45°, \pm 135°$

2 b $\tan^2\theta = 2\tan\theta$

$\Rightarrow \tan^2\theta - 2\tan\theta = 0$

$\Rightarrow \tan\theta(\tan\theta - 2) = 0$

So $\tan\theta = 0$ or $\tan\theta = 2$

(first and third quadrants)

Solutions are $(-180°, 0°, 180°)$

and $(-116.6°, 63.4°)$.

Solution set:

$-180°, -117°, 0°, 63.4°, 180°$

c $\cos^2\theta - 2\cos\theta = 1$

$\Rightarrow \cos^2\theta - 2\cos\theta - 1 = 0$

So $\cos\theta = \dfrac{2\pm\sqrt{8}}{2}$

$\Rightarrow \cos\theta = \dfrac{2-\sqrt{8}}{2}\left(\text{as } \dfrac{2+\sqrt{8}}{2} > 1\right)$

Solutions are $\pm 114°$

(second and third quadrants).

d $4\sin\theta = \tan\theta$

So $4\sin\theta = \dfrac{\sin\theta}{\cos\theta}$

$\Rightarrow 4\sin\theta\cos\theta = \sin\theta$

$\Rightarrow 4\sin\theta\cos\theta - \sin\theta = 0$

$\Rightarrow \sin\theta(4\cos\theta - 1) = 0$

So $\sin\theta = 0$ or $\cos\theta = \dfrac{1}{4}$

Solutions of $\cos\theta = \dfrac{1}{4}$ are

$\cos^{-1}\left(\dfrac{1}{4}\right)$ and $360° - \cos^{-1}\left(\dfrac{1}{4}\right)$

Solution set:

$0°,\ \pm 75.5°,\ \pm 180°$

3 a $4\sin^2\theta - 4\cos\theta = 3 - 2\cos\theta$

$\Rightarrow 4(1-\cos^2\theta) - 4\cos\theta = 3 - 2\cos\theta$

$\Rightarrow 4\cos^2\theta + 2\cos\theta - 1 = 0$

So $\cos\theta = \dfrac{-2\pm\sqrt{20}}{8} = \left(\dfrac{-1\pm\sqrt{5}}{4}\right)$

Solutions of $\cos\theta = \dfrac{-2+\sqrt{20}}{8}$ are

Solutions of $\cos\theta = \dfrac{-2-\sqrt{20}}{8}$ are

$144°, -144°$(second and third quadrants).

Solution set: $72°, 144°$

b $2\sin^2\theta = 3(1-\cos\theta)$

$\Rightarrow 2(1-\cos^2\theta) = 3(1-\cos\theta)$

$\Rightarrow 2(1-\cos\theta)(1+\cos\theta) = 3(1-\cos\theta)$

(or write as $a\cos^2\theta + b\cos\theta + c \equiv 0$)

$\Rightarrow (1-\cos\theta)(2(1+\cos\theta)-3) = 0$

$\Rightarrow (1-\cos\theta)(2\cos\theta - 1) = 0$

So $\cos\theta = 1$ or $\cos\theta = \dfrac{1}{2}$

Solution of $\cos\theta = 1$ is $0°$

Solution of $\cos\theta = \dfrac{1}{2}$ are $-60°$, $60°$

Solution set: 0°, 60’

c $4\cos^2\theta - 5\sin\theta - 5 = 0$

$\Rightarrow 4(1-\sin^2\theta) - 5\sin\theta - 5 = 0$

$\Rightarrow 4\sin^2\theta + 5\sin\theta + 1 = 0$

$\Rightarrow (4\sin\theta + 1)(\sin\theta + 1) = 0$

So $\sin\theta = -1$ or $\sin\theta = -\dfrac{1}{4}$

Solution of $\sin\theta = -1$ is $\theta = 270°$.

Solution of $\sin\theta = -\dfrac{1}{4}$ are

$\theta = 194°, 346°$ (3 s.f.)

(second and fourth quadrants).

Solution set: the empty set

None of the solutions are in the required range.

4 a $5\sin^2\theta = 4\cos^2\theta$

$\Rightarrow \tan^2\theta = \dfrac{4}{5}$ as $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

So $\tan\theta = \pm\sqrt{\dfrac{4}{5}}$

4 a There are solutions from each of the quadrants (angle to horizontal is 41.8°).

$\theta = \pm 138°, \pm 41.8°$

b $\tan\theta = \cos\theta$

$\Rightarrow \dfrac{\sin\theta}{\cos\theta} = \cos\theta$

$\Rightarrow \sin\theta = \cos^2\theta$

$\Rightarrow \sin\theta = 1 - \sin^2\theta$

$\Rightarrow \sin^2\theta + \sin\theta - 1 = 0$

So $\sin\theta = \dfrac{-1 \pm \sqrt{5}}{2}$

There are only solutions from

$\sin\theta = \dfrac{-1+\sqrt{5}}{2} \quad \left(\text{as } \dfrac{-1-\sqrt{5}}{2} < -1\right)$

Solutions are $\theta = 38.2°, 142°$

(first and second quadrants).

5 $8\sin^2 x + 6\cos x - 9 = 0$ can be written as
$8(1 - \cos^2 x) + 6\cos x - 9 = 0$
which reduces to
$8\cos^2 x - 6\cos x + 1 = 0$
So $(4\cos x - 1)(2\cos x - 1) = 0$
$\cos x = \dfrac{1}{4}$ or $\cos x = \dfrac{1}{2}$
So $x = 75.5°, 284.5°, 60°, 300°$

The solutions are
$x = 60°, 75.5°, 284.5°, 300°$

6 $\sin^2 x + 1 = \dfrac{7}{2}\cos^2 x$ can be written as

$\sin^2 x + 1 = \dfrac{7}{2}\left(1 - \sin^2 x\right)$

$2\sin^2 x + 2 = 7 - 7\sin^2 x$ which reduces to
$9\sin^2 x - 5 = 0$

$\sin^2 x = \dfrac{5}{9}$

$\sin x = \pm\dfrac{\sqrt{5}}{3}$

So $x = 48.2°, 131.8°, -48.2°, 228.2, 311.8°$.
The solutions are
$x = 48.2°, 131.8°, 228.2, 311.8°$

7 $2\cos^2 x + \cos x - 6 = 0$
$(2\cos x - 3)(\cos x + 2) = 0$

$\cos x = \dfrac{3}{2}$ or $\cos x = -2$

There are no solutions to $\cos x = \dfrac{3}{2}$ or $\cos x = -2$, so the equation has no solutions.

8 a $\cos^2 x = 2 - \sin x$ can be written as
$(1 - \sin^2 x) = 2 - \sin x$
$\sin^2 x - \sin x + 1 = 0$

b $\sin^2 x - \sin x + 1 = 0$
Using the discriminant
$b^2 - 4ac = (-1)^2 - 4 \times 1 \times 1$
$= -3$

As $b^2 - 4ac < 0$, therefore there are no real roots.

9 a $\tan^2 x - 2\tan x - 4 = 0$

$\tan x = \dfrac{2 \pm \sqrt{(-2)^2 - 4(1)(-4)}}{2(1)}$

$= \dfrac{2 \pm \sqrt{20}}{2}$

$= \dfrac{2 \pm 2\sqrt{5}}{2}$

$= 1 \pm \sqrt{5}$

$p = 1, q = 5$

b $\tan x = 1 \pm \sqrt{5}$
$x = 72.8°, 252.8°, 432.8°, -51.0°, 129.0°, 309.0°, 489.0°$

So the solutions are
$x = 72.8°, 129.0°, 252.8°, 309.0°, 432.8°, 489.0°$

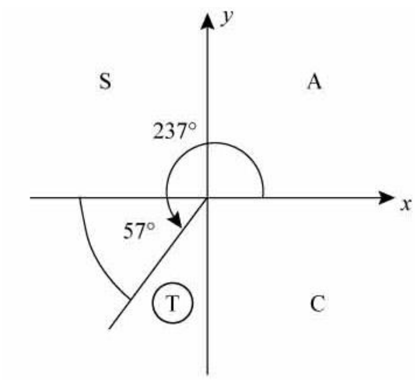
Challenge

a Let $X = 3\theta$
So $\cos^2 X - \cos X - 2 = 0$
$(\cos X + 1)(\cos X - 2) = 0$
$\cos X = -1$ or $\cos X = 2$
$\cos X = 2$ has no solutions so $\cos X = -1$
As $X = 3\theta$, then as $-180° \leq \theta \leq 180°$
So $3 \times -180° \leq X \leq 3 \times 180°$
So the interval for X is $-540° \leq X \leq 540°$.
$X = -540°, -180°, 180°, 540°$
I.e. $3\theta = -540°, -180°, 180°, 540°$
So $\theta = -180°, -60°, 60°, 180°$

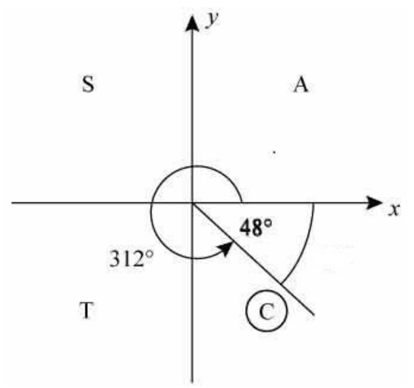
b Let $X = \theta - 45°$
So $\tan^2 X = 1$
$\tan X = \pm 1$
As $X = \theta - 45°$, then as $0 \leq \theta \leq 360°$
So $0 - 45° \leq X \leq 360° - 45°$
So the interval for X is $-45° \leq X \leq 315°$.
$X = -45°, 135°, 315°, 45°, 225°$
I.e. $\theta - 45° = -45°, 45°, 135°, 225°, 315°$
So $\theta = 0°, 90°, 180°, 270°, 360°$

Trigonometric identities and equations, Mixed exercise 10

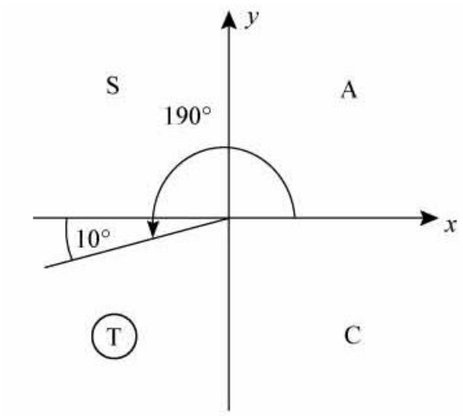
1 a 237° is in the third quadrant, so cos 237° is –ve.
The angle made with the horizontal is 57°.
So $\cos 237° = -\cos 57°$



b 312° is in the fourth quadrant so sin 312° is –ve.
The angle to the horizontal is 48°.
So $\sin 312° = -\sin 48°$



c 190° is in the third quadrant so tan 190° is +ve.
The angle to the horizontal is 10°.
So $\tan 190° = +\tan 10°$



2 a $\cos 270° = 0$

b $\sin 225° = \sin(180+45)°$
$= -\sin 45°$
$= -\dfrac{\sqrt{2}}{2}$

c $\cos 180° = -1$ $\left(\text{see graph of } y = \cos\theta\right)$

d $\tan 240° = \tan(180+60)°$
$= +\tan 60° \left(\text{third quadrant}\right)$
So $\tan 240° = +\sqrt{3}$

e $\tan 135° = -\tan 45° \left(\text{second quadrant}\right)$
So $\tan 135° = -1$

3 Using $\sin^2 A + \cos^2 A \equiv 1$

$$\sin^2 A + \left(-\sqrt{\frac{7}{11}}\right)^2 = 1$$

$$\sin^2 A = 1 - \frac{7}{11}$$

$$= \frac{4}{11}$$

$$\sin A = \pm\frac{2}{\sqrt{11}}$$

But A is in the second quadrant $\left(\text{obtuse}\right)$,
so $\sin A$ is + ve.

So $\sin A = +\dfrac{2}{\sqrt{11}}$

Using $\tan A = \dfrac{\sin A}{\cos A}$

$$\tan A = \frac{\left(\frac{2}{\sqrt{11}}\right)}{-\sqrt{\frac{7}{11}}}$$

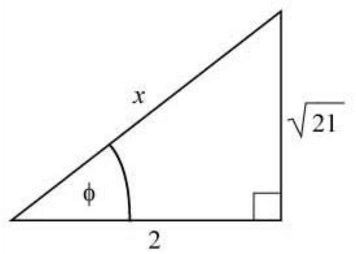
$$= -\frac{2}{\sqrt{11}} \times \frac{\sqrt{11}}{\sqrt{7}}$$

$$= -\frac{2}{\sqrt{7}}$$

$$= -\frac{2\sqrt{7}}{7}$$

$\left(\text{rationalising the denominator}\right)$

4 Draw a right-angled triangle with an angle of ϕ, where $\phi = +\dfrac{\sqrt{21}}{2}$.



Use Pythagoras' theorem to find the hypotenuse.

$$x^2 = 2^2 + \left(\sqrt{21}\right)^2$$
$$= 4 + 21$$
$$= 25$$

So $x = 5$

a $\sin\phi = \dfrac{\sqrt{21}}{5}$

As B is reflex and $\tan B$ is +ve, B is in the third quadrant.
So $\sin B = -\sin\phi$

$$= -\frac{\sqrt{21}}{5}$$

b From the diagram $\cos\phi = \dfrac{2}{5}$.

B is in the third quadrant. So
$\cos B = -\cos\phi$

$$= -\frac{2}{5}$$

5 a Factorise $\cos^4\theta - \sin^4\theta$.
(This is the difference of two squares.

$\cos^4\theta - \sin^4\theta$
$= \left(\cos^2\theta + \sin^2\theta\right)\left(\cos^2\theta - \sin^2\theta\right)$
$= (1)\left(\cos^2\theta - \sin^2\theta\right)$
$\left(\text{as} \cos^2\theta + \sin^2\theta \equiv 1\right)$

So $\cos^4\theta - \sin^4\theta = \cos^2\theta - \sin^2\theta$

b Factorise $\sin^2 3\theta - \sin^2 3\theta\cos^2 3\theta$.

$\sin^2 3\theta - \sin^2 3\theta\cos^2 3\theta$
$= \sin^2 3\theta\left(1 - \cos^2 3\theta\right)$

5 b (use $\sin^2 3\theta + \cos^2 3\theta \equiv 1$)
$\sin^2 3\theta - \sin^2 3\theta\cos^2 3\theta = \sin^2 3\theta\left(\sin^2 3\theta\right)$
$= \sin^4 3\theta$

c $\cos^4\theta + 2\sin^2\theta\cos^2\theta + \sin^4\theta$
$= \left(\cos^2\theta + \sin^2\theta\right)^2$
$= 1 \quad (\text{since } \sin^2\theta + \cos^2\theta \equiv 1)$

6 a $2(\sin x + 2\cos x) = \sin x + 5\cos x$
$\Rightarrow 2\sin x + 4\cos x = \sin x + 5\cos x$
$\Rightarrow 2\sin x - \sin x = 5\cos x - 4\cos x$
$\Rightarrow \sin x = \cos x$
(divide both sides by $\cos x$)
So $\tan x = 1$

b $\sin x\cos y + 3\cos x\sin y$
$= 2\sin x\sin y - 4\cos x\cos y$

$$\Rightarrow \frac{\sin x\cos y}{\cos x\cos y} + \frac{3\cos x\sin y}{\cos x\cos y} = \frac{2\sin x\sin y}{\cos x\cos y} - \frac{4\cos x\cos y}{\cos x\cos y}$$

$\Rightarrow \tan x + 3\tan y = 2\tan x\tan y - 4$
$\Rightarrow 2\tan x\tan y - 3\tan y = 4 + \tan x$
$\Rightarrow \tan y(2\tan x - 3) = 4 + \tan x$

So $\tan y = \dfrac{4 + \tan x}{2\tan x - 3}$

7 a $\text{LHS} = \left(1 + 2\sin\theta + \sin^2\theta\right) + \cos^2\theta$
$= 1 + 2\sin\theta + 1$ since $\sin^2\theta + \cos^2\theta \equiv 1$
$= 2 + 2\sin\theta$
$= 2(1 + \sin\theta)$
$=$ RHS

b $\text{LHS} = \cos^4\theta + \sin^2\theta$
$= \left(\cos^2\theta\right)^2 + \sin^2\theta$
$= \left(1 - \sin^2\theta\right)^2 + \sin^2\theta$
$= 1 - 2\sin^2\theta + \sin^4\theta + \sin^2\theta$
$= \left(1 - \sin^2\theta\right) + \sin^4\theta$
$= \cos^2\theta + \sin^4\theta$
(using $\sin^2\theta + \cos^2\theta \equiv 1$)
$=$ RHS

8 a $\sin\theta = \frac{3}{2}$ has no solutions as $-1 \le \sin\theta \le 1$.

b $\sin\theta = -\cos\theta$
$\Rightarrow \tan\theta = -1$
Look at the graph of $y = \tan\theta$ in the interval $0 \le \theta \le 360°$. There are two solutions.

c The minimum value of $2\sin\theta$ is –2.
The minimum value of 3 $\cos\theta$ is -3.
Each minimum value is for a different θ.
So the minimum value of
$2\sin\theta + 3\cos\theta$ is always greater than -5.
There are no solutions of
$2\sin\theta + 3\cos\theta + 6 = 0$
as the LHS can never be zero.

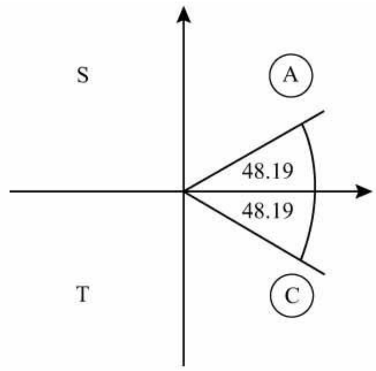
d Solving $\tan\theta + \frac{1}{\tan\theta} = 0$ is equivalent to solving $\tan^2\theta = -1$, which has no solutions.
So there are no solutions.

9 a $4xy - y^2 + 4x - y \equiv y(4x - y) + (4x - y)$
$= (4x - y)(y + 1)$

b Using **a** with $x = \sin\theta, y = \cos\theta$
$4\sin\theta\cos\theta - \cos^2\theta$
$+ 4\sin\theta - \cos\theta = 0$
So
$(4\sin\theta - \cos\theta)(\cos\theta + 1) = 0$
So $4\sin\theta - \cos\theta = 0$ or
$\cos\theta + 1 = 0$
$4\sin\theta - \cos\theta = 0$
$\Rightarrow \tan\theta = \frac{1}{4}$
The calculator solution is $\theta = 14.0°$.
$\tan\theta$ is +ve so θ is in the first and third quadrants.
So $\theta = 14.0°, 194°$
$\cos\theta + 1 = 0 \Rightarrow \cos\theta = -1$
So $\theta = +180°$ (from graph)
Solutions are $\theta = 14.0°, 180°, 194°$

10 a As $\sin(90 - \theta)° \equiv \cos\theta°$,
$\sin(90 - 3\theta)° \equiv \cos 3\theta°$
So $4\cos 3\theta° - \sin(90 - 3\theta)°$
$= 4\cos 3\theta° - \cos 3\theta$
$= 3\cos 3\theta°$

b Using **a**, $4\cos 3\theta° - \sin(90 - 3\theta)° = 2$
is equivalent to $3\cos 3\theta° = 2$
so $\cos 3\theta° = \frac{2}{3}$
Let $X = 3\theta$ and solve $\cos X° = \frac{2}{3}$
in the interval $0° \le X \le 1080°$.
The calculator solution is $X = 48.19°$
As $\cos X°$ is + ve, X is in the first and fourth quadrants.

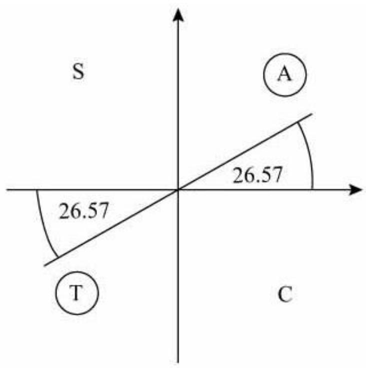


Read off all solutions in the interval $0° \le X \le 1080°$.
$X = 48.19°, 311.81°, 408.19°, 671.81°,$
$768.19°, 1031.81°$
So $\theta = \frac{X}{3} = 16.1°, 104, 136°, 224°, 256°,$
$344°$ (3 s.f.)

11 a $2\sin 2\theta = \cos 2\theta$
$\Rightarrow \frac{2\sin 2\theta}{\cos 2\theta} = 1$
$\Rightarrow 2\tan 2\theta = 1$ since $\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta}$
So $\tan 2\theta = 0.5$

b Solve $\tan 2\theta° = 0.5$ in the interval $0° \le \theta < 360°$ or $\tan X° = 0.5$ where $X = 2\theta$, $0° \le X < 720°$.

11 b The calculator solution for $\tan^{-1} 0.5$ is 26.57°.
As tan X is +ve, X is in the first and third quadrants.



Read off solutions for X in the interval $0° \le X < 720°$.

$X = 26.57°,\ 206.57°,\ 386.57°,\ 566.57°$

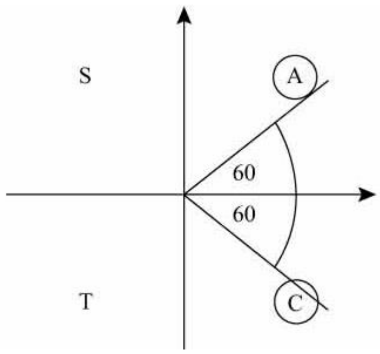
$= 2\theta$

So $\theta = \dfrac{X}{2}$

$= 13.3°, 103.3°, 193.3°, 283.3°\ (1\text{ d.p.})$

12 a $\cos(\theta + 75)° = 0.5$

Solve $\cos X° = 0.5$, where $X = \theta + 75$, $75° \le X < 435°$.
Your calculator solution for X = 60°.
As cos X is +ve, X is in the first and fourth quadrants.



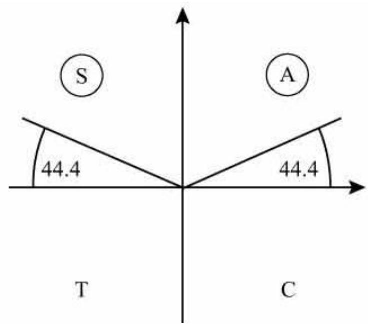
Read off all solutions in the interval $75° \le X < 435°$.

$X = 300°,\ 420°$

$\theta + 75° = 300°,\ 420°$

So $\theta = 225°,\ 345°$

12 b $\sin 2\theta° = 0.7$ in the interval $0° \le \theta < 360°$.
Solve sin $X°$ = 0.7, where $X = 2\theta,\ 0° \le X < 720°$.
The calculator solution is 44.4°.
As sin X is +ve, X is in the first and second quadrants.



Read off solutions in the interval $0° \le X < 720°$.

$X = 44.4°,\ 135.6°,\ 404.4°,\ 495.6°$

$= 2\theta$

So $\theta = \dfrac{X}{2}$

$= 22.2°, 67.8°, 202.2°, 247.8°\ (1\text{ d.p.})$

13 Multiply both sides of the equation by $(1 - \cos 2x)$, provided $\cos 2x \ne 1$.
Note: In the interval given, cos 2x is never equal to 1.

So $\cos 2x + 0.5 = 2 - 2\cos 2x$

$\Rightarrow 3\cos 2x = \dfrac{3}{2}$

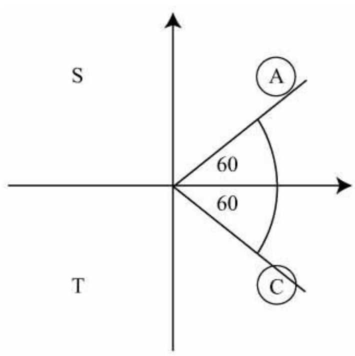
So $\cos 2x = \dfrac{1}{2}$

Solve $\cos X = \dfrac{1}{2}$ where $X = 2x$,

$0° < X < 540°$.

The calculator solution is 60°.

As $\cos X$ is + ve, X is in the first and fourth quadrants.



13 Read off solutions for X in the interval $0° < X < 540°$.

$X = 60°, 300°, 420°$

$$\text{So } x = \frac{X}{2}$$
$$= 30°, 150°, 210°$$

14 Using $\sin^2\theta + \cos^2\theta \equiv 1$

$2\cos^2\theta - \cos\theta - 1 = 1 - \cos^2\theta$

$\Rightarrow 3\cos^2\theta - \cos\theta - 2 = 0$

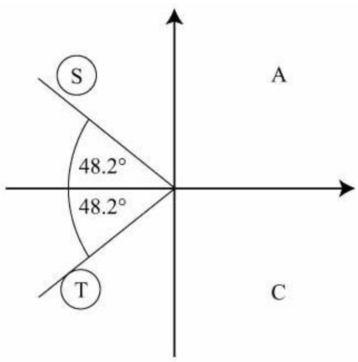
$\Rightarrow (3\cos\theta + 2)(\cos\theta - 1) = 0$

So $3\cos\theta + 2 = 0$ or $\cos\theta - 1 = 0$

For $3\cos\theta + 2 = 0$, $\cos\theta = -\frac{2}{3}$

The calculator solution is 131.8°.

As $\cos\theta$ is –ve, θ is in the second a and third quadrants.



$\theta = 131.8°, 228.2°$

For $\cos\theta = 1$, $\theta = 0°$

(See graph and check the given interval.)

So solutions are

$\theta = 0°, 131.8°, 228.2°, 360°$

15 a The student found additional solutions after dividing by three rather than before. The students has not applied the full interval for the solutions.

b Let $X = 3x$

$$\sin X = \frac{1}{2}$$

As $X = 3x$, then as $-360° \le x \le 360°$

So $3 \times -360° \le X \le 3 \times 360°$

So the interval for X is

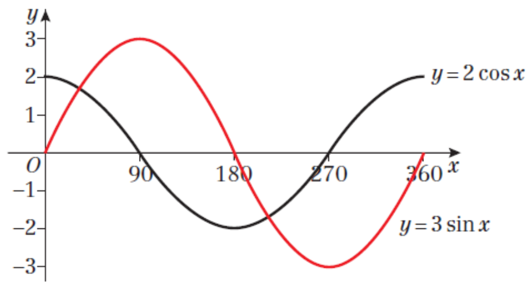
$-1080° \le X \le 1080°$

15 b $X = 30°, 150°, 390°, 510°, 750°, 870°, -210°, -330°, -570°, -690°, -930°, -1050°$

i.e. $3x = -1050°, -930°, -690°, -570°, -330°, -210°, 30°, 150°, 390°, 510°, 750°, 870°$

So $x = -350°, -310°, -230°, -190°, -110°, -70°, 10°, 50°, 130°, 170°, 250°, 290°$

16 a



b The graphs intersect at two places so there are two solutions to the equation in the given range.

c $3\sin x = 2\cos x$

$$\frac{\sin x}{\cos x} = \frac{2}{3}$$

$$\tan x = \frac{2}{3}$$

$x = 33.7°, 213.7°$

17 a Using the cosine rule

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

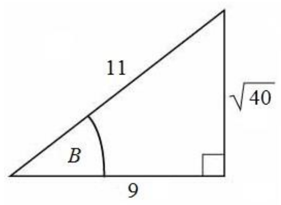
$$\cos B = \frac{6^2 + 11^2 - 7^2}{2 \times 6 \times 11}$$

$$\cos B = \frac{36 + 121 - 49}{132}$$

$$\cos B = \frac{9}{11}$$

b Using Pythagoras' theorem

$\sqrt{11^2 - 9^2} = \sqrt{40}$



$$\sin B = \frac{\sqrt{40}}{11} = \frac{2\sqrt{10}}{11}$$

18 a Using the sine rule

$$\frac{\sin Q}{q} = \frac{\sin P}{p}$$

$$\frac{\sin Q}{6} = \frac{\sin 45°}{5}$$

$$\sin Q = \frac{6\left(\sqrt{2}/2\right)}{5}$$

$$\sin Q = \frac{3\sqrt{2}}{5}$$

b Using Pythagoras' theorem or identity

$$\cos^2 x + \sin^2 x = 1$$

$\cos Q = \frac{\sqrt{7}}{5}$ for the acute angle

As Q is obtuse, it is in the second quadrant where cos Q is negative.

So $\cos Q = -\frac{\sqrt{7}}{5}$

19 a $3\sin^2 x - \cos^2 x = 2$ can be written as
$3\sin^2 x - (1 - \sin^2 x) = 2$
which reduces to
$4\sin^2 x = 3$

b $4\sin^2 x = 3$

$\sin^2 x = \frac{3}{4}$

$\sin x = \pm\frac{\sqrt{3}}{2}$

$x = 60°, 120°, -60°, -120°$
So the solutions are
$x = -120.0°, -60.0°, 60.0°, 120.0°$

20 $3\cos^2 x + 1 = 4\sin x$ can be written as
$3(1 - \sin^2 x) + 1 = 4\sin x$
which can be reduced to
$3\sin^2 x + 4\sin x - 4 = 0$
$(3\sin x - 2)(\sin x + 2) = 0$

$\sin x = \frac{2}{3}$ or $\sin x = -2$

$\sin x = -2$ has no solutions.
So $x = 41.8°, 138.2°, -221.8°, -318.2°$
So the solutions are
$x = -318.2°, -221.8°, 41.8°, 138.2°$

21 a Solving equation where $X = 2x + k$

$$3 + \sqrt{3} = 3 + 2\sin(X)$$

$$\sqrt{3} = 2\sin(X)$$

$$X = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

$$X = 60° \text{ or } 180° - 60° = 120°$$

$$k = 60° - 30° \text{ or } 120° - 30°$$

$$k = 30° \text{ or } 90°$$

b Solving equation where $X = 2x + 30$ and $30 \le X \le 750$

$$1 = 3 + 2\sin(X)$$

$$-2 = 2\sin(X)$$

$$X = \sin^{-1}(-1)$$

$$X = 270° \text{ or } 270° + 360° = 630°$$

$$x = \frac{X - 30°}{2}$$

$$x = 120° \text{ or } 300°$$

Challenge

$\tan^4 x - 3\tan^2 x + 2 = 0$
$(\tan^2 x - 2)(\tan^2 x - 1) = 0$
$\tan^2 x = 2$ or $\tan^2 x = 1$
$\tan x = \pm 1$ or $\pm\sqrt{2}$
$x = 45°, 225°, -45°, 135°, 315°, 54.7°, 234.7°, -54.7°, 125.3°, 305.3°$
So the solutions are
$x = 45°, 54.7°, 125.3°, 135°, 225°, 234.7°, 305.3°, 315°$

Review exercise 2

1 The equation of the line is:

$$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}$$
$$\frac{y-8}{6-8}=\frac{x+2}{4+2}$$
$$\frac{y-8}{-2}=\frac{x+2}{6}$$
$$3y-24=-x-2$$
$$x+3y-22=0$$

2
$$y-(-4)=\tfrac{1}{3}(x-9)$$
$$y+4=\tfrac{1}{3}(x-9)$$
$$3y+12=x-9$$
$$x-3y-21=0$$
$a = 1$, $b = -3$, $c = -21$

3 Using points A and B:

$$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}$$
$$\frac{y-3}{5-3}=\frac{x-0}{k-0}$$
$$\frac{y-3}{2}=\frac{x}{k}$$
$$ky-3k=2x$$

Substituting point C into the equation:
$$k(2k)-3k=2(10)$$
$$2k^2-3k-20=0$$
$$(2k+5)(k-4)=0$$
$$k=-\tfrac{5}{2}\text{ or }k=4$$

4 a Using points (160, 72) and (180, 81):

$$\text{Gradient}=\frac{y_2-y_1}{x_2-x_1}=\frac{81-72}{180-160}=\frac{9}{20}=0.45$$

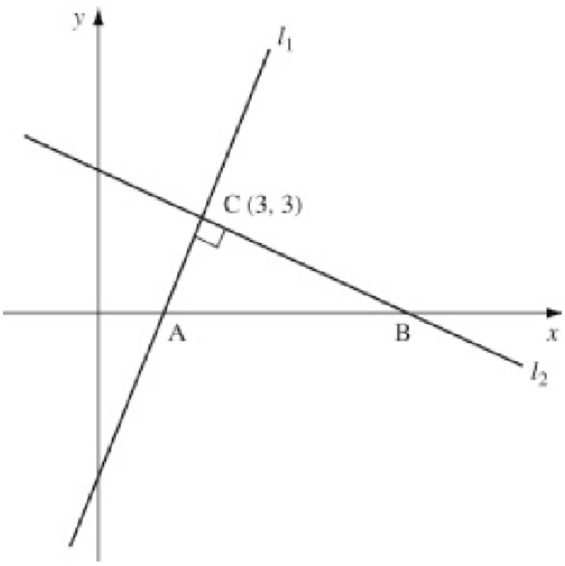
b $l = kh$, where k is the gradient.
So $l = 0.45h$

c The model may not be valid for young people/children who are still growing.

5 a The gradient of l_1 is 3.
So the gradient of l_2 is $-\frac{1}{3}$.
The equation of line l_2 is:
$$y-2=-\tfrac{1}{3}(x-6)$$
$$y-2=-\tfrac{1}{3}x+2$$
$$y=-\tfrac{1}{3}x+4$$

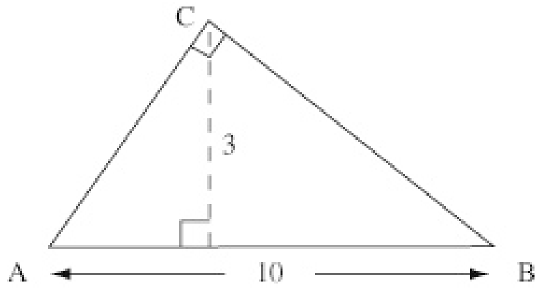
b
$$y=3x-6$$
$$y=-\tfrac{1}{3}x+4$$
$$3x-6=-\tfrac{1}{3}x+4$$
$$3x+\tfrac{1}{3}x=4+6$$
$$\tfrac{10}{3}x=10$$
$$x=3$$
$$y=(3\times3)-6=3$$
$$x=3$$
$$y=3\times3-6=3$$
The point C is (3, 3).

c



Where l_1 meets the x-axis, $y = 0$:
$$0=3x-6$$
$$3x=6$$
$$x=2$$
The point A is (2, 0).
Where l_2 meets the x-axis, $y = 0$:
$$0=-\tfrac{1}{3}x+4$$
$$\tfrac{1}{3}x=4$$
$$x=12$$
The point B is (12, 0).

5 c



$AB = 12 - 2 = 10$
The perpendicular height, using AB as the base is 3.
Area of triangle $ABC = \frac{1}{2} \times \text{base} \times \text{height}$
$= \frac{1}{2} \times 10 \times 3$
$= 15$

6 Substituting $y = 2x$ into $5y + x - 33 = 0$:
$5(2x) + x - 33 = 0$
$11x - 33 = 0$
$x = 3$
$y = 2 \times 3 = 6$
The point P is (3, 6).
Distance from origin $= \sqrt{3^2 + 6^2}$
$= \sqrt{45}$
$= 3\sqrt{5}$

7 Gradient of line $= \dfrac{y_2 - y_1}{x_2 - x_1}$
$= \dfrac{-4 - 8}{7 - 5}$
$= -6$
Gradient of the perpendicular bisector is $\frac{1}{6}$
Midpoint of line $= \left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$
$= \left(\dfrac{5 + 7}{2}, \dfrac{8 - 4}{2}\right)$
$= (6, 2)$

Equation of the perpendicular bisector is:
$y - y_1 = m(x - x_1)$
$y - 2 = \frac{1}{6}(x - 6)$
$y = \frac{1}{6}x + 1$

This line crosses the x-axis at $y = 0$:
$\frac{1}{6}x + 1 = 0$
$x = -6$

The point Q is (−6, 0).

8 Equation of circle with centre (−3, 8) and radius r:
$(x + 3)^2 + (y - 8)^2 = r^2$

r = distance from (−3, 8) to (0, 9)
$r^2 = (0 + 3)^2 + (9 - 8)^2 = 9 + 1 = 10$

The equation for C is:
$(x + 3)^2 + (y - 8)^2 = 10$

9 a Rearranging:
$x^2 - 6x + y^2 + 2y = 10$

Completing the square:
$(x - 3)^2 - 9 + (y + 1)^2 - 1 = 10$
$(x - 3)^2 + (y + 1)^2 = 20$
$a = 3,\ b = -1,\ r = \sqrt{20}$

b The circle has centre (3, −1) and radius $\sqrt{20}$.

10 a Rearranging $3x + y = 14$:
$y = 14 - 3x$

Solving simultaneously using substitution:
$(x - 2)^2 + (14 - 3x - 3)^2 = 5$
$(x - 2)^2 + (-3x + 11)^2 = 5$
$x^2 - 4x + 4 + 9x^2 - 66x + 121 - 5 = 0$
$10x^2 - 70x + 120 = 0$
$x^2 - 7x + 12 = 0$
$(x - 3)(x - 4) = 0$
So $x = 3$ and $x = 4$

$x = 3$: $y = 14 - 3 \times 3 = 5$
$x = 4$: $y = 14 - 3 \times 4 = 2$
Point A is (3, 5) and point B is (4, 2).

b Using Pythagoras' theorem:
Length $AB = \sqrt{(4-3)^2 + (2-5)^2}$
$= \sqrt{10}$

11 The equation of the circle is $x^2 + y^2 = r^2$.

Solving simultaneously using substitution:
$x^2 + (3x - 2)^2 = r^2$
$x^2 + 9x^2 - 12x + 4 - r^2 = 0$
$10x^2 - 12x + 4 - r^2 = 0$

Using the discriminant for no solutions:
$b^2 - 4ac < 0$
$(-12)^2 - 4(10)(4 - r^2) < 0$
$144 - 160 + 40r^2 < 0$

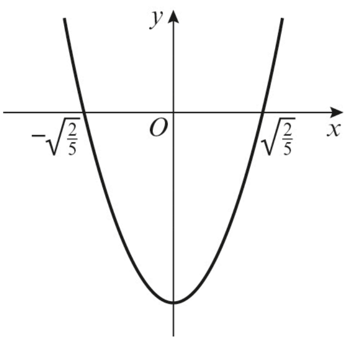
11 $40r^2 - 16 < 0$

When $40r^2 - 16 = 0$

$8(5r^2 - 2) = 0$

$r^2 = \frac{2}{5}$

$r = \pm\sqrt{\frac{2}{5}}$



$-\sqrt{\frac{2}{5}} < r < \sqrt{\frac{2}{5}}$

However, the radius cannot be negative.

So $0 < r < \sqrt{\frac{2}{5}}$

12 a Equation of circle with centre (1, 5) and radius r:

$(x - 1)^2 + (y - 5)^2 = r^2$

r = distance from (1, 5) to (4, −2)

$r^2 = (4 - 1)^2 + (-2 - 5)^2$
$= 9 + 49$
$= 58$

The equation for C is:

$(x - 1)^2 + (y - 5)^2 = 58$

b Gradient of the radius of the circle at P

$= \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2-5}{4-1} = -\frac{7}{3}$

Gradient of the tangent $= \frac{3}{7}$

Equation of the tangent at P:

$y - y_1 = m(x - x_1)$

$y + 2 = \frac{3}{7}(x - 4)$

$7y - 3x + 26 = 0$

13 a $AB^2 = (6 - 2)^2 + (5 - 1)^2$
$= 4^2 + 4^2 = 32$

$BC^2 = (8 - 6)^2 + (3 - 5)^2$
$= 2^2 + 2^2 = 8$

13 a $AC^2 = (8 - 2)^2 + (3 - 1)^2$
$= 6^2 + 2^2 = 40$

Using Pythagoras' theorem:

$AB^2 + BC^2 = 32 + 8 = 40 = AC^2$

Therefore, $\angle ABC$ is 90°.

b As triangle ABC is a right-angled triangle, AC is a diameter of the circle.

c AC is a diameter of the circle, so the midpoint of AC is the centre.

$$\text{Midpoint} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$
$$= \left(\frac{2+8}{2}, \frac{1+3}{2}\right)$$
$$= (5, 2)$$

$$\text{Radius} = \tfrac{1}{2} \times AC$$
$$= \tfrac{1}{2} \times \sqrt{40}$$
$$= \tfrac{1}{2} \times 2\sqrt{10}$$
$$= \sqrt{10}$$

The equation of the circle is:

$(x - 5)^2 + (y - 2)^2 = 10$

14
$$\frac{2x^2+20x+42}{224x+4x^2-4x^3} = \frac{x^2+10x+21}{112x+2x^2-2x^3}$$
$$= \frac{(x+3)(x+7)}{-2x(x^2-x-56)}$$
$$= \frac{(x+3)(x+7)}{-2x(x+7)(x-8)}$$
$$= \frac{(x+3)}{-2x(x-8)}$$

$a = 3, b = -2, c = -8$

15 a Using the factor theorem:

$f(\tfrac{1}{2}) = 2(\tfrac{1}{2})^3 - 7(\tfrac{1}{2})^2 - 17(\tfrac{1}{2}) + 10$
$= \tfrac{1}{4} - \tfrac{7}{4} - \tfrac{17}{2} + 10$
$= 0$

So $(2x - 1)$ is a factor of $2x^3 - 7x^2 - 17x + 10$.

15 b
$$\begin{array}{r} x^2-3x-10 \\ 2x-1\overline{)2x^3-7x^2-17x+10} \\ \underline{2x^3-x^2} \\ -6x^2-17x \\ \underline{-6x^2+3x} \\ -20x+10 \\ \underline{-20x+10} \\ 0 \end{array}$$

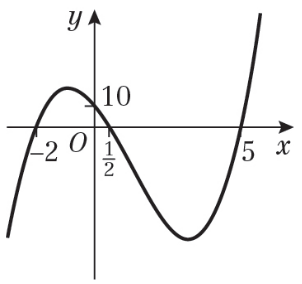
$2x^3 - 7x^2 - 17x + 10$
$= (2x - 1)(x^2 - 3x - 10)$
$= (2x - 1)(x - 5)(x + 2)$

c $(2x - 1)(x - 5)(x + 2) = 0$
So $x = \frac{1}{2}$, $x = 5$ or $x = -2$

So the curve crosses the x-axis at $(\frac{1}{2}, 0)$, $(5, 0)$ and $(-2, 0)$.

When $x = 0$, $y = -1 \times -5 \times 2 = 10$
So the curve crosses the y-axis at $(0, 10)$.

$x \to \infty, y \to \infty$
$x \to -\infty, y \to -\infty$



16 $f(x) = 3x^3 + x^2 - 38x + c$

a $f(3) = 0$
$3(3)^3 + (3)^2 - 38(3) + c = 0$
$3 \times 27 + 9 - 114 + c = 0$
$c = 24$

b $f(x) = 3x^3 + x^2 - 38x + 24$
$f(3) = 0$, so $(x - 3)$ is a factor of $3x^3 + x^2 - 38x + 24$.

16 c
$$\begin{array}{r} 3x^2+10x-8 \\ x-3\overline{)3x^3+x^2-38x+24} \\ \underline{3x^3-9x^2} \\ 10x^2-38x \\ \underline{10x^2-30x} \\ -8x-24 \\ \underline{-8x+24} \\ 0 \end{array}$$

$3x^3 + x^2 - 38x + 24$
$= (x - 3)(3x^2 + 10x - 8)$
$= (x - 3)(3x - 2)(x + 4)$

17 a $g(x) = x^3 - 13x + 12$
$g(3) = (3)^3 - 13(3) + 12$
$= 27 - 39 + 12$
$= 0$
So $(x - 3)$ is a factor of $g(x)$.

b
$$\begin{array}{r} x^2+3x-4 \\ x-3\overline{)x^3-0x^2-13x+12} \\ \underline{x^3-3x^2} \\ 3x^2-13x \\ \underline{3x^2-9x} \\ -4x+12 \\ \underline{-4x+12} \\ 0 \end{array}$$

$g(x) = x^3 - 13x + 12$
$= (x - 3)(x^2 + 3x - 4)$
$= (x - 3)(x + 4)(x - 1)$

18 a Example:
When $a = 0$ and $b = 0$, $0^2 + 0^2 = (0 + 0)^2$.

b $(a + b)^2 = a^2 + 2ab + b^2$
When $a > 0$ and $b > 0$, $2ab > 0$
Therefore $a^2 + b^2 < (a + b)^2$
When $a < 0$ and $b < 0$, $2ab > 0$
Therefore $a^2 + b^2 < (a + b)^2$
When $a > 0$ and $b < 0$, $2ab < 0$
Therefore $a^2 + b^2 > (a + b)^2$
When $a < 0$ and $b > 0$, $2ab < 0$
Therefore $a^2 + b^2 > (a + b)^2$
The conditions are $a > 0$ and $b > 0$ or $a < 0$ and $b < 0$.

19 a $p = 5$: $5^2 = 25 = 24 + 1$
$p = 7$: $7^2 = 49 = 2(24) + 1$
$p = 11$: $11^2 = 121 = 5(24) + 1$
$p = 13$: $13^2 = 169 = 7(24) + 1$
$p = 17$: $17^2 = 289 = 12(24) + 1$
$p = 19$: $19^2 = 361 = 15(24) + 1$

b $3(24) + 1 = 73$ and 73 is not a square number.

20 a Rearranging:
$x^2 - 10x + y^2 - 8y = -32$
Completing the square:
$(x - 5)^2 - 25 + (y - 4)^2 - 16 = -32$
$(x - 5)^2 + (y - 4)^2 = 9$
$(x - 5)^2 + (y - 4)^2 = 3^2$
$a = 5$, $b = 4$, $r = 3$

b Centre of circle C is (5, 4).
Centre of circle D is (0, 0).
Using Pythagoras' theorem:
Distance $= \sqrt{(5-0)^2 + (4-0)^2} = \sqrt{41}$

c Radius of circle $C = 3$
Radius of circle $D = 3$
Distance between the centres $= \sqrt{41}$
$3 + 3 < \sqrt{41}$
Therefore, the circles C and D do not touch.

21 a $(1 - 2x)^{10}$
$$= 1^{10} + \binom{10}{1}1^9(-2x) + \binom{10}{2}1^8(-2x)^2 + \binom{10}{3}1^7(-2x)^3 + \ldots$$
$$= 1 + 10(-2x) + \frac{10(9)}{2}(-2x) + \frac{10(9)(8)}{6}(-2x)^3 + \ldots$$
$= 1 - 20x + 180x^2 - 960x^3 + \ldots$

b $(0.98)^{10}$
$= (1 - 2(0.01))^{10}$
$= 1 - 20(0.01) + 180(0.01)^2 - 960(0.01)^3 + \ldots$
$= 0.817$ (3 d.p.)

22 $(1 + 2x)^5$
$$= 1^5 + \binom{5}{1}1^4(2x) + \binom{5}{2}1^3(2x)^2 + \ldots$$
$$= 1 + 5(2x) + \frac{5(4)}{2}(2x)^2 + \ldots$$
$= 1 + 10x + 40x^2 + \ldots$
$(2 - x)(1 + 2x)^5$
$= (2 - x)(1 + 10x + 40x^2 + \ldots)$
$= 2 + 20x + 80x^2 + \ldots - x - 10x^2 + \ldots$
$= 2 + 19x + 70x^2 + \ldots$
$\approx 2 + 19x + 70x^2$
$a = 2$, $b = 19$, $c = 70$

23 $(2 - 4x)^q$
$$x \text{ term} = \binom{q}{q-1}2^{q-1}(-4x)^1$$
$= q \times 2^{q-1} \times -4x$
$= -4 \times 2^{q-1}qx$
$-4 \times 2^{q-1}q = -32q$
$2^{q-1} = 8$
$q - 1 = 3$
$q = 4$

24 Using the sine rule:
$$\frac{b}{\sin B} = \frac{c}{\sin C}$$
$$\frac{b}{\sin 45°} = \frac{\sqrt{5}}{\sin 30°}$$
$$b = \frac{\sqrt{5}\sin 45°}{\sin 30°}$$
$$b = \frac{\sqrt{5} \times \frac{\sqrt{2}}{2}}{\frac{1}{2}}$$
$$b = \sqrt{10}$$
$AC = \sqrt{10}$ cm

25 a Using the cosine rule:
$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$
$$\cos 60° = \frac{(2x-3)^2 + 5^2 - (x+1)^2}{2(2x-3)(5)}$$
$$\frac{1}{2} = \frac{4x^2 - 12x + 9 + 25 - (x^2 + 2x + 1)}{10(2x-3)}$$
$5(2x - 3) = 3x^2 - 14x + 33$
$3x^2 - 24x + 48 = 0$
$x^2 - 8x + 16 = 0$

25 b $x^2 - 8x + 16 = 0$

$(x - 4)^2 = 0$

$x = 4$

c Area $= ac \sin B$

$a = 2 \times 4 - 3 = 5$

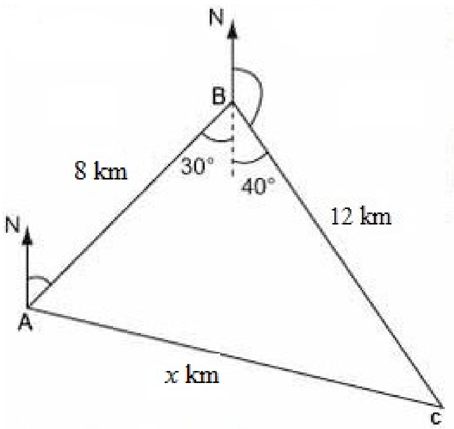
$c = 5$

Area $= \frac{1}{2} \times 5 \times 5 \times \sin 60°$

$= 10.8253...$

$= 10.8\text{ cm}^2$ (3 s.f.)

26



a Using the cosine rule

$x^2 = 8^2 + 12^2 - 2 \times 8 \times 12 \times \cos 70°$

$= 142.332\ldots$

$x = 11.93\text{ km}$

The distance of ship C from ship A is 11.93 km.

b Using the sine rule:

$$\frac{\sin 70°}{11.93} = \frac{\sin A}{12}$$

$\sin A = 0.94520\ldots$

$A = 70.9°$

The bearing of ship C from ship A is 100.9°.

27 a If triangle ABC is isosceles, then two of the sides are equal.

$AB = \sqrt{(6+2)^2 + (10-4)^2} = \sqrt{100} = 10$

$BC = \sqrt{(16-6)^2 + (10-10)^2} = \sqrt{100} = 10$

$AC = \sqrt{(16+2)^2 + (10-4)^2} = \sqrt{360}$

$= 6\sqrt{10}$

$AB = BC$

Therefore ABC is an isosceles triangle.

27 b Using the cosine rule:

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$= \frac{10^2 + 10^2 - \left(\sqrt{360}\right)^2}{2(10)(10)}$$

$$= \frac{100 + 100 - 360}{200}$$

$$= -\frac{4}{5}$$

$B = 143.13010...$

Angle $ABC = 143.1°$ (1 d.p.)

28 Using the sine rule in triangle ABD:

$$\frac{\sin \angle BDA}{4.3} = \frac{\sin 40°}{3.5}$$

$$\sin \angle BDA = \frac{4.3 \sin 40°}{3.5}$$

$= 0.78971\ldots$

$\angle BDA = 52.16°$

Using the angle sum of a triangle:

$\angle ABD = 180° - (52.16° + 40°)$

$= 87.84°$

Using the sine rule in triangle ABD:

$$\frac{AD}{\sin 87.84} = \frac{3.5}{\sin 40°}$$

$AD = 5.44\text{ cm}$

$AC = AD + DC$

$= 5.44 + 8.6$

$= 14.04\text{ cm}$

Area of triangle ABC

$= \frac{1}{2} \times 4.3 \times 14.04 \times \sin 40°$

$= 19.4\text{ cm}^2$

29 a $(x - 5)^2 + (y - 2)^2 = 5^2$

$(x - 5)^2 + (y - 2)^2 = 25$

b Substituting $x = 8$ and $y = k$ into the equation of the circle:

$(8 - 5)^2 + (k - 2)^2 = 25$

$9 + k^2 - 4k + 4 - 25 = 0$

$k^2 - 4k - 12 = 0$

$(k + 2)(k - 6) = 0$

$k = -2$ or $k = 6$

k is positive, therefore $k = 6$.

29 c $XY = \sqrt{(10-1)^2+(2+1)^2} = \sqrt{90}$

$YZ = \sqrt{(8-10)^2+(6-2)^2} = \sqrt{20}$

$XZ = \sqrt{(8-1)^2+(6+1)^2} = \sqrt{98}$

Using the cosine rule:

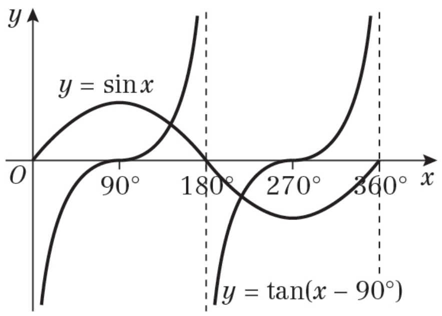
$$\cos Y = \frac{x^2+z^2-y^2}{2xz} = \frac{20+90-98}{2\sqrt{1800}} = \frac{12}{60\sqrt{2}} = \frac{1}{5\sqrt{2}}$$

Rationalising the denominator:

$$\cos Y = \frac{\sqrt{2}}{10}$$

So $\cos \angle XYZ = \frac{\sqrt{2}}{10}$

30 a



b There are two solutions in the interval $0 \le x \le 360°$.

31 a The curve $y = \sin x$ crosses the x-axis at $(-360°, 0)$, $(-180°, 0)$, $(0°, 0)$, $(180°, 0)$ and $(360°, 0)$.
$y = \sin(x + 45°)$ is a translation of $\begin{pmatrix} -45° \\ 0 \end{pmatrix}$
so subtract 45° from the x-coordinates.

The curve crosses the x-axis at $(-405°, 0)$, $(-225°, 0)$, $(-45°, 0)$, $(135°, 0)$ and $(315°, 0)$.
$(-405°, 0)$ is not in the range, so $(-225°, 0)$, $(-45°, 0)$, $(135°, 0)$ and $(315°, 0)$

31 b The curve $y = \sin(x + 45°)$ crosses the y-axis when $x = 0$.

$\sin 45° = \frac{\sqrt{2}}{2}$

$\left(0, \frac{\sqrt{2}}{2}\right)$

32 Each of the four triangular faces is an equilateral triangle.

Area of one triangle

$= \frac{1}{2}ac \sin B$

$= \frac{1}{2} \times s \times s \times \sin 60°$

$= \frac{s^2}{2} \times \frac{\sqrt{3}}{2}$

$= \frac{\sqrt{3}s^2}{4}$ cm^2

Total area

= area of 4 triangles + area of square

$= 4 \times \frac{\sqrt{3}s^2}{4} + s^2$

$= \sqrt{3}s^2 + s^2$

$= \left(\sqrt{3}+1\right)s^2$

The total surface area of the pyramid is $\left(\sqrt{3}+1\right)s^2$ cm^2.

33 a $\sin\theta = \cos\theta$

$\frac{\sin\theta}{\cos\theta} = 1$

So $\tan\theta = 1$

b When $\tan\theta = 1$
$\theta = 45°$ or $225°$
So $\sin\theta = \cos\theta$ when $\theta = 45°$ or $225°$

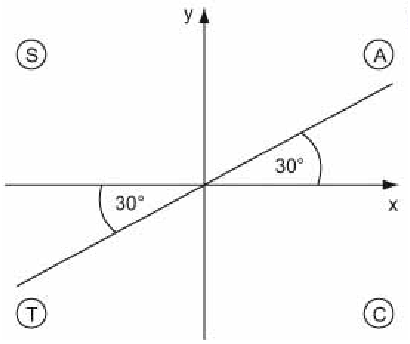
34 $3\tan^2 x = 1$

$\tan x = \pm\frac{1}{\sqrt{3}}$

For $\tan x = \frac{1}{\sqrt{3}}$

$x = 30°$

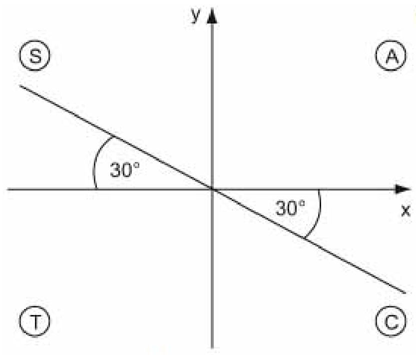
34



So $x = 30°$ or $x = 210°$

For $\tan x = -\frac{1}{\sqrt{3}}$

$x = 330°$ (or $-30°$)

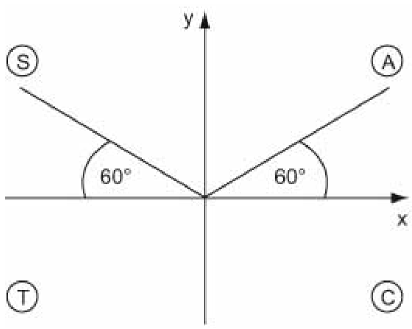


So $x = 330°$ or $x = 150°$
So $x = 30°, 150°, 210°$ or $330°$

35 $2\sin(\theta - 30°) = \sqrt{3}$

$\sin(\theta - 30°) = \frac{\sqrt{3}}{2}$

$\theta - 30° = 60°$



So $\theta - 30° = 60°$ or $\theta - 30° = 120°$
When $\theta - 30° = 60°$
$\theta = 60° + 30°$
$= 90°$
When $\theta - 30° = 120°$
$\theta = 120° + 30°$
$= 150°$
So $\theta = 90°$ or $150°$

36 a
$$2\cos^2 x = 4 - 5\sin x$$
$$2(1 - \sin^2 x) = 4 - 5\sin x$$
$$2 - 2\sin^2 x = 4 - 5\sin x$$
$$2\sin^2 x - 5\sin x + 2 = 0 \text{ (as required)}$$

b Let $\sin x = y$
$2y^2 - 5y + 2 = 0$
$(2y - 1)(y - 2) = 0$
So $y = \frac{1}{2}$ or $y = 2$
When $\sin x = \frac{1}{2}$, $x = 30°$
or $x = 180° - 30° = 150°$
$\sin x = 2$ is impossible.
$x = 30°$ or $150°$

37
$$2\tan^2 x - 4 = 5\tan x$$
$$2\tan^2 x - 5\tan x - 4 = 0$$

Using the quadratic formula:
$$\tan x = \frac{5 \pm \sqrt{(-5)^2 - 4(2)(-4)}}{2(2)}$$
$$= \frac{5 \pm \sqrt{57}}{4}$$

When $\tan x = \frac{5 + \sqrt{57}}{4}$, $x = 72.3°$
or $x = 72.3° + 180° = 252.3°$
When $\tan x = \frac{5 - \sqrt{57}}{4}$, $x = -32.5°$
or $x = -32.5° + 180° = 147.5°$
or $x = 147.5° + 180° = 327.5°$
$x = 72.3°, 147.5°, 252.3°$ or $327.5°$

38
$$5\sin^2 x = 6(1 - \cos x)$$
$$5\sin^2 x + 6\cos x - 6 = 0$$
$$5(1 - \cos^2 x) + 6\cos x - 6 = 0$$
$$5 - 5\cos^2 x + 6\cos x - 6 = 0$$
$$5\cos^2 x - 6\cos x + 1 = 0$$
$$(5\cos x - 1)(\cos x - 1) = 0$$
So $\cos x = \frac{1}{5}$ or $\cos x = 1$
When $\cos x = \frac{1}{5}$, $x = 78.5°$
or $x = 360° - 78.5° = 281.5°$
When $\cos x = 1$, $x = 0°$ or $360°$
$x = 0°, 78.5°, 281.5°$ or $360°$

39 $\text{LHS} = \cos^2 x(\tan^2 x + 1)$
$$= \cos^2 x\left(\frac{\sin^2 x}{\cos^2 x} + 1\right)$$
$$= \sin^2 x + \cos^2 x$$
$$= \text{RHS}$$

Challenge

1 a Finding points B and C using $y = 3x - 12$:
When $y = 0$, $x = 4$
When $x = 0$, $y = -12$
The point B is (4, 0) and
the point C is (0, −12).
Using Pythagoras' theorem to find the length of the square:

$$BC = \sqrt{(0-4)^2 + (-12-0)^2} = \sqrt{160}$$

Area of square $= \left(\sqrt{160}\right)^2 = 160$

b The point A is (−8, 4) and
the point D is (−12, −8).

$$\text{The gradient of line } AD = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-8-4}{-12+8} = \frac{-12}{-4} = 3$$

The equation of line AD is:
$y - y_1 = m(x - x_1)$
$y - 4 = 3(x + 8)$
$y = 3x + 28$
When $y = 0$, $x = -\frac{28}{3}$
The point S is $(-\frac{28}{3}, 0)$.

2 Rearranging $x^2 + y^2 + 8x - 10y = 59$:
$x^2 + 8x + y^2 - 10y = 59$
Completing the square:
$(x + 4)^2 - 16 + (y - 5)^2 - 25 = 59$
$(x + 4)^2 + (y - 5)^2 = 100$
Both circles have the same centre at (−4, 5). The radius of one circle is 8 and the other is 10, so
$(x + 4)^2 + (y - 5)^2 = 8^2$ lies completely inside $x^2 + y^2 + 8x - 10y = 59$.

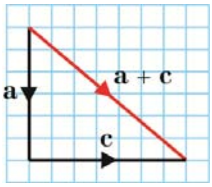
3

$$\begin{aligned}
\text{LHS} &= \binom{n}{k} + \binom{n}{k+1} \\
&= \frac{n!}{k!(n-k)!} + \frac{n!}{(k+1)!(n-k-1)!} \\
&= \frac{n!(k+1)}{(k+1)!(n-k)!} + \frac{n!(n-k)}{(k+1)!(n-k)!} \\
&= \frac{n!((k+1)+(n-k))}{(k+1)!(n-k)!} \\
&= \frac{n!(n+1)}{(k+1)!(n-k)!} \\
&= \frac{(n+1)!}{(k+1)!(n-k)!} \\
&= \binom{n+1}{k+1} \\
&= \text{RHS}
\end{aligned}$$

4 $2\sin^3 x - \sin x + 1 = \cos^2 x$
$2\sin^3 x - \sin x + 1 = 1 - \sin^2 x$
$2\sin^3 x + \sin^2 x - \sin x = 0$
$\sin x(2\sin^2 x + \sin x - 1) = 0$
$\sin x(2\sin x - 1)(\sin x + 1) = 0$
So $\sin x = 0$, $\sin x = \frac{1}{2}$ or $\sin x = -1$
When $\sin x = 0$, $x = 0°, 180°$ or $360°$
When $\sin x = \frac{1}{2}$, $x = 30°$
or $x = 180° - 30° = 150°$
When $\sin x = -1$, $x = 270°$
So $x = 0°, 30°, 150°, 180°, 270°$ or $360°$

Vectors 11A

1 a



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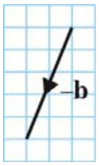
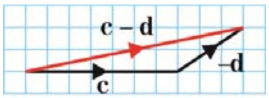


c



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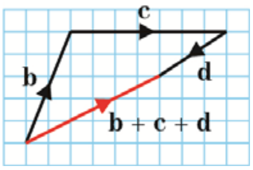
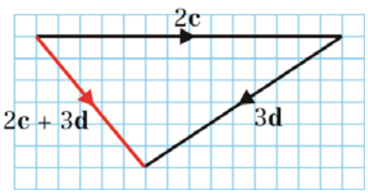
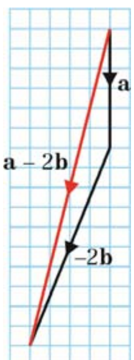


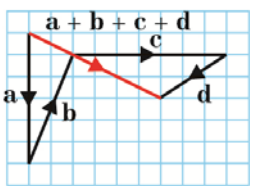
e



f



g



2 In this exercise there will usually be several correct routes to the answers because the addition law for vectors allows several options for equivalent vectors. You might reach the correct answers by a different routes to those used in these solutions.

a $\overrightarrow{AC} = \overrightarrow{AB} = 2\mathbf{b}$

b $\overrightarrow{BE} = \overrightarrow{AD}$ (parallel and equal in length)
$= \mathbf{d}$

c $\overrightarrow{HG} = \overrightarrow{BC} =$ (parallel and equal in length)
$= \overrightarrow{AB}$ (B is midpoint of $\overrightarrow{AC}$)
$= \mathbf{b}$

d $\overrightarrow{DF} = \overrightarrow{AC} =$ (parallel and equal in length)
$= 2\mathbf{b}$

e $\overrightarrow{AE} = \overrightarrow{AD} + \overrightarrow{DE}$ (triangle law of addition)
$= \overrightarrow{AD} + \overrightarrow{AB}$
($\overrightarrow{DE}$ and $\overrightarrow{AB}$ parallel and equal in length)
$= \mathbf{d} + \mathbf{b}$

f $\overrightarrow{DI} = \overrightarrow{DH} + \overrightarrow{IH}$ (triangle law of addition)
$= \overrightarrow{AD} + \overrightarrow{AB}$
($\overrightarrow{AD} = \overrightarrow{DI}$ because D is the midpoint of $\overrightarrow{AI}$, and $\overrightarrow{AB}$ is parallel and equal to $\overrightarrow{IH}$)
$= \mathbf{d} + \mathbf{b}$

g $\overrightarrow{HB} = -\overrightarrow{BH} =$ (same length, opposite direction)
$= -\overrightarrow{AI}$ (parallel and equal in length)
$= -2\mathbf{d}$

h $\overrightarrow{FE} = -\overrightarrow{EF}$ (same length, opposite direction)
$= -\overrightarrow{HG}$ (parallel and equal in length)
$= -\mathbf{b}$ (from part c)

i $\overrightarrow{AI} = \overrightarrow{AH} + \overrightarrow{IH}$ (triangle law of addition)
$= 2\mathbf{d} + \mathbf{b}$

j $\overrightarrow{BI} = \overrightarrow{BA} + \overrightarrow{AI}$

$= -\overrightarrow{AB} + \overrightarrow{AI}$

$= -\mathbf{b} + 2\mathbf{d}$

2 k $\overrightarrow{EI} = \overrightarrow{EB} + \overrightarrow{BA} + \overrightarrow{AI}$

$= -\overrightarrow{BE} - \overrightarrow{AB} - \overrightarrow{AI}$

$= -\mathbf{d} - \mathbf{b} + 2\mathbf{d}$

$= -\mathbf{b} + \mathbf{d}$

l $\overrightarrow{FB} = \overrightarrow{FD} + \overrightarrow{DA} + \overrightarrow{AB}$

$= -\overrightarrow{DF} - \overrightarrow{AD} + \overrightarrow{AB}$

$= -2\mathbf{b} - \mathbf{d} + \mathbf{b}$

$= -\mathbf{b} - \mathbf{d}$

3 a $\overrightarrow{OA} = 2\overrightarrow{OM}$ (M is the midpoint of $\overrightarrow{OA}$)

$= 2\mathbf{m}$

b $\overrightarrow{OB} = 2\overrightarrow{OP}$ (P is the mid point of $\overrightarrow{OB}$)

$= 2\mathbf{p}$

c $\overrightarrow{BN} = \frac{1}{2}\overrightarrow{BC} = \frac{1}{2}\overrightarrow{OA}$ (opposite sides parallel and equal)

$= \mathbf{m}$

d $\overrightarrow{DQ} = \overrightarrow{PD}$ ($\overrightarrow{MN}$ and $\overrightarrow{PQ}$ bisect each other)

$= \overrightarrow{OM}$ (line segments parallel and equal in length)

$= \mathbf{m}$

e $\overrightarrow{OD} = \overrightarrow{OP} + \overrightarrow{OP}$ (addition of vectors)

$= \overrightarrow{OP} + \overrightarrow{OM}$ ($\overrightarrow{PD}$ and $\overrightarrow{OM}$ are parallel and equal in length)

$= \mathbf{p} + \mathbf{m}$

f $\overrightarrow{MQ} = \overrightarrow{MO} + \overrightarrow{OP} + \overrightarrow{PQ}$ (vector addition)

$= -\overrightarrow{OM} + \overrightarrow{OP} + \overrightarrow{OA}$ ($\overrightarrow{PQ}$ and $\overrightarrow{OA}$ are parallel and equal in length)

$= -\mathbf{m} + \mathbf{p} + 2\mathbf{m}$

$= \mathbf{p} + \mathbf{m}$

g $\overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PQ}$

$= \mathbf{p} + 2\mathbf{m}$

h $\overrightarrow{AD} = \overrightarrow{AO} + \overrightarrow{OD}$ (vector addition)

$= -\overrightarrow{OA} + \overrightarrow{OD}$

$= -2\mathbf{m} + (\mathbf{p} + \mathbf{m})$

$= \mathbf{p} - \mathbf{m}$

3 i $\overrightarrow{CD} = \overrightarrow{CN} + \overrightarrow{ND}$ (vector addition)

$= \overrightarrow{MO} + \overrightarrow{PO}$ (line segments parallel and equal in length)

$= -\overrightarrow{OM} - \overrightarrow{OP}$

$= -\mathbf{m} - \mathbf{p}$

j $\overrightarrow{AP} = \overrightarrow{AO} + \overrightarrow{OP}$ (vector addition)

$= -\overrightarrow{OA} + \overrightarrow{OP}$

$= -2\mathbf{m} + \mathbf{p}$

k $\overrightarrow{BM} = \overrightarrow{BO} + \overrightarrow{OM}$ (vector addition)

$= -\overrightarrow{OB} + \overrightarrow{OM}$

$= -2\mathbf{p} + \mathbf{m}$

l $\overrightarrow{NO} = \overrightarrow{NB} + \overrightarrow{BO}$ (vector addition)

$= \overrightarrow{MO} + \overrightarrow{BO}$ ($\overrightarrow{MO}$ and $\overrightarrow{NB}$ are parallel and equal in length)

$= -\overrightarrow{OM} - \overrightarrow{OB}$

$= -\mathbf{m} - 2\mathbf{p}$

4 a $\overrightarrow{QT} = \overrightarrow{QP} + \overrightarrow{PT}$

$= -\mathbf{a} + \mathbf{d}$

$= \mathbf{d} - \mathbf{a}$

b $\overrightarrow{PR} = \overrightarrow{PQ} + \overrightarrow{QS} + \overrightarrow{SR}$

$= \mathbf{a} + \mathbf{b} + \mathbf{c}$

c $\overrightarrow{TS} = \overrightarrow{TP} + \overrightarrow{PQ} + \overrightarrow{QS}$

$= -\mathbf{d} + \mathbf{a} + \mathbf{b}$

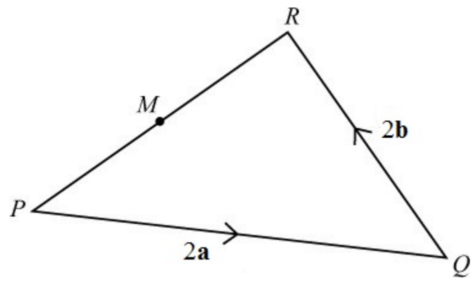
$= \mathbf{a} + \mathbf{b} - \mathbf{d}$

d $\overrightarrow{TR} = \overrightarrow{TP} + \overrightarrow{PR}$

$= -\mathbf{d} + (\mathbf{a} + \mathbf{b} + \mathbf{c})$

$= \mathbf{a} + \mathbf{b} + \mathbf{c} - \mathbf{d}$

5



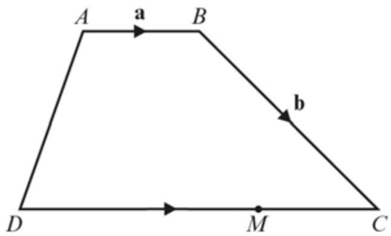
a $\overrightarrow{PR} = \overrightarrow{PQ} + \overrightarrow{QR}$

$= 2\mathbf{a} + 2\mathbf{b}$

5 b $\overrightarrow{PM} = \frac{1}{2}\overrightarrow{PR}$
$= \frac{1}{2}(2\mathbf{a} + 2\mathbf{b})$
$= \mathbf{a} + \mathbf{b}$

c $\overrightarrow{QM} = \overrightarrow{QP} + \overrightarrow{PM}$
$= -2\mathbf{a} + \mathbf{a} + \mathbf{b}$
$= -\mathbf{a} + \mathbf{b}$
$= \mathbf{b} - \mathbf{a}$

6 a



$\overrightarrow{AM} = \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CM}$
$\overrightarrow{CM} = -\mathbf{a}$
$\overrightarrow{AM} = \mathbf{a} + \mathbf{b} - \mathbf{a}$
$= \mathbf{b}$

b $\overrightarrow{BD} = \overrightarrow{BC} + \overrightarrow{CD}$
$= \mathbf{b} - 3\mathbf{a}$

c $\overrightarrow{MB} = \overrightarrow{MC} + \overrightarrow{CB}$
$= \mathbf{a} - \mathbf{b}$

d $\overrightarrow{DA} = \overrightarrow{DC} + \overrightarrow{CB} + \overrightarrow{BA}$
$= 3\mathbf{a} - \mathbf{b} - \mathbf{a}$
$= 2\mathbf{a} - \mathbf{b}$

7 a $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$
$= \mathbf{a} + \mathbf{b}$

b $\overrightarrow{OP} = \frac{5}{8}\left(\overrightarrow{OA} + \overrightarrow{AB}\right)$
$= \frac{5}{8}(\mathbf{a} + \mathbf{b})$

c $\overrightarrow{AP} = \overrightarrow{AB} + \overrightarrow{BP}$
$= \mathbf{b} - \frac{3}{8}(\mathbf{a} + \mathbf{b})$
$= \frac{5}{8}\mathbf{b} - \frac{3}{8}\mathbf{a}$

8 a $2\mathbf{a} - 6\mathbf{b} = 2(\mathbf{a} - 3\mathbf{b})$
Yes, parallel to $\mathbf{a} - 3\mathbf{b}$.

b $4\mathbf{a} - 12\mathbf{b} = 4(\mathbf{a} - 3\mathbf{b})$
Yes, parallel to $\mathbf{a} - 3\mathbf{b}$.

8 c $\mathbf{a} + 3\mathbf{b}$ is not parallel to $\mathbf{a} - 3\mathbf{b}$

d $3\mathbf{b} - \mathbf{a} = -1(\mathbf{a} - 3\mathbf{b})$
Yes, parallel to $\mathbf{a} - 3\mathbf{b}$.

e $9\mathbf{b} - 3\mathbf{a} = -3(\mathbf{a} - 3\mathbf{b})$
Yes, parallel to $\mathbf{a} - 3\mathbf{b}$.

f $\frac{1}{2}\mathbf{a} - \frac{2}{3}\mathbf{b} = \frac{1}{2}\left(\mathbf{a} - \frac{4}{3}\mathbf{b}\right)$
No, not parallel to $\mathbf{a} - 3\mathbf{b}$.

9 a i $\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AC}$
$= -\mathbf{a} + \mathbf{b}$
$= \mathbf{b} - \mathbf{a}$

ii $\overrightarrow{AP} = \frac{1}{2}\overrightarrow{AB}$
$= \frac{1}{2}\mathbf{a}$

iii $\overrightarrow{AQ} = \frac{1}{2}\overrightarrow{AC}$
$= \frac{1}{2}\mathbf{b}$

iv $\overrightarrow{PQ} = \overrightarrow{PA} + \overrightarrow{AQ}$
$= -\frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$
$= \frac{1}{2}\mathbf{b} - \frac{1}{2}\mathbf{a}$

b $\overrightarrow{PQ} = \frac{1}{2}\mathbf{b} - \frac{1}{2}\mathbf{a}$
$\overrightarrow{BC} = \mathbf{b} - \mathbf{a}$
$\mathbf{b} - \mathbf{a} = 2\left(\frac{1}{2}\mathbf{a} - \frac{1}{2}\mathbf{b}\right)$
Therefore, the vectors $\overrightarrow{PQ}$ and $\overrightarrow{BC}$ are parallel.

10 a i $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$
$= -\mathbf{a} + \mathbf{a} + 2\mathbf{b}$
$= 2\mathbf{b}$

ii $\overrightarrow{CB} = \overrightarrow{CO} + \overrightarrow{OB}$
$= -3\mathbf{b} + \mathbf{a} + 2\mathbf{b}$
$= \mathbf{a} - \mathbf{b}$

b $\overrightarrow{AB} = 2\mathbf{b}$
$\overrightarrow{OC} = 3\mathbf{b}$
$2\mathbf{b} = \frac{2}{3}(3\mathbf{b})$
Therefore, the vectors $\overrightarrow{AB}$ and $\overrightarrow{OC}$ are parallel.

11 As the vectors are parallel

$5\mathbf{a} + 3\mathbf{b} = \frac{5}{2}(2\mathbf{a} + k\mathbf{b})$

$5\mathbf{a} + 3\mathbf{b} = 5\mathbf{a} + \dfrac{5k}{2}\mathbf{b}$

$3\mathbf{b} = \dfrac{5k}{2}\mathbf{b}$

$\dfrac{5k}{2} = 3$

$k = 1.2$

Vectors 11B

1 $\mathbf{v}_1$ **i** $8\begin{pmatrix}1\\0\end{pmatrix}+0\begin{pmatrix}0\\1\end{pmatrix} = 8\mathbf{i}$

ii $8\begin{pmatrix}1\\0\end{pmatrix}+0\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}8\\0\end{pmatrix}$

$\mathbf{v}_2$ **i** $9\begin{pmatrix}1\\0\end{pmatrix}+3\begin{pmatrix}0\\1\end{pmatrix} = 9\mathbf{i} + 3\mathbf{j}$

ii $9\begin{pmatrix}1\\0\end{pmatrix}+3\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}9\\3\end{pmatrix}$

$\mathbf{v}_3$ **i** $-4\begin{pmatrix}1\\0\end{pmatrix}+2\begin{pmatrix}0\\1\end{pmatrix} = -4\mathbf{i} + 2\mathbf{j}$

ii $-4\begin{pmatrix}1\\0\end{pmatrix}+2\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-4\\2\end{pmatrix}$

$\mathbf{v}_4$ **i** $3\begin{pmatrix}1\\0\end{pmatrix}+5\begin{pmatrix}0\\1\end{pmatrix} = 3\mathbf{i} + 5\mathbf{j}$

ii $3\begin{pmatrix}1\\0\end{pmatrix}+5\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}3\\5\end{pmatrix}$

$\mathbf{v}_5$ **i** $-3\begin{pmatrix}1\\0\end{pmatrix}-2\begin{pmatrix}0\\1\end{pmatrix} = -3\mathbf{i} - 2\mathbf{j}$

ii $-3\begin{pmatrix}1\\0\end{pmatrix}-2\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-3\\-2\end{pmatrix}$

$\mathbf{v}_6$ **i** $0\begin{pmatrix}1\\0\end{pmatrix}-5\begin{pmatrix}0\\1\end{pmatrix} = -5\mathbf{j}$

ii $0\begin{pmatrix}1\\0\end{pmatrix}-5\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\-5\end{pmatrix}$

2 **a** $4\mathbf{a} = 4(2\mathbf{i} + 3\mathbf{j})$
$= 8\mathbf{i} + 12\mathbf{j}$

b $\frac{1}{2}\mathbf{a} = \frac{1}{2}(2\mathbf{i} + 3\mathbf{j})$
$= \mathbf{i} + \frac{1}{2}\mathbf{j}$

2 **c** $-\mathbf{b} = -(4\mathbf{i} - \mathbf{j})$
$= -4\mathbf{i} + \mathbf{j}$

d $2\mathbf{b} + \mathbf{a} = 2(4\mathbf{i} - \mathbf{j}) + (2\mathbf{i} + 3\mathbf{j})$
$= (8\mathbf{i} - 2\mathbf{j}) + (2\mathbf{i} + 3\mathbf{j})$
$= (8 + 2)\mathbf{i} + (-2 + 3)\mathbf{j}$
$= 10\mathbf{i} + \mathbf{j}$

e $3\mathbf{a} - 2\mathbf{b} = 3(2\mathbf{i} + 3\mathbf{j}) - 2(4\mathbf{i} - \mathbf{j})$
$= (6\mathbf{i} + 9\mathbf{j}) - (8\mathbf{i} - 2\mathbf{j})$
$= (6 - 8)\mathbf{i} + (9 + 2)\mathbf{j}$
$= -2\mathbf{i} + 11\mathbf{j}$

f $\mathbf{b} - 3\mathbf{a} = (4\mathbf{i} - \mathbf{j}) - 3(2\mathbf{i} + 3\mathbf{j})$
$= (4\mathbf{i} - \mathbf{j}) - (6\mathbf{i} + 9\mathbf{j})$
$= (4 - 6)\mathbf{i} + (-1 - 9)\mathbf{j}$
$= -2\mathbf{i} - 10\mathbf{j}$

g $4\mathbf{b} - \mathbf{a} = 4(4\mathbf{i} - \mathbf{j}) - (2\mathbf{i} + 3\mathbf{j})$
$= (16\mathbf{i} - 4\mathbf{j}) - (2\mathbf{i} + 3\mathbf{j})$
$= (16 - 2)\mathbf{i} + (-4 - 3)\mathbf{j}$
$= 14\mathbf{i} - 7\mathbf{j}$

h $2\mathbf{a} - 3\mathbf{b} = 2(2\mathbf{i} + 3\mathbf{j}) - 3(4\mathbf{i} - \mathbf{j})$
$= (4\mathbf{i} + 6\mathbf{j}) - (12\mathbf{i} - 3\mathbf{j})$
$= (4 - 12)\mathbf{i} + (6 + 3)\mathbf{j}$
$= -8\mathbf{i} + 9\mathbf{j}$

3 **a** $5\mathbf{a} = 5\begin{pmatrix}9\\7\end{pmatrix}$
$= \begin{pmatrix}45\\35\end{pmatrix}$

b $-\frac{1}{2}\mathbf{c} = -\frac{1}{2}\begin{pmatrix}-8\\-1\end{pmatrix}$
$= \begin{pmatrix}4\\0.5\end{pmatrix}$

c $\mathbf{a} + \mathbf{b} + \mathbf{c} = \begin{pmatrix}9\\7\end{pmatrix}+\begin{pmatrix}11\\-3\end{pmatrix}+\begin{pmatrix}-8\\-1\end{pmatrix}$
$= \begin{pmatrix}12\\3\end{pmatrix}$

d $2\mathbf{a} - \mathbf{b} + \mathbf{c} = 2\begin{pmatrix}9\\7\end{pmatrix}-\begin{pmatrix}11\\-3\end{pmatrix}+\begin{pmatrix}-8\\-1\end{pmatrix}$
$= \begin{pmatrix}-1\\16\end{pmatrix}$

3 e $2\mathbf{b} + 2\mathbf{c} - 3\mathbf{a} = 2\begin{pmatrix}11\\-3\end{pmatrix} + 2\begin{pmatrix}-8\\-1\end{pmatrix} - 3\begin{pmatrix}9\\7\end{pmatrix}$

$= \begin{pmatrix}-21\\-29\end{pmatrix}$

f $\frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b} = \frac{1}{2}\begin{pmatrix}9\\7\end{pmatrix} + \frac{1}{2}\begin{pmatrix}11\\-3\end{pmatrix}$

$= \begin{pmatrix}10\\2\end{pmatrix}$

4 a $\mathbf{a} + \lambda\mathbf{b} = (2\mathbf{i} + 5\mathbf{j}) + \lambda(3\mathbf{i} - \mathbf{j})$

$= (2 + 3\lambda)\mathbf{i} + (5 - \lambda)\mathbf{j}$

Parallel to $\mathbf{i}$, so $5 - \lambda = 0$, $\lambda = 5$.

b $\mu\mathbf{a} + \mathbf{b} = \mu(2\mathbf{i} + 5\mathbf{j}) + (3\mathbf{i} - \mathbf{j})$

$= (2\mu + 3)\mathbf{i} + (5\mu - 1)\mathbf{j}$

Parallel to $\mathbf{j}$, so $2\mu + 3 = 0$, $\mu = -\frac{3}{2}$

5 a $\mathbf{c} + \lambda\mathbf{d} = (3\mathbf{i} + 4\mathbf{j}) + \lambda(\mathbf{i} - 2\mathbf{j})$

$= (3 + \lambda)\mathbf{i} + (4 - 2\lambda)\mathbf{j}$

Parallel to $\mathbf{i} + \mathbf{j}$, so $3 + \lambda = 4 - 2\lambda$

$3\lambda = 1$, $\lambda = \frac{1}{3}$

b $\mu\mathbf{c} + \mathbf{d} = \mu(3\mathbf{i} + 4\mathbf{j}) + (\mathbf{i} - 2\mathbf{j})$

$= (3\mu + 1)\mathbf{i} + (4\mu - 2)\mathbf{j}$

Parallel to $\mathbf{i} + 3\mathbf{j}$, so $4\mu - 2 = 3(3\mu + 1)$

$4\mu - 2 = 9\mu + 3$

$5\mu = -5$, $\mu = -1$

c $\mathbf{c} - s\mathbf{d} = (3\mathbf{i} + 4\mathbf{j}) - s(\mathbf{i} - 2\mathbf{j})$

$= (3 - s)\mathbf{i} + (4 + 2s)\mathbf{j}$

Parallel to $2\mathbf{i} + \mathbf{j}$, so

$3 - s = 2(4 + 2s)$

$3 - \mathbf{s} = 8 + 4\mathrm{i}$

$-5 = 5s$, $s = -1$

d $\mathbf{d} - t\mathbf{c} = (\mathbf{i} - 2\mathbf{j}) - t(3\mathbf{i} + 4\mathbf{j})$

$= (1 - 3t)\,\mathbf{i} + (-2 - 4t)\mathbf{j}$

Parallel to $-2\mathbf{i} + 3\mathbf{j}$, so

$-2(-2 - 4t) = 3(1 - 3t)$

$4 + 8t = 3 - 9t$

$1 = -17t$, $t = -\frac{1}{17}$

6 $\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AC}$

$= -(4\mathbf{i} + 3\mathbf{j}) + 5\mathbf{i} + 2\mathbf{j}$

$= \mathbf{i} - \mathbf{j}$

7 a $\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC}$

$= -(2\mathbf{i} + 4\mathbf{j}) + 7\mathbf{i}$

$= 5\mathbf{i} - 4\mathbf{j}$

$= 5\begin{pmatrix}1\\0\end{pmatrix} - 4\begin{pmatrix}0\\1\end{pmatrix}$

$= \begin{pmatrix}5\\-4\end{pmatrix}$

b $\overrightarrow{AP} = \frac{3}{5}\overrightarrow{AC}$

$= \frac{3}{5}(5\mathbf{i} - 4\mathbf{j})$

$= 3\mathbf{i} - \frac{12}{5}\mathbf{j}$

$= 3\begin{pmatrix}1\\0\end{pmatrix} - \frac{12}{5}\begin{pmatrix}0\\1\end{pmatrix}$

$= \begin{pmatrix}3\\-\frac{12}{5}\end{pmatrix}$

c $\overrightarrow{OP} = \overrightarrow{OA} + \frac{3}{5}\overrightarrow{AC}$

$= 2\mathbf{i} + 4\mathbf{j} + \frac{3}{5}(5\mathbf{i} - 4\mathbf{j})$

$= 5\mathbf{i} + \frac{8}{5}\mathbf{j}$

$= 5\begin{pmatrix}1\\0\end{pmatrix} + \frac{8}{5}\begin{pmatrix}0\\1\end{pmatrix}$

$= \begin{pmatrix}5\\\frac{8}{5}\end{pmatrix}$

8 $\begin{pmatrix}10\\k\end{pmatrix} - 2\begin{pmatrix}j\\3\end{pmatrix} = \begin{pmatrix}2\\5\end{pmatrix}$

So $10 - 2j = 2$ and $k - 6 = 5$

$j = 4$ and $k = 11$

9 $\begin{pmatrix}p\\-q\end{pmatrix} + 2\begin{pmatrix}q\\p\end{pmatrix} = \begin{pmatrix}7\\4\end{pmatrix}$

So $p + 2q = 7$ **(1)**

and $-q + 2p = 4$ **(2)**

Solve the two equations simultaneously.

Multiply equation **(2)** by 2 to give

$-2q + 4p = 8$ **(3)**

Add equations **(1)** and **(3)**

$5p = 15$

$p = 3$ and $q = 2$

10 a The resultant vector

$\mathbf{a} + \mathbf{b} = 3\mathbf{i} - 2\mathbf{j} + p\mathbf{i} - 2p\mathbf{j}$

$\quad = (3 + p)\mathbf{i} - (2 + 2p)\mathbf{j}$

$(3 + p)\mathbf{i} - (2 + 2p)\mathbf{j}$ is parallel to $2\mathbf{i} - 3\mathbf{j}$

so $(3 + p)\mathbf{i} - (2 + 2p)\mathbf{j} = \lambda(2\mathbf{i} - 3\mathbf{j})$

so $3 + p = 2\lambda$ **(1)**

and $2 + 2p = 3\lambda$ **(2)**

Solve the two equations simultaneously.

Multiply equation **(1)** by 2 to give

$6 + 2p = 4\lambda$

$\lambda = 4$

$p = 5$

b When $p = 5$

$(3 + p)\mathbf{i} - (2 + 2p)\mathbf{j} = 8\mathbf{i} - 12\mathbf{j}$

Vectors 11C

1 a $|3\mathbf{i}+4\mathbf{j}| = \sqrt{3^2+4^2}$
$= \sqrt{9+16}$
$= \sqrt{25}$
$= 5$

b $|6\mathbf{i}-8\mathbf{j}| = \sqrt{6^2+8^2}$
$= \sqrt{36+64}$
$= \sqrt{100}$
$= 10$

c $|5\mathbf{i}+12\mathbf{j}| = \sqrt{5^2+12^2}$
$= \sqrt{25+144}$
$= \sqrt{169}$
$= 13$

d $|2\mathbf{i}+4\mathbf{j}| = \sqrt{2^2+4^2}$
$= \sqrt{4+16}$
$= \sqrt{20}$
$= 4.47\ (3\text{ s.f.})$

e $|3\mathbf{i}-5\mathbf{j}| = \sqrt{3^2+5^2}$
$= \sqrt{9+25}$
$= \sqrt{34}$
$= 5.83\ (3\text{ s.f.})$

f $|4\mathbf{i}+7\mathbf{j}| = \sqrt{4^2+7^2}$
$= \sqrt{16+49}$
$= \sqrt{65}$
$= 8.06\ (3\text{ s.f.})$

g $|-3\mathbf{i}+5\mathbf{j}| = \sqrt{3^2+5^2}$
$= \sqrt{9+25}$
$= \sqrt{34}$
$= 5.83\ (3\text{ s.f.})$

h $|-4\mathbf{i}+-\mathbf{j}| = \sqrt{4^2+1^2}$
$= \sqrt{16+1}$
$= \sqrt{17}$
$= 4.12\ (3\text{ s.f.})$

2 a $\mathbf{a} = \begin{pmatrix}2\\3\end{pmatrix}$, $\mathbf{b} = \begin{pmatrix}3\\-4\end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix}5\\-1\end{pmatrix}$

$\mathbf{a}+\mathbf{b} = \begin{pmatrix}2\\3\end{pmatrix} + \begin{pmatrix}3\\-4\end{pmatrix}$
$= \begin{pmatrix}5\\-1\end{pmatrix}$
$|\mathbf{a}+\mathbf{b}| = \sqrt{5^2+(-1)^2}$
$= \sqrt{26}$

b $2\mathbf{a}-\mathbf{c} = 2\begin{pmatrix}2\\3\end{pmatrix} - \begin{pmatrix}5\\-1\end{pmatrix}$
$= \begin{pmatrix}-1\\7\end{pmatrix}$
$|2\mathbf{a}-\mathbf{c}| = \sqrt{(-1)^2+7^2}$
$= \sqrt{50}$
$= 5\sqrt{2}$

c $3\mathbf{b}-2\mathbf{c} = 3\begin{pmatrix}3\\-4\end{pmatrix} - 2\begin{pmatrix}5\\-1\end{pmatrix}$
$= \begin{pmatrix}-1\\-10\end{pmatrix}$
$|3\mathbf{b}-2\mathbf{c}| = \sqrt{(-1)^2+(-10)^2}$
$= \sqrt{101}$

3 a a unit vector is $\dfrac{\mathbf{a}}{|\mathbf{a}|}$

$\mathbf{a} = \begin{pmatrix}4\\3\end{pmatrix}$
$|\mathbf{a}| = \sqrt{4^2+3^2}$
$= \sqrt{25}$
$= 5$
$\dfrac{\mathbf{a}}{|\mathbf{a}|} = \dfrac{1}{5}\begin{pmatrix}4\\3\end{pmatrix} = \begin{pmatrix}0.8\\0.6\end{pmatrix}$

3 b a unit vector is $\frac{\mathbf{b}}{|\mathbf{b}|}$

$$\mathbf{b} = \begin{pmatrix} 5 \\ -12 \end{pmatrix}$$

$$|\mathbf{b}| = \sqrt{5^2 + (-12)^2} = \sqrt{169} = 13$$

$$\frac{\mathbf{b}}{|\mathbf{b}|} = \frac{1}{13}\begin{pmatrix} 5 \\ -12 \end{pmatrix} = \begin{pmatrix} \frac{5}{13} \\ -\frac{12}{13} \end{pmatrix}$$

c a unit vector is $\frac{\mathbf{c}}{|\mathbf{c}|}$

$$\mathbf{c} = \begin{pmatrix} -7 \\ 24 \end{pmatrix}$$

$$|\mathbf{c}| = \sqrt{(-7)^2 + 24^2} = \sqrt{625} = 25$$

$$\frac{\mathbf{c}}{|\mathbf{c}|} = \frac{1}{25}\begin{pmatrix} -7 \\ 24 \end{pmatrix} = \begin{pmatrix} -0.28 \\ 0.96 \end{pmatrix}$$

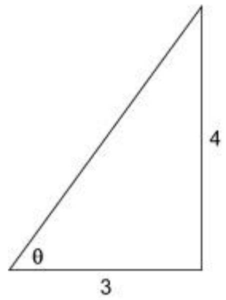
d a unit vector is $\frac{\mathbf{d}}{|\mathbf{d}|}$

$$\mathbf{d} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

$$|\mathbf{d}| = \sqrt{1^2 + (-3)^2} = \sqrt{10}$$

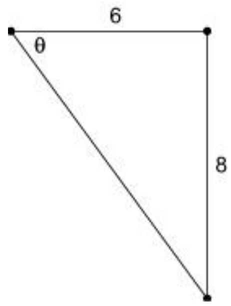
$$\frac{\mathbf{d}}{|\mathbf{d}|} = \frac{1}{\sqrt{10}}\begin{pmatrix} 1 \\ -3 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{10}}{10} \\ -\frac{3\sqrt{10}}{10} \end{pmatrix}$$

4 a $3\mathbf{i} + 4\mathbf{j}$



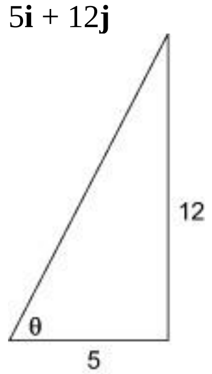
$\tan^{-1}\left(\frac{4}{3}\right) = 53.1°$ above (3 s.f.)

b $6\mathbf{i} - 8\mathbf{j}$



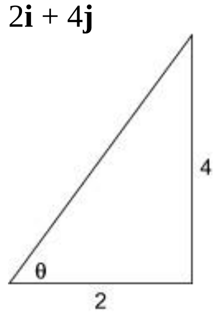
$\tan^{-1}\left(\frac{8}{6}\right) = 53.1°$ below (3 s.f.)

c $5\mathbf{i} + 12\mathbf{j}$



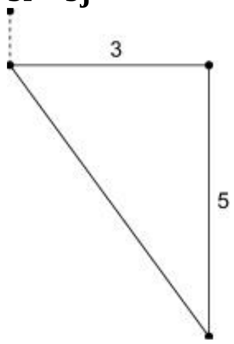
$\tan^{-1}\left(\frac{12}{5}\right) = 67.4°$ above (3 s.f.)

d $2\mathbf{i} + 4\mathbf{j}$



$\tan^{-1}\left(\frac{4}{2}\right) = 63.4°$ above (3 s.f.)

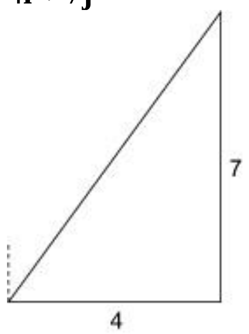
5 a $3\mathbf{i} - 5\mathbf{j}$



$90° + \tan^{-1}\left(\frac{5}{3}\right) = 90° + 59°$

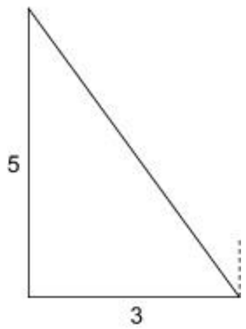
$= 149° \text{ (3 s.f.) to the right}$

b $4\mathbf{i} + 7\mathbf{j}$



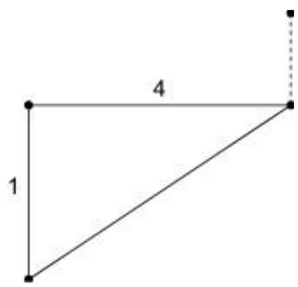
$\tan^{-1}\left(\frac{4}{7}\right) = 29.7° \text{ (3 s.f.) to the right}$

c $-3\mathbf{i} + 5\mathbf{j}$



$\tan^{-1}\left(\frac{3}{5}\right) = 31.0° \text{ (3 s.f.) to the left}$

d $-4\mathbf{i} - \mathbf{j}$



5 d $90° + \tan^{-1}\left(\frac{1}{4}\right) = 90° + 14°$

$= 104° \text{ (3 s.f.) to the left}$

6 a $\cos 45° = \dfrac{x}{15}$

$x = 15\cos 45°$

$= \dfrac{15\sqrt{2}}{2}$

$\sin 45° = \dfrac{y}{15}$

$y = 15\sin 45°$

$= \dfrac{15\sqrt{2}}{2}$

The vector is $\dfrac{15\sqrt{2}}{2}\mathbf{i} + \dfrac{15\sqrt{2}}{2}\mathbf{j}$

or $\begin{pmatrix} \dfrac{15\sqrt{2}}{2} \\ \dfrac{15\sqrt{2}}{2} \end{pmatrix}$

b $\cos 20° = \dfrac{x}{8}$

$x = 8\cos 20°$

$= 7.52$

$\sin 20° = \dfrac{y}{8}$

$y = 8\sin 20°$

$= 2.74$

The vector is $7.52\mathbf{i} + 2.74\mathbf{j}$

or $\begin{pmatrix} 7.52 \\ 2.74 \end{pmatrix}$

c $\cos 25° = \dfrac{x}{20}$

$x = 20\cos 25°$

$= 18.1$

$\sin 25° = \dfrac{y}{20}$

$y = 20\sin 25°$

$= 8.45$

The vector is $18.1\mathbf{i} - 8.45\mathbf{j}$

or $\begin{pmatrix} 18.1 \\ -8.45 \end{pmatrix}$

6 d $\cos 30° = \dfrac{x}{5}$

$x = 5\cos 30°$

$= \dfrac{5\sqrt{3}}{2}$

$\sin 30° = \dfrac{y}{5}$

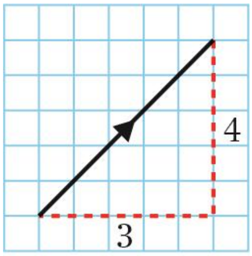
$y = 5\sin 30°$

$= 2.5$

The vector is $\dfrac{5\sqrt{3}}{2}\mathbf{i} - 2.5\mathbf{j}$

or $\begin{pmatrix} \dfrac{5\sqrt{3}}{2} \\ -2.5 \end{pmatrix}$

7 a



magnitude $= \sqrt{3^2 + 4^2}$

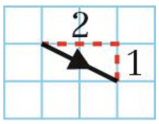
$= \sqrt{25} = 5$

$\tan\theta = \frac{4}{3}$

$\theta = \tan^{-1}\frac{4}{3}$

$= 53.1°$ above the positive x-axis

b



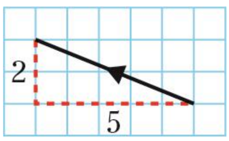
magnitude $= \sqrt{2^2 + (-1)^2} = \sqrt{5}$

$\tan\theta = \frac{1}{2}$

$\theta = \tan^{-1}\left(\frac{1}{2}\right)$

$= 26.6°$ below the positive x-axis

c



magnitude $= \sqrt{(-5)^2 + 2^2} = \sqrt{29}$

$\tan\theta = \frac{2}{5}$

7 c $\theta = \tan^{-1}\left(\frac{2}{5}\right)$

$= 21.8°$ above the negative x-axis

$= 158.2°$ above the positive x-axis

8 $|2\mathbf{i} - k\mathbf{j}| = \sqrt{2^2 + (-k)^2} = \sqrt{4 + k^2}$

$\sqrt{4 + k^2} = 2\sqrt{10} = \sqrt{40}$

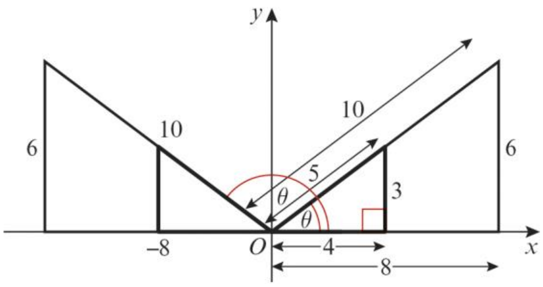
$4 + k^2 = 40$

$k^2 = 36$

$k = \pm 6$

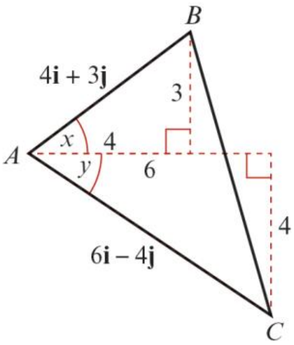
9 $|p\mathbf{i} + q\mathbf{j}| = 10$

Adding the information and using Pythagoras' theorem



$p = \pm 8$ and $q = 6$

10



a $\tan x = \frac{3}{4}$

$x = \tan^{-1}\frac{3}{4}$

$= 36.8699° = 36.9°$ (3 s.f.)

b $\tan y = \frac{2}{3}$

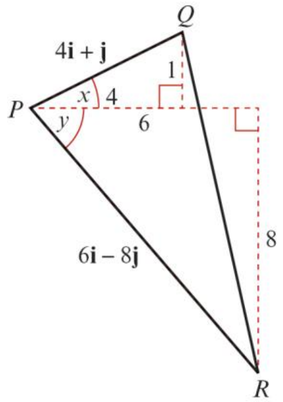
$y = \tan^{-1}\frac{2}{3}$

$= 33.6901° = 33.7°$ (3 s.f.)

c Angle $BAC = x + y$

$= 70.6°$ (3 s.f.)

11



a Angle $QPR = x + y$

$\tan x = \frac{1}{4}$

$x = \tan^{-1} \frac{1}{4}$

$= 14.0362...$

$\tan y = \frac{4}{3}$

$y = \tan^{-1} \frac{4}{3}$

$= 53.1301...$

Angle $QPR = 67.2°$ (1 d.p.)

b Area $= \frac{1}{2} rq \sin P$

$r = \sqrt{4^2 + 1^2}$

$= \sqrt{17}$

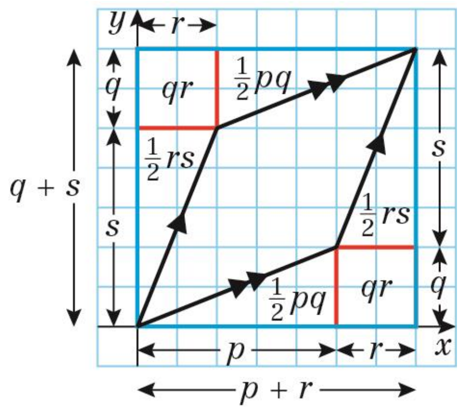
$q = \sqrt{6^2 + 8^2}$

$= \sqrt{100} = 10$

Area $= \frac{1}{2} \times \sqrt{17} \times 10 \times \sin 67.2°$

$= 19.0$ units^2 (3 s.f.)

Challenge



Area of parallelogram

= area of large blue rectangle − 2(area of small red rectangle) − 2(area of triangle 1) − 2(area of triangle 2)

$= (p + r)(q + s) - 2(qr) - 2(\frac{1}{2}pq) - 2(\frac{1}{2}rs)$

$= pq + ps + qr + rs - 2qr - pq - rs$

$= ps - qr$

Vectors 11D

1 a i $\overrightarrow{OA} = 3\mathbf{i} - \mathbf{j}$, $\overrightarrow{OB} = 4\mathbf{i} + 5\mathbf{j}$,
$\overrightarrow{OC} = -2\mathbf{i} + 6\mathbf{j}$

ii $\overrightarrow{AB} = B - A$
$= (4\mathbf{i} + 5\mathbf{j}) - (3\mathbf{i} - \mathbf{j})$
$= 4\mathbf{i} + 5\mathbf{j} - 3\mathbf{i} + \mathbf{j}$
$= \mathbf{i} + 6\mathbf{j}$

iii $\overrightarrow{AC} = C - A$
$= (-2\mathbf{i} + 6\mathbf{j}) - (3\mathbf{i} - \mathbf{j})$
$= -2\mathbf{i} + 6\mathbf{j} - 3\mathbf{i} + \mathbf{j}$
$= -5\mathbf{i} + 7\mathbf{j}$

b i $|\overrightarrow{OC}| = |-2\mathbf{i} + 6\mathbf{j}|$
$= \sqrt{(-2)^2 + 6^2}$
$= \sqrt{40}$
$= \sqrt{4}\sqrt{10}$
$= 2\sqrt{10}$

ii $|\overrightarrow{AB}| = |\mathbf{i} + 6\mathbf{j}|$
$= \sqrt{1^2 + 6^2}$
$= \sqrt{37}$

iii $|\overrightarrow{AC}| = |-5\mathbf{i} + 7\mathbf{j}|$
$= \sqrt{(-5)^2 + 7^2}$
$= \sqrt{74}$

2 a $\overrightarrow{PQ} = -\overrightarrow{OP} + \overrightarrow{OQ}$
$= -(4\mathbf{i} - 3\mathbf{j}) + 3\mathbf{i} + 2\mathbf{j}$
$= -\mathbf{i} + 5\mathbf{j}$
or
$-\begin{pmatrix} 4 \\ -3 \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$

b i $|\overrightarrow{OP}| = \sqrt{4^2 + (-3)^2}$
$= \sqrt{25} = 5$

ii $|\overrightarrow{OQ}| = \sqrt{3^2 + 2^2}$
$= \sqrt{13}$

2 b iii $|\overrightarrow{PQ}| = \sqrt{(-1)^2 + 5^2}$
$= \sqrt{26}$

3 a $\overrightarrow{PQ} = -\overrightarrow{OP} + \overrightarrow{OQ}$
$= 5\mathbf{i} + 6\mathbf{j}$
$= -\overrightarrow{OP} + (4\mathbf{i} - 3\mathbf{j})$

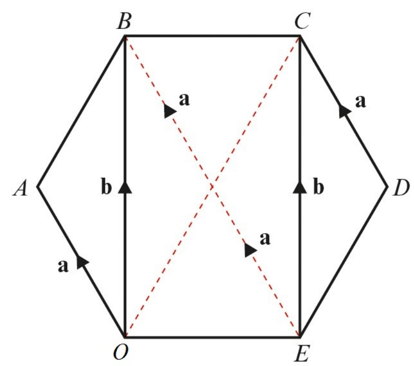
$-\overrightarrow{OP} = 5\mathbf{i} + 6\mathbf{j} - (4\mathbf{i} - 3\mathbf{j})$
$= \mathbf{i} + 9\mathbf{j}$
$\overrightarrow{OP} = -\mathbf{i} - 9\mathbf{j}$
or
$\overrightarrow{OP} = \begin{pmatrix} -1 \\ -9 \end{pmatrix}$

b i $|\overrightarrow{OP}| = \sqrt{(-1)^2 + (-9)^2}$
$= \sqrt{82}$

ii $|\overrightarrow{OQ}| = \sqrt{4^2 + (-3)^2}$
$= \sqrt{25}$
$= 5$

iii $|\overrightarrow{PQ}| = \sqrt{5^2 + 6^2}$
$= \sqrt{61}$

4

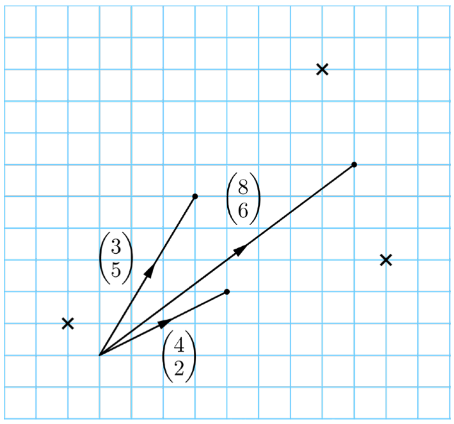


a $\overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{BE} + \overrightarrow{EC}$
$= \mathbf{b} - 2\mathbf{a} + \mathbf{b}$
$= -2\mathbf{a} + 2\mathbf{b}$

b $\overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{CD}$
$= -2\mathbf{a} + 2\mathbf{b} - \mathbf{a}$
$= -3\mathbf{a} + 2\mathbf{b}$

4 c $\overrightarrow{OE} = \overrightarrow{OB} + \overrightarrow{BE}$
$= \mathbf{b} - 2\mathbf{a}$
$= -2\mathbf{a} + \mathbf{b}$

5 The sketch shows the three possible positions of the fourth vertex.



$$\begin{pmatrix}4\\2\end{pmatrix} - \begin{pmatrix}3\\5\end{pmatrix} = \begin{pmatrix}1\\-3\end{pmatrix}$$
$$\begin{pmatrix}8\\6\end{pmatrix} + \begin{pmatrix}1\\-3\end{pmatrix} = \begin{pmatrix}9\\3\end{pmatrix}$$
or
$$-\begin{pmatrix}4\\2\end{pmatrix} + \begin{pmatrix}3\\5\end{pmatrix} = \begin{pmatrix}-1\\3\end{pmatrix}$$
$$\begin{pmatrix}8\\6\end{pmatrix} + \begin{pmatrix}-1\\3\end{pmatrix} = \begin{pmatrix}7\\9\end{pmatrix}$$
or
$$\begin{pmatrix}3\\5\end{pmatrix} - \begin{pmatrix}8\\6\end{pmatrix} = \begin{pmatrix}-5\\-1\end{pmatrix}$$
$$\begin{pmatrix}4\\2\end{pmatrix} + \begin{pmatrix}-5\\-1\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix}$$

So $\begin{pmatrix}9\\3\end{pmatrix}$ or $\begin{pmatrix}7\\9\end{pmatrix}$ or $\begin{pmatrix}-1\\1\end{pmatrix}$

6 a $\overrightarrow{AB} = -(4\mathbf{i} - 5\mathbf{j}) + 6\mathbf{i} + 3\mathbf{j}$
$= 2\mathbf{i} + 8\mathbf{j}$

b $|\overrightarrow{AB}| = \sqrt{2^2 + 8^2}$
$= \sqrt{68}$
$= 2\sqrt{17}$

7 Using the radius of the circle
$|\overrightarrow{OA}| = 3$
Using the position vector
$|\overrightarrow{OA}| = \sqrt{4k^2 + k^2}$
$= \sqrt{5k^2}$
$= \sqrt{5}k$
$\sqrt{5}k = 3$
$|k| = \dfrac{3}{\pm\sqrt{5}}$
Rationalising the denominator
$k = \pm\dfrac{3\sqrt{5}}{5}$

Challenge

Using Pythagoras' theorem
$x^2 + y^2 = 13$
Solve the equations simultaneously.
Substitute $y = 6 - \frac{3}{2}x$ into $x^2 + y^2 = 13$:
$x^2 + (6 - \frac{3}{2}x)^2 = 13$
$x^2 + 36 - 18x + \frac{9}{4}x^2 - 13 = 0$
$13x^2 - 72x + 92 = 0$
$(13x - 46)(x - 2) = 0$
$x = \frac{46}{13}$ or $x = 2$
When $x = \frac{46}{13}$, $y = \frac{9}{13}$
When $x = 2$, $y = 3$
$\overrightarrow{OB} = \frac{46}{13}\mathbf{i} + \frac{9}{13}\mathbf{j}$ or
$\overrightarrow{OB} = 2\mathbf{i} + 3\mathbf{j}$

Vectors 11E

1 $\overrightarrow{XY} = \overrightarrow{XW} + \overrightarrow{WY} = \mathbf{b} - \mathbf{a}$

$\overrightarrow{YZ} = \overrightarrow{YW} + \overrightarrow{WZ} = \mathbf{c} - \mathbf{b}$

Since $\overrightarrow{XY} = \overrightarrow{YZ}$:

$\mathbf{b} - \mathbf{a} = \mathbf{c} - \mathbf{b}$

$\mathbf{b} + \mathbf{b} = \mathbf{a} + \mathbf{c}$

$\mathbf{a} + \mathbf{c} = 2\mathbf{b}$

2 a i $\overrightarrow{OB} = 2\overrightarrow{OR}$

$= 2\mathbf{r}$

ii $\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ}$ (addition of vectors)

$= -\overrightarrow{OP} + \overrightarrow{OQ}$

$\overrightarrow{OQ} = \overrightarrow{OA} + \overrightarrow{AQ}$ (addition of vectors)

$\overrightarrow{AQ} = \frac{1}{2}\overrightarrow{AB}$

$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$ (addition of vectors)

$= -\overrightarrow{OA} + \overrightarrow{OB}$

$= -2\mathbf{p} + 2\mathbf{r}$

$\therefore \overrightarrow{AQ} = \frac{1}{2}(-2\mathbf{p} + 2\mathbf{r})$

$= -\mathbf{p} + \mathbf{r}$

$\therefore \overrightarrow{OQ} = 2\mathbf{p} + (-\mathbf{p} + \mathbf{r})$

$= \mathbf{p} + \mathbf{r}$

$\therefore \overrightarrow{PQ} = -\mathbf{p} + (\mathbf{p} + \mathbf{r})$

$= \mathbf{r}$

b $\overrightarrow{OB} = 2\mathbf{r}$ and $\overrightarrow{PQ} = \mathbf{r}$

$\Rightarrow \overrightarrow{OB}$ and $\overrightarrow{PQ}$ are parallel.

$\Rightarrow \angle AOB = \angle APQ$ and $\angle ABO = \angle AQP$

(corresponding angles, parallel lines).

Angle A is common to both triangles.

$\Rightarrow \angle PAQ$ and $\angle OAB$ are similar (three equal angles)

3 a M divides OA in the ratio $2:1$.

$\Rightarrow \overrightarrow{OM} = \frac{2}{3}\mathbf{a}$

Using vector addition:

$\overrightarrow{ON} = \overrightarrow{OA} + \overrightarrow{AN}$

$\overrightarrow{AN} = \lambda\overrightarrow{AB}$ (N lies on AB, so $\overrightarrow{AN} = \lambda\overrightarrow{AB}$)

$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$

$\overrightarrow{ON} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$

3 $\overrightarrow{ON} = \overrightarrow{OM} + \overrightarrow{MN}$

$= \overrightarrow{OM} + \mu\overrightarrow{OB}$ ($\overrightarrow{MN}$ is parallel to $\overrightarrow{OB}$)

$= \frac{2}{3}\mathbf{a} + \mu\mathbf{b}$

But $\overrightarrow{ON} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$

So:

$\mathbf{a} + \lambda(\mathbf{b} - \mathbf{a}) = \frac{2}{3}\mathbf{a} + \mu\mathbf{b}$

$\mathbf{a}(1-\lambda) + \lambda\mathbf{b} = \frac{2}{3}\mathbf{a} + \mu\mathbf{b}$

$\Rightarrow$ (comparing coefficients of **a** and **b**):

$1 - \lambda = \frac{2}{3}$ and $\lambda = \mu$

so $\lambda = \mu = \frac{1}{3}$ and $\overrightarrow{ON} = \frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}$

b $\overrightarrow{AN} = \frac{1}{3}(\mathbf{b} - \mathbf{a})$

$\Rightarrow \overrightarrow{NB} = \frac{2}{3}(\mathbf{b} - \mathbf{a})$

$\Rightarrow AN : NB = 1:2$

4 a M is the mid-point of OA, so:

$\overrightarrow{OM} = \frac{1}{2}\overrightarrow{OA}$

$= \frac{1}{2}\mathbf{a}$

Using vector addition:

$\overrightarrow{MQ} = \overrightarrow{MA} + \overrightarrow{AB} + \overrightarrow{BQ}$

$= \overrightarrow{MA} + \overrightarrow{AB} + \frac{1}{4}\overrightarrow{BC}$

$= \frac{1}{2}\mathbf{a} + \mathbf{c} - \frac{1}{4}\mathbf{a}$

$= \frac{1}{4}\mathbf{a} + \mathbf{c}$

and:

$\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC}$

$= -\mathbf{a} + \mathbf{c}$

$= \mathbf{c} - \mathbf{a}$

P lies on AC and MQ, so:

$\overrightarrow{OP} = \overrightarrow{OM} + \lambda\overrightarrow{MQ}$

$= \frac{1}{2}\mathbf{a} + \lambda\left(\frac{1}{4}\mathbf{a} + \mathbf{c}\right)$

and $\overrightarrow{OP} = \overrightarrow{OA} + \mu\overrightarrow{AC}$

$= \mathbf{a} + \mu(\mathbf{c} - \mathbf{a})$

Comparing coefficients of **a** and **c**:

$\frac{1}{2} + \frac{1}{4}\lambda = 1 - \mu$ and $\lambda = \mu$

$\Rightarrow \frac{5}{4}\lambda = \frac{1}{2}$

$\lambda = \mu = \frac{2}{5}$

$\overrightarrow{OP} = \frac{3}{5}\mathbf{a} + \frac{2}{5}\mathbf{c}$

4 b $\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA}$

$= \frac{3}{5}\mathbf{a} + \frac{2}{5}\mathbf{c} - \mathbf{a}$

$= -\frac{2}{5}\mathbf{a} + \frac{2}{5}\mathbf{c} = \frac{2}{5}(\mathbf{c} - \mathbf{a})$

$\overrightarrow{AC} = \frac{3}{5}(\mathbf{c} - \mathbf{a})$

$\Rightarrow AP{:}PC = 2{:}3$

5 a $\overrightarrow{AB} = -\overrightarrow{OA} + \overrightarrow{OB}$

$= -\begin{pmatrix} 5 \\ 8 \end{pmatrix} + \begin{pmatrix} 4 \\ 3 \end{pmatrix}$

$= \begin{pmatrix} -1 \\ -5 \end{pmatrix}$

$|\overrightarrow{AB}| = \sqrt{(-1)^2 + (-5)^2}$

$= \sqrt{26}$

b $\overrightarrow{AC} = -\overrightarrow{OA} + \overrightarrow{OC}$

$= -\begin{pmatrix} 5 \\ 8 \end{pmatrix} + \begin{pmatrix} 7 \\ 6 \end{pmatrix}$

$= \begin{pmatrix} 2 \\ -2 \end{pmatrix}$

$|\overrightarrow{AC}| = \sqrt{2^2 + (-2)^2}$

$= \sqrt{8}$

$= 2\sqrt{2}$

c $\overrightarrow{BC} = -\overrightarrow{OB} + \overrightarrow{OC}$

$= -\begin{pmatrix} 4 \\ 3 \end{pmatrix} + \begin{pmatrix} 7 \\ 6 \end{pmatrix}$

$= \begin{pmatrix} 3 \\ 3 \end{pmatrix}$

$|\overrightarrow{BC}| = \sqrt{3^2 + 3^2}$

$= \sqrt{18}$

$= 3\sqrt{2}$

5 d Using the cosine rule:

$\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

$\cos A = \dfrac{(2\sqrt{2})^2 + (\sqrt{26})^2 - (3\sqrt{2})^2}{2(2\sqrt{2})(\sqrt{26})}$

$\cos A = \dfrac{8 + 26 - 18}{4\sqrt{52}}$

$\cos A = \dfrac{16}{8\sqrt{13}}$

$\cos A = \dfrac{2}{\sqrt{13}}$

$A = 56.3...°$

Using the sine rule:

$\dfrac{\sin B}{b} = \dfrac{\sin A}{a}$

$\dfrac{\sin B}{2\sqrt{2}} = \dfrac{\sin 56.3°}{3\sqrt{2}}$

$\sin B = \dfrac{2\sqrt{2}\sin 56.3°}{3\sqrt{2}}$

$B = 33.68...°$

$C = 180° - 56° - 34° = 90°$

The angles are 56°, 34° and 90°.

6 a $\overrightarrow{OS} = \overrightarrow{OP} + \overrightarrow{PR} + \overrightarrow{RS}$

$\overrightarrow{OP} = \mathbf{a}$

$\overrightarrow{PR} = \frac{1}{3}(-\mathbf{a} + \mathbf{b})$

$\overrightarrow{RS} = 2\overrightarrow{OR} = 2(\overrightarrow{OP} + \overrightarrow{PR})$

$= 2(\mathbf{a} + \frac{1}{3}(-\mathbf{a} + \mathbf{b}))$

$= 2\mathbf{a} - \frac{2}{3}\mathbf{a} + \frac{2}{3}\mathbf{b}$

$= \frac{4}{3}\mathbf{a} + \frac{2}{3}\mathbf{b}$

So $\overrightarrow{OS} = \mathbf{a} + \frac{1}{3}(-\mathbf{a} + \mathbf{b}) + \frac{4}{3}\mathbf{a} + \frac{2}{3}\mathbf{b}$

$= 2\mathbf{a} + \mathbf{b}$

b $\overrightarrow{TP} = \overrightarrow{TO} + \overrightarrow{OP}$

$= \mathbf{b} + \mathbf{a} = \mathbf{a} + \mathbf{b}$

$\overrightarrow{PS} = \overrightarrow{PR} + \overrightarrow{RS}$

$= \frac{1}{3}(-\mathbf{a} + \mathbf{b}) + \frac{4}{3}\mathbf{a} + \frac{2}{3}\mathbf{b} = \mathbf{a} + \mathbf{b}$

$\overrightarrow{TP}$ is parallel and equal to $\overrightarrow{PS}$ and point P is common to both lines, so T, P and S lie on a straight line.

Challenge

a $\overrightarrow{PX} = j\overrightarrow{PR}$

$\overrightarrow{PR} = \overrightarrow{PO} + \overrightarrow{OR}$

$= -\mathbf{a} + \mathbf{b}$

$\overrightarrow{PX} = j(-\mathbf{a} + \mathbf{b})$

$= -j\mathbf{a} + j\mathbf{b}$

b $\overrightarrow{PX} = \overrightarrow{PO} + \overrightarrow{OX}$

$\overrightarrow{OX} = k\overrightarrow{ON}$

$\overrightarrow{ON} = \mathbf{a} + \frac{1}{2}\mathbf{b}$

$\overrightarrow{PX} = -\mathbf{a} + k\left(\mathbf{a} + \frac{1}{2}\mathbf{b}\right)$

$= (k-1)\mathbf{a} + \frac{1}{2}k\mathbf{b}$

as $\overrightarrow{PX} = -j\mathbf{a} + j\mathbf{b}$

and $\overrightarrow{PX} = (k-1)\mathbf{a} + \frac{1}{2}k\mathbf{b}$

then $-j\mathbf{a} + j\mathbf{b} = (k-1)\mathbf{a} + \frac{1}{2}k\mathbf{b}$

c The coefficients of **a** and **b** must be the same, so: $k - 1 = -j$ and $\frac{1}{2}k = j$.

d Solving the equation simultaneously and using substitution:

$k - 1 = -\frac{1}{2}k$

$k = \frac{2}{3}$

$j = \frac{1}{3}$

e $\overrightarrow{PX} = -\frac{1}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}$

As $OPQR$ is a parallelogram, $\overrightarrow{YR} = \overrightarrow{PX}$.
Therefore $\overrightarrow{PX} = \overrightarrow{XY} = \overrightarrow{YR}$, so the line PR is divided into three equal parts.
Therefore, the lines ON and OM divide the diagonal PR into three equal parts.

Vectors 11F

1 a Speed $= |3\mathbf{i}+4\mathbf{j}|$
$= \sqrt{3^2+4^2}$
$= \sqrt{9+16}$
$= \sqrt{25}$
$= 5 \text{ m s}^{-1}$

b Speed $= |24\mathbf{i}+7\mathbf{j}|$
$= \sqrt{24^2+(-7)^2}$
$= \sqrt{576+49}$
$= \sqrt{625}$
$= 25 \text{ km h}^{-1}$

c Speed $= |5\mathbf{i}+2\mathbf{j}|$
$= \sqrt{5^2+2^2}$
$= \sqrt{25+4}$
$= \sqrt{29}$
$= 5.39 \text{ m s}^{-1} (3 \text{ s.f.})$

d Speed $= |-7\mathbf{i}+4\mathbf{j}|$
$= \sqrt{(-7)^2+4^2}$
$= \sqrt{49+16}$
$= \sqrt{65}$
$= 8.06 \text{ cm s}^{-1} (3 \text{ s.f.})$

2 a Distance = speed × time
$= \sqrt{8^2+6^2} \times 5$
$= 5 \times \sqrt{64+36}$
$= 5 \times \sqrt{100}$
$= 50 \text{ km}$

b Distance = speed × time
$= \sqrt{5^2+(-1)^2} \times 10$
$= 10 \times \sqrt{25+1}$
$= 10 \times \sqrt{26}$
$= 51.0 \text{ m} (3 \text{ s.f.})$

c Distance = speed × time
$= \sqrt{6^2+2^2} \times 0.75$
$= 0.75 \times \sqrt{36+4}$
$= 0.75 \times \sqrt{40}$
$= 4.74 \text{ km} (3 \text{ s.f.})$

d Distance = speed × time
$= \sqrt{(-4)^2+(-7)^2} \times 120$
$= 120 \times \sqrt{16+49}$
$= 120 \times \sqrt{65}$
$= 967 \text{ cm} (3 \text{ s.f.})$

3 a Speed $= \sqrt{(-3)^2+4^2}$
$= \sqrt{9+16}$
$= \sqrt{25}$
$= 5 \text{ m s}^{-1}$

Distance $= 5 \times 15 = 75 \text{ m}$

b Speed $= \sqrt{2^2+5^2}$
$= \sqrt{4+25}$
$= \sqrt{29}$
$= 5.39 \text{ m s}^{-1} (3 \text{ s.f.})$

Distance $= 3 \times 5.39 = 16.2 \text{ m} (3 \text{ s.f.})$

c Speed $= \sqrt{5^2+(-2)^2}$
$= \sqrt{25+4}$
$= \sqrt{29}$
$= 5.39 \text{ km h}^{-1} (3 \text{ s.f.})$

Distance $= 3 \times 5.39 = 16.2 \text{ km} (3 \text{ s.f.})$

3 d Speed $= \sqrt{12^2 + (-5)^2}$

$= \sqrt{144 + 25}$

$= \sqrt{169}$

$= 13 \text{ km h}^{-1}$

Distance $= 0.5 \times 13 = 6.5$ km

4 a $= \dfrac{(16\mathbf{i} - 5\mathbf{j}) - (2\mathbf{i} + 3\mathbf{j})}{5}$

$= \dfrac{14\mathbf{i} - 8\mathbf{j}}{5}$

$= 2.8\mathbf{i} - 1.6\mathbf{j} \text{ m s}^{-2}$

5 a $\tan\theta = \frac{7}{5}$

$\theta = \tan^{-1}\frac{7}{5}$

$= 54.5°$ (3 s.f.)

b magnitude of $\mathbf{F} = |m\mathbf{a}|$

$= 0.3\sqrt{5^2 + 7^2}$

$= 0.3\sqrt{74}$

6 a $\tan\theta = \frac{1}{2}$

$\theta = \tan^{-1}\frac{1}{2}$

$= 26.6°$ below

b $3\mathbf{i} - 4\mathbf{j} + p\mathbf{i} + q\mathbf{j} = \lambda(2\mathbf{i} - \mathbf{j})$

$(3 + p)\mathbf{i} + (q - 4)\mathbf{j} = 2\lambda\mathbf{i} - \lambda\mathbf{j}$

$3 + p = 2\lambda$ and $q - 4 = -\lambda$

Multiplying the second equation by 2:

$2q - 8 = -2\lambda$

Solving simultaneously:

$3 + p = -2q + 8$

$p + 2q = 5$

c When $p = 1, \lambda = 2$

$\mathbf{R} = 2(2\mathbf{i} - \mathbf{j})$

$= 4\mathbf{i} - 2\mathbf{j}$

$|\mathbf{R}| = \sqrt{4^2 + 2^2}$

$= \sqrt{20}$

$= 2\sqrt{5}$ N

7 a $\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AC}$

$= -(30\mathbf{i} + 40\mathbf{j}) + 40\mathbf{i} - 60\mathbf{j}$

$= 10\mathbf{i} - 100\mathbf{j}$

7 b $AB = \sqrt{30^2 + 40^2} = \sqrt{2500} = 50$

$AC = \sqrt{40^2 + (-60)^2} = \sqrt{5200}$

$BC = \sqrt{10^2 + (-100)^2} = \sqrt{10\,100}$

Using the cosine rule:

$\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

$\cos A = \dfrac{\sqrt{5200}^2 + 50^2 - \sqrt{10\,100}^2}{2(\sqrt{5200})(50)}$

$\cos A = \dfrac{5200 + 2500 - 10\,100}{1000\sqrt{52}}$

$\cos A = \dfrac{-2400}{1000\sqrt{52}}$

$A = 109.440\,03\ldots$

So $\angle BAC = 109.4°$

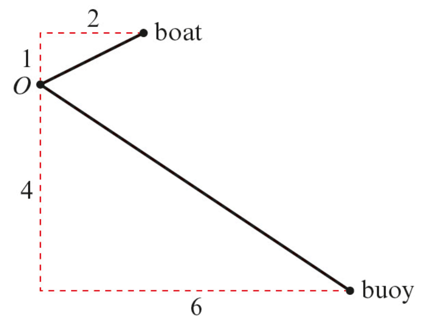
c Area $= \frac{1}{2}bc\sin A$

$= \frac{1}{2} \times \sqrt{5200} \times 50 \times \sin 109.4°$

$= 1700.418\ldots$

$= 1700 \text{ m}^2$ (3 s.f.)

8 a



Distance $= \sqrt{(6-2)^2 + (-4-1)^2}$

$= \sqrt{16 + 25}$

$= \sqrt{41}$ km

b Bearing $= 270° + \tan^{-1}\frac{5}{4}$

$= 321.3°$

c The vector of the boat to the buoy

$= -(2\mathbf{i} + \mathbf{j}) + 6\mathbf{i} - 4\mathbf{j} = 4\mathbf{i} - 5\mathbf{j}$

Velocity $= 8\mathbf{i} - 10\mathbf{j}$

so $\lambda(8\mathbf{i} - 10\mathbf{j}) = 4\mathbf{i} - 5\mathbf{j}$

$\lambda = \frac{1}{2}$

Therefore, the boat is travelling directly towards the buoy.

8 d Speed $= \sqrt{8^2 + 10^2}$

$= \sqrt{164}$

$= 2\sqrt{41}\ \text{km h}^{-1}$

e Time $= \dfrac{\text{distance}}{\text{speed}}$

$= \dfrac{\sqrt{41}}{2\sqrt{41}}$

$= \dfrac{1}{2}$ hour

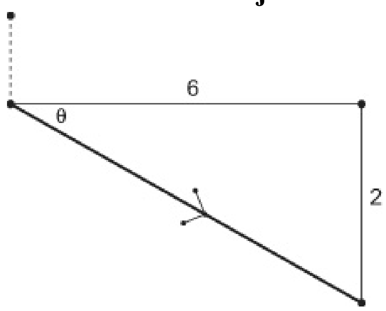
$= 30$ minutes

Vectors, 11 Mixed Exercise

1 a $\mathbf{R} = -3\mathbf{i} + 7\mathbf{j} + \mathbf{i} - \mathbf{j}$
$= -2\mathbf{i} + 6\mathbf{j}$
$|\mathbf{R}| = \sqrt{2^2 + 6^2}$
$= \sqrt{40}$
$= 2\sqrt{10}$
The magnitude of **R** is $2\sqrt{10}$ N

b $\tan\theta = \frac{1}{3}$
$\theta = \tan^{-1}\frac{1}{3}$
$= 18°$ (nearest degree)

2 a $(\text{Path of } S) = (4\mathbf{i} - 6\mathbf{j}) - (-2\mathbf{i} - 4\mathbf{j})$
$= 6\mathbf{i} - 2\mathbf{j}$



$\tan\theta = \frac{1}{3} \Rightarrow \theta = 18.43\ldots°$
$\text{Bearing} = 90° + \theta = 108°$

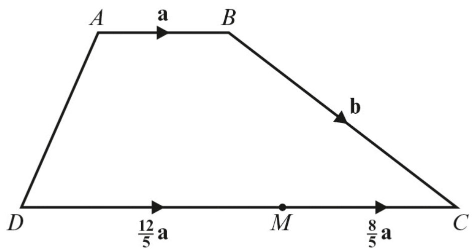
b Expressing velocity, **v**, in km h^{-1}:
$\mathbf{v} = (6\mathbf{i} - 2\mathbf{j}) \times \frac{60}{40}$
$\mathbf{v} = 9\mathbf{i} - 3\mathbf{j}$
Then the speed is:
$\sqrt{9^2 + (-3)^2} = \sqrt{90} = 3\sqrt{10}$
Speed is 9.49 km h^{-1} (3 s.f.) to the left.

3 a Speed $= \sqrt{9^2 + 4^2}$
$= \sqrt{97}$
$= 9.85 \text{ m s}^{-1}$ (3 s.f.)

b Distance = speed × time
$= \sqrt{97} \times 6$
$= 59.1$ m

c This model becomes less accurate as t increases because it ignores friction and air resistance.

4



a $\overrightarrow{AM} = \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CM}$
$= \mathbf{a} + \mathbf{b} - \frac{8}{5}\mathbf{a}$
$= \mathbf{b} - \frac{3}{5}\mathbf{a}$

b $\overrightarrow{BD} = \overrightarrow{BC} + \overrightarrow{CD}$
$= \mathbf{b} - 4\mathbf{a}$

c $\overrightarrow{MB} = \overrightarrow{MC} + \overrightarrow{CB}$
$= \frac{8}{5}\mathbf{a} - \mathbf{b}$

d $\overrightarrow{DA} = \overrightarrow{DC} + \overrightarrow{CB} + \overrightarrow{BA}$
$= 4\mathbf{a} - \mathbf{b} - \mathbf{a}$
$= 3\mathbf{a} - \mathbf{b}$

5 As the vectors are parallel
$5\mathbf{a} + k\mathbf{b} = \frac{5}{8}(8\mathbf{a} + 2\mathbf{b})$
$k = \frac{5}{8} \times 2$
$= 1.25$

6 a $\mathbf{a} + \mathbf{b} + \mathbf{c} = \begin{pmatrix} 7 \\ 4 \end{pmatrix} + \begin{pmatrix} 10 \\ -2 \end{pmatrix} + \begin{pmatrix} -5 \\ -3 \end{pmatrix}$
$= \begin{pmatrix} 12 \\ -1 \end{pmatrix}$

b $\mathbf{a} - 2\mathbf{b} + \mathbf{c} = \begin{pmatrix} 7 \\ 4 \end{pmatrix} - 2\begin{pmatrix} 10 \\ -2 \end{pmatrix} + \begin{pmatrix} -5 \\ -3 \end{pmatrix}$
$= \begin{pmatrix} -18 \\ 5 \end{pmatrix}$

c $2\mathbf{a} + 2\mathbf{b} - 3\mathbf{c} = 2\begin{pmatrix} 7 \\ 4 \end{pmatrix} + 2\begin{pmatrix} 10 \\ -2 \end{pmatrix} - 3\begin{pmatrix} -5 \\ -3 \end{pmatrix}$
$= \begin{pmatrix} 49 \\ 13 \end{pmatrix}$

7 a $\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AC}$
$= -(3\mathbf{i} + 5\mathbf{j}) + 6\mathbf{i} + 3\mathbf{j}$
$= 3\mathbf{i} - 2\mathbf{j}$

b $\tan x = \frac{5}{3}$

$x = \tan^{-1}\frac{5}{3}$

$= 59.036...$

$\tan y = \frac{1}{2}$

$y = \tan^{-1}\frac{1}{2}$

$= 26.565...$

$\angle BAC = 59.036... - 26.565...$
$= 32.5°$ (3 s.f.)

c Area $= \frac{1}{2}bc\sin A$

$b = \sqrt{6^2 + 3^2} = \sqrt{45}$

$c = \sqrt{3^2 + 5^2} = \sqrt{34}$

Area $= \frac{1}{2} \times \sqrt{45} \times \sqrt{34} \times \sin 32.5°$

$= 10.508...$

$= 10.5$ units^2 (3 s.f.)

8 a $4\mathbf{i} - 3\mathbf{j} + 2p\mathbf{i} - p\mathbf{j} = \lambda(2\mathbf{i} - 3\mathbf{j})$
$(4 + 2p)\mathbf{i} - (3 + p)\mathbf{j} = 2\lambda\mathbf{i} - 3\lambda\mathbf{j}$
Equating coefficients:
$4 + 2p = 2\lambda$ and $3 + p = 3\lambda$
Solving simultaneously:
Rearranging the $3 + p = 3\lambda$
$p = 3\lambda - 3$
Using substitution:
$4 + 2(3\lambda - 3) = 2\lambda$
$4 + 6\lambda - 6 = 2\lambda$
$4\lambda = 2$
$\lambda = \frac{1}{2}$
$p = -1.5$

b $\mathbf{a} + \mathbf{b} = 4\mathbf{i} - 3\mathbf{j} - 3\mathbf{i} + 1.5\mathbf{j}$
$= \mathbf{i} - 1.5\mathbf{j}$

9 a i a unit vector is $\frac{\mathbf{a}}{|\mathbf{a}|}$

$\mathbf{a} = \begin{pmatrix} 8 \\ 15 \end{pmatrix}$

$|\mathbf{a}| = \sqrt{8^2 + 15^2} = \sqrt{289} = 17$

9 a i $\frac{\mathbf{a}}{|\mathbf{a}|} = \frac{1}{17}\begin{pmatrix} 8 \\ 15 \end{pmatrix} = \frac{1}{17}(8\mathbf{i} + 15\mathbf{j})$

ii $\tan\theta = \frac{15}{8}$

$\theta = \tan^{-1}\frac{15}{8}$

$= 61.9°$ (3 s.f.) above

b i a unit vector is $\frac{\mathbf{b}}{|\mathbf{b}|}$

$\mathbf{b} = \begin{pmatrix} 24 \\ -7 \end{pmatrix}$

$|\mathbf{b}| = \sqrt{24^2 + 7^2}$

$= \sqrt{625}$

$= 25$

$\frac{\mathbf{b}}{|\mathbf{b}|} = \frac{1}{25}\begin{pmatrix} 25 \\ -7 \end{pmatrix}$

$= \frac{1}{25}(24\mathbf{i} - 7\mathbf{j})$

ii $\tan\theta = \frac{7}{24}$

$\theta = \tan^{-1}\frac{7}{24}$

$= 16.3°$ (3 s.f.) below

c i a unit vector is $\frac{\mathbf{c}}{|\mathbf{c}|}$

$\mathbf{c} = \begin{pmatrix} -9 \\ 40 \end{pmatrix}$

$|\mathbf{c}| = \sqrt{9^2 + 40^2}$

$= \sqrt{1681}$

$= 41$

$\frac{\mathbf{c}}{|\mathbf{c}|} = \frac{1}{41}\begin{pmatrix} -9 \\ 40 \end{pmatrix}$

$= \frac{1}{41}(-9\mathbf{i} + 40\mathbf{j})$

ii $\tan x = \frac{40}{9}$

$x = \tan^{-1}\frac{40}{9}$

$= 77.3°$ (3 s.f.)

$\theta = 180° - 77.3°$
$= 102.7°$ above

9 d i a unit vector is $\dfrac{\mathbf{d}}{|\mathbf{d}|}$

$$\mathbf{d}=\begin{pmatrix}3\\-2\end{pmatrix}$$

$$|\mathbf{d}|=\sqrt{3^2+2^2}$$
$$=\sqrt{13}$$
$$\frac{\mathbf{d}}{|\mathbf{d}|}=\frac{1}{\sqrt{13}}\begin{pmatrix}3\\-2\end{pmatrix}$$
$$=\frac{1}{\sqrt{13}}(3\mathbf{i}-2\mathbf{j})$$

ii $\tan\theta=\dfrac{2}{3}$

$$\theta=\tan^{-1}\frac{2}{3}$$
$= 33.7°$ (3 s.f.) below

10 $\cos 55° = \dfrac{p}{15}$

$p = 15\cos 55°$
$p = 8.6$

Using Pythagoras' theorem:
$$q=\sqrt{15^2-8.6^2}$$
$$=12.3$$
$p = 8.6$ and $q = 12.3$

11 $|3\mathbf{i} - k\mathbf{j}| = \sqrt{3^2+k^2}$
$$=\sqrt{9+k^2}$$
$$=3\sqrt{5}$$
$$\sqrt{9+k^2}=\sqrt{45}$$
$k^2 + 9 = 45$
$k^2 = 36$
$k = \pm 6$

Q12 has been replaced in the summer 2020 impression of the book onwards. This is the answer to the updated question.

12 a $\overrightarrow{AB}=\mathbf{b}-\mathbf{a}$, $\overrightarrow{AP}=3\overrightarrow{AB}=3\mathbf{b}-3\mathbf{a}$
$$\overrightarrow{MP}=\overrightarrow{MA}+\overrightarrow{AP}$$
$$=\frac{1}{2}\overrightarrow{OA}+\overrightarrow{AP}$$
$$=\frac{1}{2}\mathbf{a}+3\mathbf{b}-3\mathbf{a}$$
$$=3\mathbf{b}-\frac{5}{2}\mathbf{a}$$

b $\overrightarrow{ON}=\overrightarrow{OM}+\overrightarrow{MN}$
$$=\frac{1}{2}\mathbf{a}+k\left(3\mathbf{b}-\frac{5}{2}\mathbf{a}\right)$$
$$=\left(\frac{1}{2}-\frac{5}{2}k\right)\mathbf{a}+3k\mathbf{b}$$

c Since $\overrightarrow{ON}$ is parallel to **b**, component of **a** is 0:
$\dfrac{1}{2}-\dfrac{5}{2}k=0\Rightarrow k=\dfrac{1}{5}$ so $\overrightarrow{ON}=\dfrac{3}{5}\overrightarrow{OB}$,
so $ON : NB = 3 : 2$ as required.

This is the answer to Q12 from older impressions of the book.

12 a Using $\overrightarrow{ON}=\overrightarrow{OM}+\overrightarrow{MN}$
$$\overrightarrow{ON}=\frac{3}{5}\mathbf{a}+\lambda\mathbf{b}$$
$$(1-\lambda)\mathbf{a}+\lambda\mathbf{b}=\frac{3}{5}\mathbf{a}+\lambda\mathbf{b}$$
Equating coefficients
$$1-\lambda=\frac{3}{5}$$
$$\lambda=\frac{2}{5}$$
$$\overrightarrow{ON}=\frac{3}{5}\mathbf{a}+\frac{2}{5}\mathbf{b}$$

b $\overrightarrow{MN}=\lambda\mathbf{b}$
$$=\frac{2}{5}\mathbf{b}$$

c $\overrightarrow{AN}=\dfrac{2}{5}(-\mathbf{a}+\mathbf{b})$
$$\overrightarrow{AB}=-\mathbf{a}+\mathbf{b}$$
Therefore, $AN : NB = 2 : 3$

13 a $\tan\theta = \frac{1}{3}$

$\theta = \tan^{-1}\frac{1}{3}$

$= 18.4°$ below

b $4\mathbf{i} - 5\mathbf{j} + p\mathbf{i} + q\mathbf{j} = \lambda(3\mathbf{i} - \mathbf{j})$
$(4 + p)\mathbf{i} + (q - 5)\mathbf{j} = 3\lambda\mathbf{i} - \lambda\mathbf{j}$
$4 + p = 3\lambda$ and $q - 5 = -\lambda$

Multiplying the second equation by 3:
$3q - 15 = -3\lambda$

Solving simultaneously:
$4 + p = -3q + 15$
$p + 3q = 11$

c When $p = 2$, $\lambda = 2$.
$\mathbf{R} = 2(3\mathbf{i} - \mathbf{j})$
$= 6\mathbf{i} - 2\mathbf{j}$
$|\mathbf{R}| = \sqrt{6^2 + 2^2}$
$= \sqrt{40}$
$= 2\sqrt{10}$ N

14 $\mathbf{v} - \mathbf{u} = (15\mathbf{i} - 3\mathbf{j}) - (3\mathbf{i} + 4\mathbf{j})$
$= 12\mathbf{i} - 7\mathbf{j}$
$|\mathbf{a}| = \frac{\sqrt{12^2 + 7^2}}{2}$
$= \frac{\sqrt{193}}{2}$

Challenge

$y = 5 - \frac{5}{3}x$

Using Pythagoras' theorem:
$x^2 + y^2 = \frac{17}{2}$

Solve the equations simultaneously.
Substitute $y = 5 - \frac{5}{3}x$ into $x^2 + y^2 = \frac{17}{2}$.

$$x^2 + \left(5 - \tfrac{5}{3}x\right)^2 = \tfrac{17}{2}$$
$$x^2 + 25 - \tfrac{50}{3}x + \tfrac{25}{9}x^2 - \tfrac{17}{2} = 0$$
$$18x^2 + 450 - 300x + 50x^2 - 153 = 0$$
$$68x^2 - 300x + 297 = 0$$

Using the quadratic formula:
$x =$

$x = \frac{300 \pm \sqrt{9216}}{136}$

$x = \frac{300 \pm 96}{136}$

$x = \frac{99}{34}$ or $x = \frac{51}{34}$

When $x = \frac{99}{34}$, $y = \frac{5}{34}$

When $x = \frac{51}{34}$, $y = \frac{5}{2}$

$\overrightarrow{OB} = \frac{99}{34}\mathbf{i} + \frac{5}{34}\mathbf{j}$ or $\overrightarrow{OB} = \frac{51}{34}\mathbf{i} + \frac{5}{2}\mathbf{j}$

Differentiation 12A

1 a Examples of estimates of gradients:
Gradient of tangent at $x = -1$ is

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{3 \; -1}{(-1)-(-0.5)} = -4$$

Gradient of tangent at $x = 0$ is

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{1-(-1)}{(-0.5)-(0.5)} = -2$$

Gradient of tangent at $x = 1$ is

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{(-1)-(-1)}{2-0} = 0$$

Gradient of tangent at $x = 2$ is

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{(-1)-1}{1.5-2.5} = 2$$

Gradient of tangent at $x = 3$ is

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{3-1}{3-2.5} = 4$$

x-coordinate	−1	0	1	2	3
Estimate for gradient of curve	−4	−2	0	2	4

b The gradient of the curve at the point where $x = p$ is $2p - 2$.

c Gradient of tangent at $x = 1.5$ is

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{(-1.7)-0.3}{0.5-2.5} = 1$$

$2p - 2 = 2(1.5) - 2 = 1$

2 a Substituting $x = 0.6$ into $y = \sqrt{1-x^2}$:
$y = \sqrt{1-0.6^2} = \sqrt{0.64} = 0.8$, therefore the point A (0.6, 0.8) lies on the curve.

b Gradient of tangent at $x = 0.6$ is

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{1.1-0.8}{0.2-0.6} = -0.75$$

2 c i Gradient of AD $= \dfrac{y_2 - y_1}{x_2 - x_1}$

$$= \frac{0.8-\sqrt{0.19}}{0.6-0.9} = -1.21 \text{ (3 s.f.)}$$

ii Gradient of AC $= \dfrac{y_2 - y_1}{x_2 - x_1}$

$$= \frac{0.8-0.6}{0.6-0.8} = -1$$

iii Gradient of AB $= \dfrac{y_2 - y_1}{x_2 - x_1}$

$$= \frac{0.8-\sqrt{0.51}}{0.6-0.7} = -0.859 \text{ (3 s.f.)}$$

d As the points move closer to A, the gradient tends to -0.75.

3 a i Gradient $= \dfrac{16-9}{4-3} = \dfrac{7}{1} = 7$

ii Gradient $= \dfrac{12.25-9}{3.5-3} = \dfrac{3.25}{0.5} = 6.5$

iii Gradient $= \dfrac{9.61-9}{3.1-3} = \dfrac{0.61}{0.1} = 6.1$

iv Gradient $= \dfrac{9.0601-9}{3.01-3} = \dfrac{0.0601}{0.01} = 6.01$

v Gradient $= \dfrac{(3+h)^2-9}{(3+h)-3}$

$$= \frac{6h+h^2}{h} = \frac{h(6+h)}{h} = 6 + h$$

3 b When h is small, the gradient of the chord is close to the gradient of the tangent, and $6 + h$ is close to the value 6.
So the gradient of the tangent at (3, 9) is 6.

4 a i Gradient = $\dfrac{25-16}{5-4} = \dfrac{9}{1} = 9$

ii Gradient = $\dfrac{20.25-16}{4.5-4} = \dfrac{4.25}{0.5} = 8.5$

iii Gradient = $\dfrac{16.81-16}{4.1-4} = \dfrac{0.81}{0.1} = 8.1$

iv Gradient = $\dfrac{16.0801-16}{4.01-4}$

$$= \frac{0.0801}{0.01} = 8.01$$

v Gradient $= \dfrac{(4+h)^2-16}{4+h-4}$

$$= \frac{16+8h+h^2-16}{h}$$

$$= \frac{8h+h^2}{h}$$

$$= \frac{h(8+h)}{h}$$

$$= 8+h$$

b When h is small, the gradient of the chord is close to the gradient of the tangent, and $8 + h$ is close to the value 8.
So the gradient of the tangent at (4, 16) is 8.

Differentiation 12B

1 a $f(x) = x^2$

$$f'(2) = \lim_{h\to0}\frac{f(2+h)-f(2)}{h}$$
$$= \lim_{h\to0}\frac{(2+h)^2-2^2}{h}$$
$$= \lim_{h\to0}\frac{4+4h+h^2-4}{h}$$
$$= \lim_{h\to0}\frac{4h+h^2}{h}$$
$$= \lim_{h\to0}\frac{h(4+h)}{h}$$
$$= \lim_{h\to0}(4+h)$$

As $h \to 0$, $4 + h \to 4$.
So $f'(2) = 4$

b $$f'(-3) = \lim_{h\to0}\frac{f(-3+h)-f(-3)}{h}$$
$$= \lim_{h\to0}\frac{(-3+h)^2-(-3)^2}{h}$$
$$= \lim_{h\to0}\frac{9-6h+h^2-9}{h}$$
$$= \lim_{h\to0}\frac{-6h+h^2}{h}$$
$$= \lim_{h\to0}\frac{h(-6+h)}{h}$$
$$= \lim_{h\to0}(-6+h)$$

As $h \to 0$, $-6 + h \to -6$.
So $f'(-3) = -6$

c $$f'(0) = \lim_{h\to0}\frac{f(0+h)-f(0)}{h}$$
$$= \lim_{h\to0}\frac{(0+h)^2-0^2}{h}$$
$$= \lim_{h\to0}\frac{h^2}{h}$$
$$= \lim_{h\to0} h$$

$f'(0) = 0$

d $$f'(50) = \lim_{h\to0}\frac{f(50+h)-f(50)}{h}$$
$$= \lim_{h\to0}\frac{(50+h)^2-50^2}{h}$$
$$= \lim_{h\to0}\frac{2500+100h+h^2-2500}{h}$$
$$= \lim_{h\to0}\frac{100h+h^2}{h}$$
$$= \lim_{h\to0}\frac{h(100+h)}{h}$$
$$= \lim_{h\to0}(100+h)$$

As $h \to 0$, $100 + h \to 100$.
So $f'(50) = 100$

2 a $f(x) = x^2$

$$f'(x) = \lim_{h\to0}\frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to0}\frac{(x+h)^2-x^2}{h}$$
$$= \lim_{h\to0}\frac{x^2+2xh+h^2-x^2}{h}$$
$$= \lim_{h\to0}\frac{2xh+h^2}{h}$$
$$= \lim_{h\to0}\frac{h(2x+h)}{h}$$
$$= \lim_{h\to0}(2x+h)$$

b As $h \to 0$, $2x + h \to 2x$.
So $f'(x) = 2x$

3 a $y = x^3$, therefore $f(x) = x^3$

$$g = \lim_{h\to0}\frac{f(-2+h)-f(-2)}{h}$$
$$= \lim_{h\to0}\frac{(-2+h)^3-(-2)^3}{h}$$
$$= \lim_{h\to0}\frac{-8+3(-2)^2h+3(-2)h^2+h^3+8}{h}$$
$$= \lim_{h\to0}\frac{12h-6h^2+h^3}{h}$$
$$= \lim_{h\to0}\frac{h(12-6h+h^2)}{h}$$
$$= \lim_{h\to0}(12-6h+h^2)$$

3 b As $h \to 0$, $12 - 6h + h^2 \to 12$.
So $g = 12$

4 a y-coordinate of point B
$= (-1 + h)^3 - 5(-1 + h)$
Gradient of AB

$$= \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{(-1+h)^3 - 5(-1+h) - 4}{(-1+h) - (-1)}$$
$$= \frac{-1 + 3h - 3h^2 + h^3 + 5 - 5h - 4}{h}$$
$$= \frac{h^3 - 3h^2 - 2h}{h}$$
$$= h^2 - 3h - 2$$

b At point A, as $h \to 0$, $h^2 - 3h - 2 \to -2$.
So gradient $= -2$

5 $f(x) = 6x$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h\to 0}\frac{6(x+h) - 6x}{h}$$
$$= \lim_{h\to 0}\frac{6x + 6h - 6x}{h}$$
$$= \lim_{h\to 0}\frac{6h}{h}$$
$$= \lim_{h\to 0} 6$$

So $f'(x) = 6$

6 $f(x) = 4x^2$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h\to 0}\frac{4(x+h)^2 - 4x^2}{h}$$
$$= \lim_{h\to 0}\frac{4x^2 + 8xh + 4h^2 - 4x^2}{h}$$
$$= \lim_{h\to 0}\frac{8xh + 4h^2}{h}$$
$$= \lim_{h\to 0}\frac{h(8x + 4h)}{h}$$
$$= \lim_{h\to 0}(8x + 4h)$$

As $h \to 0$, $8x + 4h \to 8x$.
So $f'(x) = 8x$

7 $f(z) = az^2$

$$f'(z) = \lim_{h\to 0}\frac{f(z+h) - f(z)}{h}$$
$$= \lim_{h\to 0}\frac{a(z+h)^2 - az^2}{h}$$
$$= \lim_{h\to 0}\frac{az^2 + 2azh + ah^2 - az^2}{h}$$
$$= \lim_{h\to 0}\frac{2azh + ah^2}{h}$$
$$= \lim_{h\to 0}\frac{h(2az + ah)}{h}$$
$$= \lim_{h\to 0}(2az + ah)$$

As $h \to 0$, $2az + ah \to 2az$.
So $f'(z) = 2az$

Challenge

a $f(x) = \dfrac{1}{x}$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h\to 0}\frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$
$$= \lim_{h\to 0}\frac{\frac{x - (x+h)}{x(x+h)}}{h}$$
$$= \lim_{h\to 0}\frac{-h}{xh(x+h)}$$
$$= \lim_{h\to 0}\frac{-1}{x(x+h)}$$
$$= \lim_{h\to 0}\frac{-1}{x^2 + hx}$$

b As $h \to 0$, $\dfrac{-1}{x^2 + hx} \to -\dfrac{1}{x^2}$.

So $f'(x) = -\dfrac{1}{x^2}$

Differentiation 12C

1 a $f(x) = x^7$
$f'(x) = 7x^6$

b $f(x) = x^8$
$f'(x) = 8x^7$

c $f(x) = x^4$
$f'(x) = 4x^3$

d $f(x) = x^{\frac{1}{3}}$

$$f'(x) = \frac{1}{3}x^{\frac{1}{3}-1} = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3x^{\frac{2}{3}}}$$

e $f(x) = x^{\frac{1}{4}}$

$$f'(x) = \frac{1}{4}x^{\frac{1}{4}-1} = \frac{1}{4}x^{-\frac{3}{4}} = \frac{1}{4x^{\frac{3}{4}}}$$

f $f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$

$$f'(x) = \frac{1}{3}x^{\frac{1}{3}-1} = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3x^{\frac{2}{3}}}$$

g $f(x) = x^{-3}$
$f'(x) = -3x^{-3-1} = -3x^{-4}$

h $f(x) = x^{-4}$
$f'(x) = -4x^{-4-1} = -4x^{-5}$

i $f(x) = \frac{1}{x^2} = x^{-2}$

$$f'(x) = -2x^{-3-1} = -2x^{-3} = -\frac{2}{x^3}$$

j $f(x) = \frac{1}{x^5} = x^{-5}$

$$f'(x) = -5x^{-5-1} = -5x^{-6} = -\frac{5}{x^6}$$

k $f(x) = \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$

$$f'(x) = -\frac{1}{2}x^{-\frac{1}{2}-1} = -\frac{1}{2}x^{-\frac{3}{2}} = -\frac{1}{2x^{\frac{3}{2}}}$$

1 l $f(x) = \frac{1}{\sqrt[3]{x}} = x^{-\frac{1}{3}}$

$$f'(x) = -\frac{1}{3}x^{-\frac{1}{3}-1} = -\frac{1}{3}x^{-\frac{4}{3}} = -\frac{1}{3x^{\frac{4}{3}}}$$

m $f(x) = x^3 \times x^6 = x^{3+6} = x^9$
$f'(x) = 9x^8$

n $f(x) = x^2 \times x^3 = x^5$
$f'(x) = 5x^4$

o $f(x) = x \times x^2 = x^3$
$f'(x) = 3x^2$

p $f(x) = \frac{x^2}{x^4} = x^{-2}$

$$f'(x) = -2x^{-3-1} = -2x^{-3} = -\frac{2}{x^3}$$

q $f(x) = \frac{x^3}{x^2} = x$
$f'(x) = 1x^0 = 1$

r $f(x) = \frac{x^6}{x^3} = x^3$
$f'(x) = 3x^2$

2 a $y = 3x^2$

$$\frac{dy}{dx} = 2 \times 3x^{2-1} = 6x$$

b $y = 6x^9$

$$\frac{dy}{dx} = 9 \times 6x^{9-1} = 54x^8$$

c $y = \frac{1}{2}x^4$

$$\frac{dy}{dx} = 4 \times \frac{1}{2}x^{4-1} = 2x^3$$

d $y = 20x^{\frac{1}{4}}$

$$\frac{dy}{dx} = \frac{1}{4} \times 20x^{\frac{1}{4}-1} = 5x^{-\frac{3}{4}} = \frac{5}{x^{\frac{3}{4}}}$$

2 e $y = 6x^{\frac{5}{4}}$

$\frac{dy}{dx} = \frac{5}{4} \times 6x^{\frac{5}{4}-1} = \frac{15}{2}x^{\frac{1}{4}}$

f $y = 10x^{-1}$

$\frac{dy}{dx} = -1 \times 10x^{-1-1} = -10x^{-2}$

g $y = \frac{4x^6}{2x^3} = 2x^3$

$\frac{dy}{dx} = 3 \times 2x^{3-1} = 6x^2$

h $y = \frac{x}{8x^5} = \frac{1}{8}x^{-4}$

$\frac{dy}{dx} = -4 \times \frac{1}{8}x^{-4-1} = -\frac{1}{2}x^{-5} = -\frac{1}{2x^5}$

i $y = -\frac{2}{\sqrt{x}} = -2x^{-\frac{1}{2}}$

$\frac{dy}{dx} = \left(-\frac{1}{2}\right) \times (-2)x^{-\frac{1}{2}-1} = x^{-\frac{3}{2}} = \frac{1}{x^{\frac{3}{2}}}$

j $y = \sqrt{\frac{5x^4 \times 10x}{2x^2}} = 5x^{\frac{3}{2}}$

$\frac{dy}{dx} = \frac{3}{2} \times 5x^{\frac{3}{2}-1} = \frac{15}{2}x^{\frac{1}{2}} = \frac{15\sqrt{x}}{2}$

3 a $y = 3\sqrt{x} = 3x^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2} \times 3x^{\frac{1}{2}-1} = \frac{3}{2}x^{-\frac{1}{2}} = \frac{3}{2\sqrt{x}}$

When $x = 4$, $\frac{dy}{dx} = \frac{3}{2\sqrt{4}} = \frac{3}{4}$

b When $x = 9$, $\frac{dy}{dx} = \frac{3}{2\sqrt{9}} = \frac{3}{6} = \frac{1}{2}$

c When $x = \frac{1}{4}$, $\frac{dy}{dx} = \frac{3}{2\sqrt{\frac{1}{4}}} = \frac{3}{1} = 3$

d When $x = \frac{9}{16}$, $\frac{dy}{dx} = \frac{3}{2\sqrt{\frac{9}{16}}} = \frac{3}{\frac{3}{2}} = 2$

4 $2y^2 - x^3 = 0$

$2y^2 = x^3$

$y^2 = \frac{1}{2}x^3$

$y = \frac{1}{\sqrt{2}}x^{\frac{3}{2}}$

$\frac{dy}{dx} = \frac{3}{2} \times \frac{1}{\sqrt{2}}x^{\frac{3}{2}-1} = \frac{3}{2\sqrt{2}}x^{\frac{1}{2}} = \frac{3}{2}\sqrt{\frac{x}{2}}$

Differentiation 12D

1 a $y = 2x^2 - 6x + 3$

$\frac{dy}{dx} = 2(2x) - 6(1) + 0 = 4x - 6$

b $y = \frac{1}{2}x^2 + 12x$

$\frac{dy}{dx} = \frac{1}{2}(2x) + 12(1) = x + 12$

c $y = 4x^2 - 6$

$\frac{dy}{dx} = 4(2x) - 0 = 8x$

d $y = 8x^2 + 7x + 12$

$\frac{dy}{dx} = 8(2x) + 7(1) + 0 = 16x + 7$

e $y = 5 + 4x - 5x^2$

$\frac{dy}{dx} = 0 + 4(1) - 5(2x) = 4 - 10x$

2 a $y = 3x^2$

$\frac{dy}{dx} = 6x$

At the point (2, 12), $x = 2$

Substituting $x = 2$ into $\frac{dy}{dx} = 6x$ gives:

Gradient = 6 × 2 = 12

b $y = x^2 + 4x$

$\frac{dy}{dx} = 2x + 4$

At the point (1, 5), $x = 1$

Substituting $x = 1$ into $\frac{dy}{dx} = 2x + 4$ gives:

Gradient = 2 × 1 + 4 = 6

c $y = 2x^2 - x - 1$

$\frac{dy}{dx} = 4x - 1$

At the point (2, 5), $x = 2$

Substituting $x = 2$ into $\frac{dy}{dx} = 4x - 1$ gives:

Gradient = 4 × 2 − 1 = 7

2 d $y = \frac{1}{2}x^2 + \frac{3}{2}x$

$\frac{dy}{dx} = x + \frac{3}{2}$

At the point (1, 2), $x = 1$

Substituting $x = 1$ into $\frac{dy}{dx} = x + \frac{3}{2}$ gives:

Gradient $= 1 + \frac{3}{2} = 2\frac{1}{2}$

e $y = 3 - x^2$

$\frac{dy}{dx} = -2x$

At the point (1, 2), $x = 1$

Substituting $x = 1$ into $\frac{dy}{dx} = -2x$ gives:

Gradient = −2 × 1 = −2

f $y = 4 - 2x^2$

$\frac{dy}{dx} = -4x$

At the point (−1, 2), $x = -1$

Substituting $x = -1$ into $\frac{dy}{dx} = -4x$ gives:

Gradient = −4 × −1 = 4

3 $y = 3 + 2x - x^2$

When $x = 1$, $y = 3 + 2 - 1$

$\Rightarrow y = 4$ when $x = 1$

$\frac{dy}{dx} = 2 - 2x$

When $x = 1, \frac{dy}{dx} = 2 - 2$

$\Rightarrow \frac{dy}{dx} = 0$ when $x = 1$

Therefore, the y-coordinate is 4 and the gradient is 0 when the x-coordinate is 1 on the given curve.

4 $y = x^2 + 5x - 4$

$\frac{dy}{dx} = 2x + 5$

$2x + 5 = 3$

$2x = -2$

$x = -1$

4 Substituting $x = -1$ into $y = x^2 + 5x - 4$:
$y = (-1)^2 + 5(-1) - 4 = 1 - 5 - 4 = -8$
So $(-1, -8)$ is the point where the gradient is 3.

5 The curve $y = x^2 - 5x + 10$ meets the line $y = 4$ when:
$x^2 - 5x + 10 = 4$
$x^2 - 5x + 6 = 0$
$(x - 3)(x - 2) = 0$
$x = 3$ or $x = 2$

Gradient of curve $= \dfrac{dy}{dx} = 2x - 5$

When $x = 3$, $\dfrac{dy}{dx} = 2 \times 3 - 5 = 1$

When $x = 2$, $\dfrac{dy}{dx} = 2 \times 2 - 5 = -1$

So the gradient is -1 at $(2, 4)$ and 1 at $(3, 4)$.

6 The curve $y = 2x^2$ meets the line $y = x + 3$ when
$2x^2 = x + 3$
$2x^2 - x - 3 = 0$
$(2x - 3)(x + 1) = 0$
$x = 1.5$ or -1

Gradient of curve $= \dfrac{dy}{dx} = 4x$

When $x = -1$, $\dfrac{dy}{dx} = 4 \times -1 = -4$

When $x = 1.5$, $\dfrac{dy}{dx} = 4 \times 1.5 = 6$

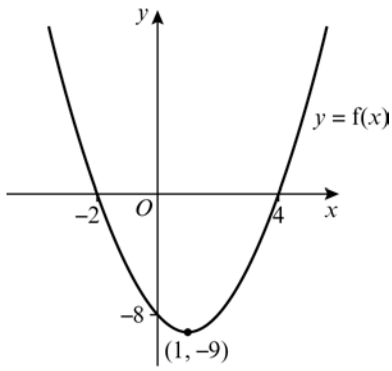
So the gradient is -4 at $(-1, 2)$ and 6 at $(1.5, 4.5)$.

7 a $y = f(x) = x^2 - 2x - 8$
As $a = 1$ is positive, the graph has a $\bigvee$ shape and a minimum point.

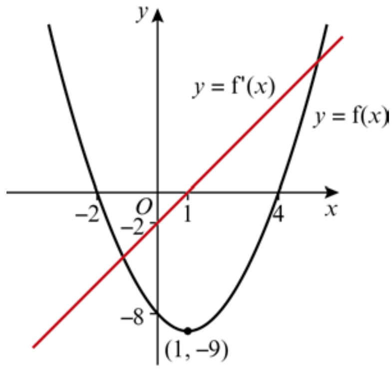
When $x = 0$, $y = -8$, so the graph crosses the y-axis at $(0, -8)$.
When $y = 0$,
$x^2 - 2x - 8 = 0$
$(x + 2)(x - 4) = 0$
$x = -2$ or $x = 4$, so the graph crosses the x-axis at $(-2, 0)$ and $(4, 0)$.

Completing the square:
$x^2 - 2x - 8 = (x - 1)^2 - 1 - 8$
$= (x - 1)^2 - 9$
So the minimum point has coordinates $(1, -9)$.

7 a The sketch of the graph is:



b $f'(x) = 2x - 2 + 0 = 2x - 2$



c At the turning point the gradient of $y = f(x)$ is zero, i.e. $f'(x) = 0$.

Differentiation 12E

1 a Let $y = x^4 + x^{-1}$

$$\frac{dy}{dx} = 4x^3 + (-1)x^{-2}$$
$$= 4x^3 - x^{-2}$$

b Let $y = 2x^5 + 3x^{-2}$

$$\frac{dy}{dx} = 5 \times 2x^{5-1} + (-2) \times 3x^{-2-1}$$
$$= 10x^4 - 6x^{-3}$$

c Let $y = 6x^{\frac{3}{2}} + 2x^{-\frac{1}{2}} + 4$

$$\frac{dy}{dx} = \frac{3}{2} \times 6x^{\frac{3}{2}-1} + \left(-\frac{1}{2}\right) \times 2x^{-\frac{1}{2}-1} + 0$$
$$= 9x^{\frac{1}{2}} - x^{-\frac{3}{2}}$$

2 a $f(x) = x^3 - 3x + 2$
$f'(x) = 3x^2 - 3$
At $(-1, 4)$, $x = -1$
$f'(-1) = 3(-1)^2 - 3 = 0$
The gradient at $(-1, 4)$ is 0.

b $f(x) = 3x^2 + 2x^{-1}$
$f'(x) = 6x + 2(-1)x^{-2} = 6x - 2x^{-2}$
At $(2, 13)$, $x = 2$
$f'(2) = 6(2) - 2(2)^{-2} = 12 - \frac{2}{4} = 11\frac{1}{2}$
The gradient at $(2, 13)$ is $11\frac{1}{2}$.

3 a $f(x) = x^2 - 5x$
$f'(x) = 2x - 5$
When gradient is zero, $f'(x) = 0$.
$2x - 5 = 0$
$x = 2.5$
When $x = 2.5$, $y = f(2.5)$
$= (2.5)^2 - 5(2.5)$
$= -6.25$
Therefore, the gradient is zero at $(2.5, -6.25)$.

b $f(x) = x^3 - 9x^2 + 24x - 20$
$f'(x) = 3x^2 - 18x + 24$
When gradient is zero, $f'(x) = 0$.
$3x^2 - 18x + 24 = 0$
$3(x^2 - 6x + 8) = 0$
$3(x - 4)(x - 2) = 0$
$x = 4$ or $x = 2$

When $x = 4$, $y = f(4)$
$= 4^3 - 9 \times 4^2 + 24 \times 4 - 20$
$= -4$
When $x = 2$, $y = f(2)$
$= 2^3 - 9 \times 2^2 + 24 \times 2 - 20$
$= 0$
Therefore, the gradient is zero at $(4, -4)$ and $(2, 0)$.

c $f(x) = x^{\frac{3}{2}} - 6x + 1$.
$f'(x) = \frac{3}{2}x^{\frac{1}{2}} - 6$
When gradient is zero, $f'(x) = 0$.
$\frac{3}{2}x^{\frac{1}{2}} - 6 = 0$
$x^{\frac{1}{2}} = 4$
$x = 16$
When $x = 16$, $y = f(16)$
$= 16^{\frac{3}{2}} - 6 \times 16 + 1$
$= -31$
Therefore, the gradient is zero at $(16, -31)$.

d $f(x) = x^{-1} + 4x$
$f'(x) = -x^{-2} + 4$
For zero gradient, $f'(x) = 0$.
$-x^{-2} + 4 = 0$
$\frac{1}{x^2} = 4$
$x = \pm\frac{1}{2}$

When $x = \frac{1}{2}$, $y = f\left(\frac{1}{2}\right)$
$y = \left(\frac{1}{2}\right)^{-1} + 4\left(\frac{1}{2}\right)$
$= 2 + 2$
$= 4$
When $x = -\frac{1}{2}$, $y = f\left(-\frac{1}{2}\right)$
$y = \left(-\frac{1}{2}\right)^{-1} + 4\left(-\frac{1}{2}\right)$
$= -2 - 2$
Therefore, the gradient is zero at $\left(\frac{1}{2}, 4\right)$ and $\left(-\frac{1}{2}, -4\right)$

4 a Let $y = 2\sqrt{x}$

$= 2x^{\frac{1}{2}}$

$\frac{dy}{dx} = 2\left(\frac{1}{2}\right)x^{-\frac{1}{2}}$

$= x^{-\frac{1}{2}}$

$= \frac{1}{\sqrt{x}}$

b Let $y = \frac{3}{x^2}$

$= 3x^{-2}$

$\frac{dy}{dx} = 3(-2)x^{-3}$

$= -6x^{-3}$

$= -\frac{6}{x^3}$

c Let $y = \frac{1}{3x^3}$

$= \frac{1}{3}x^{-3}$

$\frac{dy}{dx} = \frac{1}{3}(-3)x^{-4}$

$= -x^{-4}$

$= -\frac{1}{x^4}$

d Let $y = \frac{1}{3}x^3(x-2)$

$= \frac{1}{3}x^4 - \frac{2}{3}x^3$

$\frac{dy}{dx} = \frac{4}{3}x^3 - \frac{2}{3} \times 3x^2$

$= \frac{4}{3}x^3 - 2x^2$

e Let $y = \frac{2}{x^3} + \sqrt{x}$

$= 2x^{-3} + x^{\frac{1}{2}}$

$\frac{dy}{dx} = -6x^{-4} + \frac{1}{2}x^{-\frac{1}{2}}$

$= -\frac{6}{x^4} + \frac{1}{2\sqrt{x}}$

f Let $y = \sqrt[3]{x} + \frac{1}{2x}$

$= x^{\frac{1}{3}} + \frac{1}{2}x^{-1}$

$\frac{dy}{dx} = \frac{1}{3}x^{-\frac{2}{3}} - \frac{1}{2}x^{-2}$

g Let $y = \frac{2x+3}{x}$

$= \frac{2x}{x} + \frac{3}{x}$

$= 2 + 3x^{-1}$

$\frac{dy}{dx} = 0 - 3x^{-2}$

$= -\frac{3}{x^2}$

h Let $y = \frac{3x^2 - 6}{x}$

$= \frac{3x^2}{x} - \frac{6}{x}$

$= 3x - 6x^{-1}$

$\frac{dy}{dx} = 3 + 6x^{-2}$

$= 3 + \frac{6}{x^2}$

i Let $y = \frac{2x^3 + 3x}{\sqrt{x}}$

$= \frac{2x^3}{x^{\frac{1}{2}}} + \frac{3x}{x^{\frac{1}{2}}}$

$= 2x^{\frac{5}{2}} + 3x^{\frac{1}{2}}$

$\frac{dy}{dx} = 5x^{\frac{3}{2}} + \frac{3}{2}x^{-\frac{1}{2}}$

j Let $y = x(x^2 - x + 2)$

$= x^3 - x^2 + 2x$

$\frac{dy}{dx} = 3x^2 - 2x + 2$

k Let $y = 3x^2(x^2 + 2x)$

$= 3x^4 + 6x^3$

$\frac{dy}{dx} = 12x^3 + 18x^2$

4 l Let $y = (3x-2)\left(4x+\frac{1}{x}\right)$

$= 12x^2 - 8x + 3 - \frac{2}{x}$

$= 12x^2 - 8x + 3 - 2x^{-1}$

$\frac{dy}{dx} = 24x - 8 + 2x^{-2}$

$= 24x - 8 + \frac{2}{x^2}$

5 a $f(x) = x(x+1)$

$= x^2 + x$

$f'(x) = 2x + 1$

Gradient at $(0, 0) = f'(0) = 1$

b $f(x) = \frac{2x-6}{x^2}$

$= \frac{2x}{x^2} - \frac{6}{x^2}$

$= 2x^{-1} - 6x^{-2}$

$f'(x) = -2x^{-2} + 12x^{-3}$

$= -\frac{2}{x^2} + \frac{12}{x^3}$

Gradient at $(3, 0) = f'(3) = -\frac{2}{3^2} + \frac{12}{3^3}$

$= -\frac{2}{9} + \frac{12}{27}$

$= \frac{2}{9}$

c $f(x) = \frac{1}{\sqrt{x}}$

$= x^{-\frac{1}{2}}$

$f'(x) = -\frac{1}{2}x^{-\frac{3}{2}}$

Gradient at $\left(\frac{1}{4}, 2\right) = f'\left(\frac{1}{4}\right) = -\frac{1}{2}\left(\frac{1}{4}\right)^{-\frac{3}{2}}$

$= -\frac{1}{2} \times 2^3$

$= -4$

5 d $f(x) = 3x - \frac{4}{x^2}$

$= 3x - 4x^{-2}$

$f'(x) = 3 + 8x^{-3}$

Gradient at $(2, 5) = f'(2) = 3 + 8(2)^{-3}$

$= 3 + \frac{8}{8} = 4$

6 $f(x) = \frac{12}{p\sqrt{x}} + x$

$= \frac{12}{p}x^{-\frac{1}{2}} + x$, $f'(2) = 3$

$f'(x) = -\frac{1}{2} \times \frac{12}{p}x^{-\frac{1}{2}-1} + 1$

$= -\frac{6}{p}x^{-\frac{3}{2}} + 1$

$f'(2) = -\frac{6}{p}(2)^{-\frac{3}{2}} + 1$

$= -\frac{6}{2p\sqrt{2}} + 1$

$-\frac{6}{2p\sqrt{2}} + 1 = 3$

$-\frac{6}{2p\sqrt{2}} = 2$

$p = -\frac{3}{2\sqrt{2}}$

$= -\frac{3}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$

$= -\frac{3}{4}\sqrt{2}$

7 a $f(x) = (2-x)^9$

$= 2^9 + \binom{9}{1}2^8(-x) + \binom{9}{2}2^7(-x)^2 + \ldots$

$= 512 - 2304x + 4608x^2$

b $f(x) \approx 512 - 2304x + 4608x^2$

$f'(x) \approx -2304 + 2 \times 4608x^{2-1}$

$= 9216x - 2304$

Differentiation 12F

1 a $y = x^2 - 7x + 10$

$\frac{dy}{dx} = 2x - 7$

When $x = 2$, gradient $= 2 \times 2 - 7 = -3$

So the equation of the tangent at (2, 0) is

$y - 0 = -3(x - 2)$

$y = -3x + 6$

$y + 3x - 6 = 0$

b $y = x + \frac{1}{x} = x + x^{-1}$

$\frac{dy}{dx} = 1 - x^{-2}$

When $x = 2$, gradient $= 1 - 2^{-2} =$

So the equation of the tangent at $\left(2, 2\frac{1}{2}\right)$ is

$y - 2\frac{1}{2} = \frac{3}{4}(x - 2)$

$4y - 10 = 3x - 6$

$4y - 3x - 4 = 0$

c $y = 4\sqrt{x} = 4x^{\frac{1}{2}}$

$\frac{dy}{dx} = 2x^{-\frac{1}{2}}$

When $x = 9$, gradient $= 2 \times 9^{-\frac{1}{2}} = \frac{2}{3}$

So the equation of the tangent at (9, 12) is

$y - 12 = \frac{2}{3}(x - 9)$

$3y - 36 = 2x - 18$

$3y - 2x - 18 = 0$

d $y = \frac{2x - 1}{x} = \frac{2x}{x} - \frac{1}{x} = 2 - x^{-1}$

$\frac{dy}{dx} = 0 + x^{-2} = x^{-2}$

When $x = 1$, gradient $= 1^{-2} = 1$

So the equation of the tangent at (1, 1) is

$y - 1 = 1 \times (x - 1)$

$y = x$

e $y = 2x^3 + 6x + 10$

$\frac{dy}{dx} = 6x^2 + 6$

When $x = -1$, gradient $= 6(-1)^2 + 6 = 12$

e So the equation of the tangent at (−1, 2) is

$y - 2 = 12(x - (-1))\frac{3}{4}$

$y - 2 = 12x + 12$

$y = 12x + 14$

f $y = x^2 - \frac{7}{x^2} = x^2 - 7x^{-2}$

$\frac{dy}{dx} = 2x + 14x^{-3}$

When $x = 1$, gradient $= 2 + 14 = 16$

So the equation of the tangent at (1, −6) is

$y - (-6) = 16(x - 1)$

$y + 6 = 16x - 16$

$y = 16x - 22$

2 a $y = x^2 - 5x$

$\frac{dy}{dx} = 2x - 5$

When $x = 6$, gradient of curve $= 2 \times 6 - 5$

$= 7$

So gradient of normal is $-\frac{1}{7}$.

The equation of the normal at (6, 6) is

$y - 6 = -\frac{1}{7}(x - 6)$

$7y - 42 = -x + 6$

$7y + x - 48 = 0$

b $y = x^2 - \frac{8}{\sqrt{x}} = x^2 - 8x^{-\frac{1}{2}}$

$\frac{dy}{dx} = 2x + 4x^{-\frac{3}{2}}$

When $x = 4$, gradient of curve

$= 2 \times 4 + 4(4)^{-\frac{3}{2}} = 8 + \frac{4}{8} = \frac{17}{2}$

So gradient of normal is $-\frac{2}{17}$.

The equation of the normal at (4, 12) is

$y - 12 = -\frac{2}{17}(x - 4)$

$17y - 204 = -2x + 8$

$17y + 2x - 212 = 0$

3 $y = x^2 + 1$

$\frac{dy}{dx} = 2x$

When $x = 2$ and $\frac{dy}{dx} = 4$

So the equation of the tangent at (2, 5) is

$y - 5 = 4(x - 2)$

$y = 4x - 3$

When $x = 1$, gradient of curve = 2

So gradient of normal is $-\frac{1}{2}$.

The equation of the normal is

$$y - 2 = -\frac{1}{2}(x - 1)$$

$$y = -\frac{1}{2}x + 2\frac{1}{2}$$

Tangent at (2, 5) and normal at (1, 2) meet when

$4x - 3 = -\frac{1}{2}x + 2\frac{1}{2}$

$8x - 6 = -x + 5$

$9x = 11$

$x = \frac{11}{9}$

$y = 4 \times \frac{11}{9} - 3 = \frac{17}{9}$

So the tangent at (2, 5) meets the normal at (1, 2) at $\left(\frac{11}{9}, \frac{17}{9}\right)$.

4 $y = x + x^3$

$\frac{dy}{dx} = 1 + 3x^2$

When $x = 0$, gradient of curve $= 1 + 3 \times 0^2$
$= 1$

So gradient of normal is $-\frac{1}{1} = -1$.

The equation of the normal at (0, 0) is

$y - 0 = -1(x - 0)$

$y = -x$

When $x = 1$, gradient of curve $= 1 + 3 \times 1^2$
$= 4$

So gradient of normal is $-\frac{1}{4}$.

4 The equation of the normal at (1, 2) is

$y - 2 = -\frac{1}{4}(x - 1)$

$4y - 8 = -x + 1$

$4y + x - 9 = 0$

Normals at (0, 0) and (1, 2) meet when

$4(-x) + x - 9 = 0$

$3x = -9$

$x = -3$

$y = 3$

The normals meet at (−3, 3).

5 $y = f(x) = 12 - 4x + 2x^2$

$\frac{dy}{dx} = 0 - 4 + 4x = 4x - 1$

When $x = -1$, $y = 12 - 4(-1) + 2(-1)^2$
$= 18$

$\frac{dy}{dx} = 4(-1) - 4 = -8$

The tangent at (−1, 18) has gradient −8.

So its equation is

$y - 18 = -8(x + 1)$

$y - 18 = -8x - 8$

$y = -8x + 10$

The normal at (−1, 18) has gradient $\frac{-1}{-8} = \frac{1}{8}$. So its equation is

$y - 18 = \frac{1}{8}(x + 1)$

$8y - 144 = x + 1$

$8y - x - 145 = 0$

6 $y = 2x^2$

$\frac{dy}{dx} = 4x$

When $x = \frac{1}{2}$, $y = 2 \times \left(\frac{1}{2}\right)^2 = \frac{1}{2}$

$\frac{dy}{dx} = 4 \times \frac{1}{2} = 2$

So gradient of normal is $-\frac{1}{2}$.

The equation of the normal at $\left(\frac{1}{2}, \frac{1}{2}\right)$ is

$y - \frac{1}{2} = -\frac{1}{2}\left(x - \frac{1}{2}\right)$

$y = -\frac{1}{2}x + \frac{3}{4}$

6 The normal intersects the curve when

$2x^2 = -\frac{1}{2}x + \frac{3}{4}$

$8x^2 + 2x - 3 = 0$

$(4x + 3)(2x - 1) = 0$

$x = -\frac{3}{4}$ or $\frac{1}{2}$

$x = \frac{1}{2}$ is point P,

so $x = -\frac{3}{4}$ must be point Q.

When $x = -\frac{3}{4}$, $y = -\frac{1}{2}\left(-\frac{3}{4}\right) + \frac{3}{4} = \frac{9}{8}$

Point Q is $\left(-\frac{3}{4}, \frac{9}{8}\right)$.

Challenge

$y = 4x^2 + 1$

$\frac{dy}{dx} = 8x$

Gradient of line $L = 8x$

Equation of line L:

$y = 8x(x) + c$

$= 8x^2 + c$

Line L passes through the point $(0, -8)$,

so $c = -8$

$y = 8x^2 - 8$

Line L meets the curve when

$4x^2 + 1 = 8x^2 - 8$

$4x^2 = 9$

$x^2 = \frac{9}{4}$

$x = \pm\frac{3}{2}$

As the gradient is positive, $x = \frac{3}{2}$

$y = 8x(x) - 8$

$y = 8\left(\frac{3}{2}\right)x - 8$

$y = 12x - 8$

Differentiation 12G

1 a $f(x) = 3x^2 + 8x + 2$
$f'(x) = 6x + 8$
If $f'(x) \geq 0$ then
$6x + 8 \quad 0$
$6x \geq -8$
$x \quad -\frac{4}{3}$
So $f(x)$ is increasing for $x \geq -\frac{4}{3}$.

b $f(x) = 4x - 3x^2$
$f'(x) = 4 - 6x$
If $f'(x) \geq 0$ then
$4 - 6x \geq 0$
$4 \geq 6x$
$x \leq \frac{4}{6}$
$x \leq \frac{2}{3}$
So $f(x)$ is increasing for $x \leq \frac{2}{3}$.

c $f(x) = 5 - 8x - 2x^2$
$f'(x) = -8 - 4x$
If $f'(x) \geq 0$ then
$-8 - 4x \geq 0$
$-8 \geq 4x$
$x \leq -2$
So $f(x)$ is increasing for $x \leq -2$.

d $f(x) = 2x^3 - 15x^2 + 36x$
$f'(x) = 6x^2 - 30x + 36$
If $f'(x) \geq 0$ then
$6x^2 - 30x + 36 \geq 0$
$6(x^2 - 5x + 6) \geq 0$
$6(x - 2)(x - 3) \geq 0$
Considering the 3 regions:

	$x \leq 2$	$2 \leq x \leq 3$	$x \geq 3$
$6(x - 2)(x - 3)$	+ve	−ve	+ve

So $x \leq 2$ or $x \geq 3$
So $f(x)$ is increasing for $x \leq 2$ and $x \geq 3$.

e $f(x) = 3 + 3x - 3x^2 + x^3$
$f'(x) = 3 - 6x + 3x^2$
If $f'(x) \geq 0$ then
$3 - 6x + 3x^2 \geq 0$
$3(1 - 2x + x^2) \geq 0$
$3(1 - x)^2 \geq 0$
So $f(x)$ is increasing for $x \in \mathbb{R}$.

f $f(x) = 5x^3 + 12x$
$f'(x) = 15x^2 + 12$
If $f'(x) \geq 0$ then
$15x^2 + 12 \geq 0$
This is true for all real values of x.
So $f(x)$ is increasing for $x \in \mathbb{R}$.

$\geq$

g $f(x) = x^4 + 2x^2$
$f'(x) = 4x^3 + 4x$
If $f'(x) \geq 0$ then
$4x^3 + 4x \geq 0$
$4x(x^2 + 1) \geq 0$
$x \geq 0$
So $f(x)$ is increasing for $x \geq 0$.

h $f(x) = x^4 - 8x^3 \geq$
$f'(x) = 4x^3 - 24x^2$
If $f'(x) \geq 0$ then
$4x^3 - 24x^2 \geq 0$
$4x^2(x - 6) \geq 0$
$x \geq 6$
So $f(x)$ is increasing for $x \geq 6$.

2 a $f(x) = x^2 - 9x$
$f'(x) = 2x - 9$
If $f'(x) \leq 0$ then
$2x - 9 \leq 0$
$2x \leq 9$
$x \leq \frac{9}{2}$
So $f(x)$ is decreasing for $x \leq \frac{9}{2}$.

b $f(x) = 5x - x^2$
$f'(x) = 5 - 2x$
If $f'(x) \leq 0$ then
$5 - 2x \leq 0$
$2x \geq 5$
$x \geq \frac{5}{2}$
So $f(x)$ is decreasing for $x \geq \frac{5}{2}$.

c $f(x) = 4 - 2x - x^2$
$f'(x) = -2 - 2x$
If $f'(x) \leq 0$ then
$-2 - 2x \leq 0$
$2x \geq -2$
$x \geq -1$
So $f(x)$ is decreasing for $x \geq -1$.

2 d $f(x) = 2x^3 - 3x^2 - 12x$

$f'(x) = 6x^2 - 6x - 12$

If $f'(x) \leq 0$ then

$6x^2 - 6x - 12 \leq 0$

$6(x^2 - x - 2) \leq 0$

$6(x - 2)(x + 1) \leq 0$

Considering the 3 regions:

	$x \leq -1$	$-1 \leq x \leq 2$	$x \geq 2$
$6(x-2)(x+1)$	+ve	−ve	+ve

So $-1 \leq x \leq 2$

So $f(x)$ is decreasing on the interval $[-1, 2]$.

e $f(x) = 1 - 27x + x^3$

$f'(x) = -27 + 3x^2$

If $f'(x) \leq 0$ then

$-27 + 3x^2 \leq 0$

$3x^2 \leq 27$

$x^2 \leq 9$

$-3 \leq x \leq 3$

So $f(x)$ is decreasing on the interval $[-3, 3]$.

f $f(x) = x + 25x^{-1}$

$f'(x) = 1 - \frac{25}{x^2}$

If $f'(x) \leq 0$ then

$1 - \frac{25}{x^2} \leq 0$

$1 \leq \frac{25}{x^2}$

$x^2 \leq 25$

$-5 \leq x \leq 5$

$f(x)$ is not defined for $x = 0$.

So $f(x)$ is decreasing on the intervals $[-5, 0)$ and $(0, 5]$.

g $f(x) = x^{\frac{1}{2}} + 9x^{-\frac{1}{2}}$

$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} - 9 \times \frac{1}{2}x^{-\frac{3}{2}}$

If $f'(x) \leq 0$ then

$\frac{1}{2}x^{-\frac{1}{2}} - 9 \times \frac{1}{2}x^{-\frac{3}{2}} \leq 0$

$\frac{1}{2}x^{-\frac{1}{2}} - \frac{9}{2}x^{-\frac{3}{2}} \leq 0$

$\frac{x^{-\frac{3}{2}}}{2}(x - 9) \leq 0$

$f(x)$ is defined for $x > 0$.

$f'(x) \leq 0$ for $x \leq 9$, so $f(x)$ is decreasing on the interval $(0, 9]$.

2 h $f(x) = x^2(x + 3)$

$= x^3 + 3x^2$

$f'(x) = 3x^2 + 6x$

If $f'(x) \leq 0$ then

$3x^2 + 6x \leq 0$

$3x(x + 2) \leq 0$

Considering the 3 regions:

	$x \leq -2$	$-2 \leq x \leq 0$	$x \geq 0$
$3x(x+2)$	+ve	−ve	+ve

So $f(x)$ is decreasing on the interval $[-2, 0]$.

3 $f(x) = 4 - x(2x^2 + 3) = 4 - 2x^3 - 3x$

$f'(x) = -6x^2 - 3$

$x^2 \geq 0$ for all $x \in \mathbb{R}$, so $-6x^2 - 3 \leq 0$ for all $x \in \mathbb{R}$.

Therefore, $f(x)$ is decreasing for all $x \in \mathbb{R}$.

4 a $f(x) = x^2 + px$

$f'(x) = 2x + p \geq 0$ when $-1 \leq x \leq 1$

When $x = -1$, $f'(x) = -2 + p \geq 0$, so $p \geq 2$

So for $f'(x) \geq 0$, $p \geq 2$ e.g. $p = 3$

When $x = 1$, $f'(x) = 2 + p \geq 0$, so $p \geq -2$

However, $p \geq 2$ to work with $x = -1$.

b Using the proof from part **a**, any value $p \geq 2$ will work.

Differentiation 12H

1 a $y = 12x^2 + 3x + 8$

$\frac{dy}{dx} = 24x + 3$

$\frac{d^2y}{dx^2} = 24$

b $y = 15x + 6 + \frac{3}{x}$

$= 15x + 6 + 3x^{-1}$

$\frac{dy}{dx} = 15 - 3x^{-2}$

$\frac{d^2y}{dx^2} = 6x^{-3}$

c $y = 9\sqrt{x} - \frac{3}{x^2}$

$= 9x^{\frac{1}{2}} - 3x^{-2}$

$\frac{dy}{dx} = \frac{9}{2}x^{-\frac{1}{2}} + 6x^{-3}$

$\frac{d^2y}{dx^2} = -\frac{9}{4}x^{-\frac{3}{2}} - 18x^{-4}$

$\frac{d^2y}{dx^2} = -\frac{9}{4\left(\sqrt{x}\right)^3} - \frac{18}{x^4}$

d $y = (5x + 4)(3x - 2)$

$= 15x^2 + 2x - 8$

$\frac{dy}{dx} = 30x + 2$

$\frac{d^2y}{dx^2} = 30$

e $y = \frac{3x + 8}{x^2}$

$= \frac{3x}{x^2} + \frac{8}{x^2}$

$= \frac{3}{x} + 8x^{-2}$

$= 3x^{-1} + 8x^{-2}$

$\frac{dy}{dx} = -3x^{-2} - 16x^{-3}$

$\frac{d^2y}{dx^2} = 6x^{-3} + 48x^{-4}$

2 $f(t) = \frac{t^2 + 2}{\sqrt{t}} = t^{\frac{3}{2}} + 2t^{-\frac{1}{2}}$

$f'(t) = \frac{3}{2}t^{\frac{1}{2}} - t^{-\frac{3}{2}}$

Acceleration = $f''(t) = \frac{3}{4\sqrt{t}} + \frac{3}{2\left(\sqrt{t}\right)^5}$

3 $y = (2x - 3)^3$

$= 8x^3 - 36x^2 + 54x - 27$

$\frac{dy}{dx} = 24x^2 - 72x + 54$

$\frac{d^2y}{dx^2} = 48x - 72$

$48x - 72 = 0$

$x = \frac{3}{2}$

4 $f(x) = px^3 - 3px^2 + x^2 - 4$

$f'(x) = 3px^2 - 6px + 2x$

$f''(x) = 6px - 6p + 2$

$f''(2) = -1$

$12p - 6p + 2 = -1$

$6p = -3$

$p = -\frac{1}{2}$

Differentiation 12I

1 a $f(x) = x^2 - 12x + 8$
$f'(x) = 2x - 12$
Putting $f'(x) = 0$
$2x - 12 = 0$
$x = 6$
$f(6) = 6^2 - 12 \times 6 + 8 = -28$
The least value of $f(x)$ is -28.

b $f(x) = x^2 - 8x - 1$
$f'(x) = 2x - 8$
Putting $f'(x) = 0$
$2x - 8 = 0$
$x = 4$
$f(4) = 4^2 - 8 \times 4 - 1 = -17$
The least value of $f(x)$ is -17.

c $f(x) = 5x^2 + 2x$
$f'(x) = 10x + 2$
Putting $f'(x) = 0$
$10x + 2 = 0$
$x = -\frac{2}{10} = -\frac{1}{5}$
$f\left(-\frac{1}{5}\right) = 5\left(-\frac{1}{5}\right)^2 + 2\left(-\frac{1}{5}\right) = \frac{5}{25} - \frac{2}{5} = -\frac{1}{5}$
The least value of $f(x)$ is $-\frac{1}{5}$.

2 a $f(x) = 10 - 5x^2$
$f'(x) = -10x$
Putting $f'(x) = 0$
$-10x = 0$
$x = 0$
$f(0) = 10 - 5 \times 0^2 = 10$
The greatest value of $f(x)$ is 10.

b $f(x) = 3 + 2x - x^2$
$f'(x) = 2 - 2x$
Putting $f'(x) = 0$
$2 - 2x = 0$
$x = 1$
$f(1) = 3 + 2 - 1 = 4$
The greatest value of $f(x)$ is 4.

c $f(x) = (6 + x)(1 - x) = 6 - 5x - x^2$
$f'(x) = -5 - 2x$
Putting $f'(x) = 0$
$-5 - 2x = 0$

2 c $x = -\frac{5}{2}$
$f\left(-\frac{5}{2}\right) = \frac{7}{2} \times \frac{7}{2} = \frac{49}{4} = 12\frac{1}{4}$
The greatest value of $f(x)$ is $12\frac{1}{4}$.

3 a $y = 4x^2 + 6x$
$\frac{dy}{dx} = 8x + 6$
Putting $8x + 6 = 0$
$x = -\frac{6}{8} = -\frac{3}{4}$
When $x = -\frac{3}{4}$,
$y = 4\left(-\frac{3}{4}\right)^2 + 6\left(-\frac{3}{4}\right) = \frac{9}{4} - \frac{9}{2} = -\frac{9}{4}$
So $\left(-\frac{3}{4}, -\frac{9}{4}\right)$ is a stationary point.
$\frac{d^2y}{dx^2} = 8 > 0$
So $\left(-\frac{3}{4}, -\frac{9}{4}\right)$ is a minimum point.

b $y = 9 + x - x^2$
$\frac{dy}{dx} = 1 - 2x$
Putting $1 - 2x = 0$
$x = \frac{1}{2}$
When $x = \frac{1}{2}$,
$y = 9 + \frac{1}{2} - \left(\frac{1}{2}\right)^2$
$y = 9\frac{1}{4}$
So $\left(\frac{1}{2}, 9\frac{1}{4}\right)$ is a stationary point.
$\frac{d^2y}{dx^2} = -2 < 0$
So $\left(\frac{1}{2}, 9\frac{1}{4}\right)$ is a maximum point.

c $y = x^3 - x^2 - x + 1$
$\frac{dy}{dx} = 3x^2 - 2x - 1$
Putting $3x^2 - 2x - 1 = 0$
$(3x + 1)(x - 1) = 0$

3 c So $x = -\frac{1}{3}$ or $x = 1$

When $x = -\frac{1}{3}$,

$y = \left(-\frac{1}{3}\right)^3 - \left(-\frac{1}{3}\right)^2 - \left(-\frac{1}{3}\right) + 1$

$= 1\frac{5}{27}$

When $x = 1$,

$y = 1^3 - 1^2 - 1 + 1$

$= 0$

So $\left(-\frac{1}{3}, 1\frac{5}{27}\right)$ and $(1, 0)$ are stationary points.

$\frac{d^2y}{dx^2} = 6x - 2$

When $x = -\frac{1}{3}$, $\frac{d^2y}{dx^2} = 6\left(-\frac{1}{3}\right) - 2 = -4 < 0$

So $\left(-\frac{1}{3}, 1\frac{5}{27}\right)$ is a maximum point.

When $x = 1$, $\frac{d^2y}{dx^2} = 6(1) - 2 = 4 > 0$

So $(1, 0)$ is a minimum point.

d $y = x(x^2 - 4x - 3) = x^3 - 4x^2 - 3x$

$\frac{dy}{dx} = 3x^2 - 8x - 3$

Putting $3x^2 - 8x - 3 = 0$

$(3x + 1)(x - 3) = 0$

So $x = -\frac{1}{3}$ or $x = 3$

When $x = -\frac{1}{3}$,

$y = \left(-\frac{1}{3}\right)^3 - 4\left(-\frac{1}{3}\right)^2 - 3\left(-\frac{1}{3}\right)$

$= \frac{14}{27}$

When $x = 3$,

$y = 3^3 - 4(3)^2 - 3(3)$

$= -18$

So $\left(-\frac{1}{3}, \frac{14}{27}\right)$ and $(3, -18)$ are stationary points.

$\frac{d^2y}{dx^2} = 6x - 8$

3 d When $x = -\frac{1}{3}$, $\frac{d^2y}{dx^2} = 6\left(-\frac{1}{3}\right) - 8$

$= -10 < 0$

So $\left(-\frac{1}{3}, \frac{14}{27}\right)$ is a maximum point.

When $x = 3$, $\frac{d^2y}{dx^2} = 6(3) - 8$

$= 10 > 0$

So $(3, -18)$ is a minimum point.

e $y = x + \frac{1}{x} = x + x^{-1}$

$\frac{dy}{dx} = 1 - x^{-2}$

Putting $1 - x^{-2} = 0$

$x^2 = 1$

$x = \pm 1$

When $x = 1$,

$y = 1 + \frac{1}{1}$

$= 2$

When $x = -1$,

$y = -1 + \frac{1}{-1}$

$= -2$

So $(1, 2)$ and $(-1, -2)$ are stationary points.

$\frac{d^2y}{dx^2} = 2x^{-3}$

When $x = 1$, $\frac{d^2y}{dx^2} = 2 > 0$

So $(1, 2)$ is a minimum point.

When $x = -1$, $\frac{d^2y}{dx^2} = -2 < 0$

So $(-1, -2)$ is a maximum point.

f $y = x^2 + \frac{54}{x} = x^2 + 54x^{-1}$

$\frac{dy}{dx} = 2x - 54x^{-2}$

Putting $2x - 54x^{-2} = 0$

$x = \frac{27}{x^2}$

$x^3 = 27$

$x = 3$

3 f When $x = 3$,

$y = 3^2 + \frac{54}{3}$

$= 27$

So (3, 27) is a stationary point.

$\frac{d^2y}{dx^2} = 2 + 108x^{-3}$

When $x = 3$, $\frac{d^2y}{dx^2} = 2 + \frac{108}{3^3} = 6 > 0$

So (3, 27) is a minimum point.

g $y = x - 3\sqrt{x} = x - 3x^{\frac{1}{2}}$

$\frac{dy}{dx} = 1 - \frac{3}{2}x^{-\frac{1}{2}}$

Putting $1 - \frac{3}{2}x^{-\frac{1}{2}} = 0$

$1 = \frac{3}{2\sqrt{x}}$

$\sqrt{x} = \frac{3}{2}$

$x = \frac{9}{4}$

When $x = \frac{9}{4}$,

$y = \frac{9}{4} - 3\sqrt{\frac{9}{4}}$

$= -\frac{9}{4}$

So $\left(\frac{9}{4}, -\frac{9}{4}\right)$ is a stationary point.

$\frac{d^2y}{dx^2} = \frac{3}{4}x^{-\frac{3}{2}}$

When $x = \frac{9}{4}$, $\frac{d^2y}{dx^2} = \frac{3}{4} \times \left(\frac{9}{4}\right)^{-\frac{3}{2}}$

$= \frac{3}{4} \times \left(\frac{2}{3}\right)^3$

$= \frac{2}{9} > 0$

So $\left(\frac{9}{4}, -\frac{9}{4}\right)$ is a minimum point.

h $y = x^{\frac{1}{2}}(x - 6) = x^{\frac{3}{2}} - 6x^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}} - 3x^{-\frac{1}{2}}$

Putting $\frac{3}{2}x^{\frac{1}{2}} - 3x^{-\frac{1}{2}} = 0$

$\frac{3}{2}x^{\frac{1}{2}} = \frac{3}{x^{\frac{1}{2}}}$

$\frac{3}{2}x = 3$

$x = 2$

When $x = 2$,

$y = 2^{\frac{1}{2}}(-4)$

$= -4\sqrt{2}$

So $(2, -4\sqrt{2})$ is a stationary point.

$\frac{d^2y}{dx^2} = \frac{3}{4}x^{-\frac{1}{2}} + \frac{3}{2}x^{-\frac{3}{2}}$

When $x = 2$, $\frac{d^2y}{dx^2} = \frac{3}{4\sqrt{2}} + \frac{3}{4\sqrt{2}} > 0$

So $(2, -4\sqrt{2})$ is a minimum point.

i $y = x^4 - 12x^2$

$\frac{dy}{dx} = 4x^3 - 24x$

Putting $4x^3 - 24x = 0$

$4x(x^2 - 6) = 0$

$x = 0$ or $x = \pm\sqrt{6}$

When $x = 0$, $y = 0$

When $x = \pm\sqrt{6}$, $y = -36$

So (0, 0), $(\sqrt{6}, -36)$ and $(-\sqrt{6}, -36)$ are stationary points.

$\frac{d^2y}{dx^2} = 12x^2 - 24$

When $x = 0$, $\frac{d^2y}{dx^2} = -24 < 0$

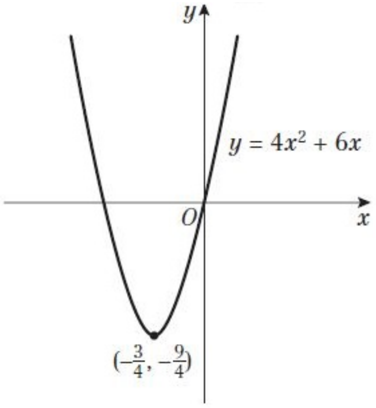
So (0, 0) is a maximum point.

When $x = \pm\sqrt{6}$,

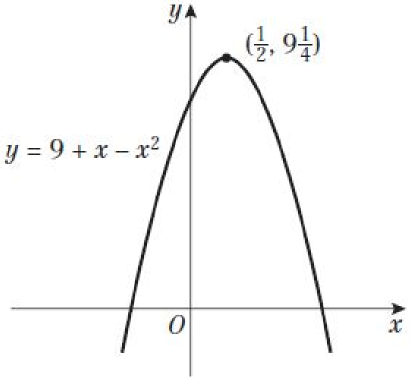
$\frac{d^2y}{dx^2} = 12 \times 6 - 24 = 48 > 0$

So $(\sqrt{6}, -36)$ and $(-\sqrt{6}, -36)$ are minimum points.

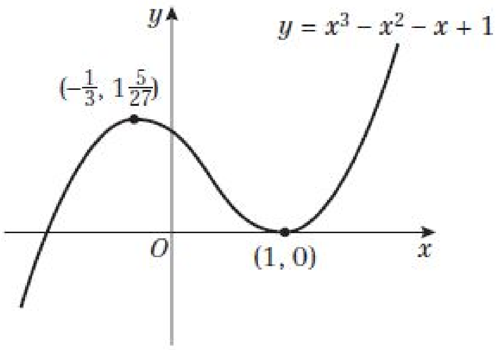
4 a



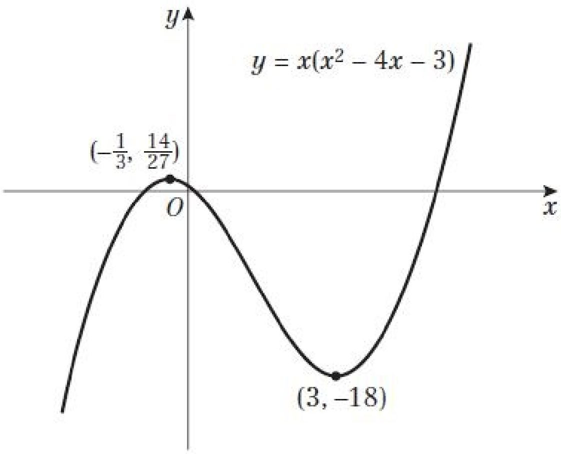
b



c



d



5 $y = x^3 - 3x^2 + 3x$

$\frac{dy}{dx} = 3x^2 - 6x + 3$

Putting $3x^2 - 6x + 3 = 0$

$3(x^2 - 2x + 1) = 0$

$3(x - 1)^2 = 0$

$x = 1$

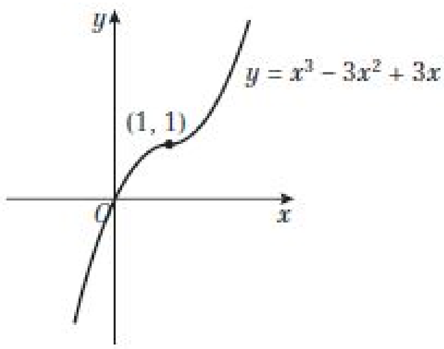
When $x = 1$, $y = 1$

So (1, 1) is a stationary point.

Considering points near to (1, 1):

x	0.9	1	1.1
$\frac{dy}{dx}$	0.03	0	0.03
	+ve	0	+ve
Shape	/	—	/

The gradient on either side of (1, 1) is positive, so (1, 1) is a point of inflection.



6 $f(x) = 27 - 2x^4$

$f'(x) = -8x^3$

Putting $-8x^3 = 0$

$x = 0$

When $x = 0$, $y = 27$

So (0, 27) is a stationary point.

$f''(x) = -24x^2$

When $x = 0$, $f''(x) = 0$, so not conclusive

Considering points near to (0, 27):

x	−0.1	0	0.1
$f'(x)$	0.008	0	−0.008
	+ve	0	−ve
Shape	/	—	/

So (0, 27) is a maximum point.

So the maximum value of f(x) is 27 and the range of values is $f(x) \leq 27$.

7 a $f(x) = x^4 + 3x^3 - 5x^2 - 3x + 1$

$f'(x) = 4x^3 + 9x^2 - 10x - 3$

Putting $4x^3 + 9x^2 - 10x - 3 = 0$

Using the factor theorem: $f'(1) = 0$,

so dividing $4x^3 + 9x^2 - 10x - 3$ by $x - 1$:

$$\begin{array}{r}
4x^2 + 13x + 3 \\
x-1 \overline{\big) 4x^3 + 9x^2 - 10x - 3} \\
\underline{4x^3 - 4x^2} \qquad\qquad \\
13x^2 - 10x \qquad \\
\underline{13x^2 - 13x} \qquad \\
3x - 3 \\
\underline{3x - 3} \\
0
\end{array}$$

$(x - 1)(4x^2 + 13x + 3) = 0$

$(x - 1)(4x + 1)(x + 3) = 0$

$x = 1, x = -\frac{1}{4}$ or $x = -3$

When $x = 1$,

$y = (1)^4 + 3(1)^3 - 5(1)^2 - 3(1) + 1$

$= -3$

When $x = -\frac{1}{4}$,

$$y = \left(-\frac{1}{4}\right)^4 + 3\left(-\frac{1}{4}\right)^3 - 5\left(-\frac{1}{4}\right)^2 - 3\left(-\frac{1}{4}\right) + 1$$

$$= \frac{357}{256}$$

When $x = -3$,

$y = (-3)^4 + 3(-3)^3 - 5(-3)^2 - 3(-3) + 1$

$= -35$

So $(1, -3)$, $(-3, -35)$ and $(-\frac{1}{4}, \frac{357}{256})$ are stationary points.

$f''(x) = 12x^2 + 18x - 10$

When $x = 1$, $f''(x) = 20 > 0$

So $(1, -3)$ is a minimum point.

When $x = -3$,

$f''(x) = 12(-3)^2 + 18(-3) - 10 = 44 > 0$

So $(-3, -35)$ is a minimum point.

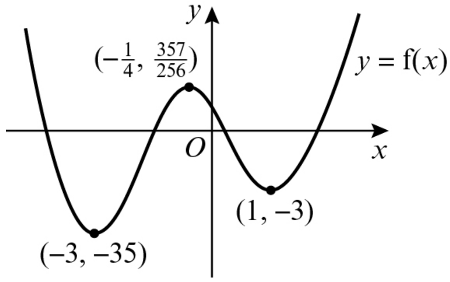
When $x = -\frac{1}{4}$,

$$f''(x) = 12\left(-\frac{1}{4}\right)^2 + 18\left(-\frac{1}{4}\right) - 10$$

$$= - -\frac{55}{4} < 0$$

So $(-\frac{1}{4}, \frac{357}{256})$ is a maximum point.

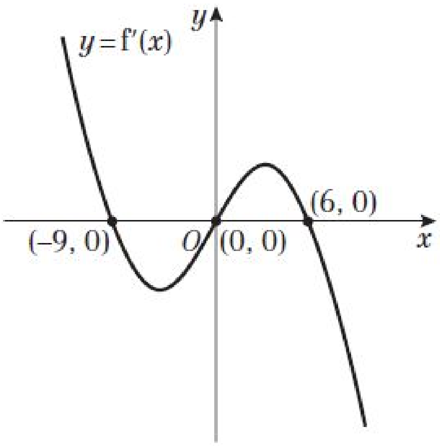
7 b



Differentiation 12J

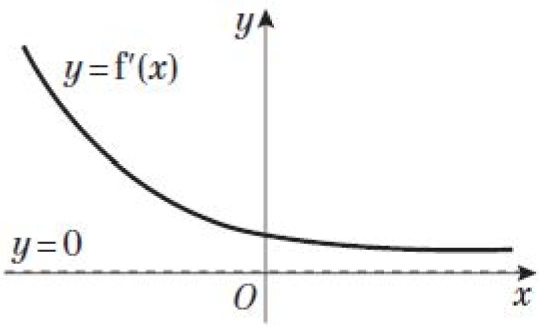
1 a

x	$y = f(x)$	$y = f'(x)$
$x < -9$	Positive gradient	Above x-axis
$x = -9$	Maximum	Cuts x-axis
$-9 < x < 0$	Negative gradient	Below x-axis
$x = 0$	Minimum	Cuts x-axis
$0 < x < 6$	Positive gradient	Above x-axis
$x = 6$	Maximum	Cuts x-axis
$x > 6$	Negative gradient	Below x-axis



b

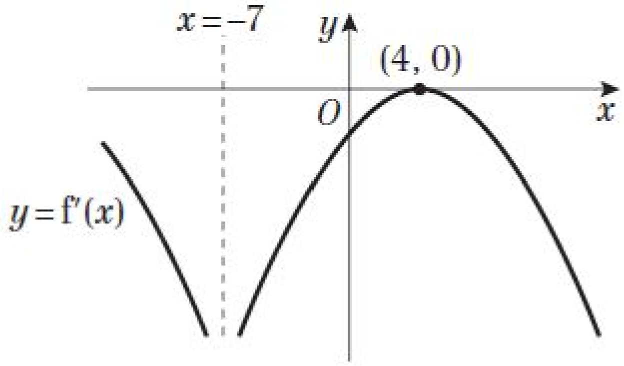
x	$y = f(x)$	$y = f'(x)$
All values of x	Positive gradient	Above x-axis with asymptote at $y = 0$



c

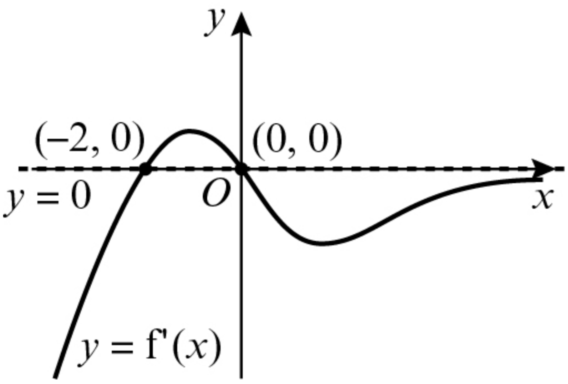
x	$y = f(x)$	$y = f'(x)$
$x < -7$	Negative gradient	Below x-axis with asymptote at $x = -7$
$-7 < x < 4$	Negative gradient	Below x-axis
$x = 4$	Point of inflection	Touches x-axis
$x > 4$	Negative gradient	Below x-axis

1 c



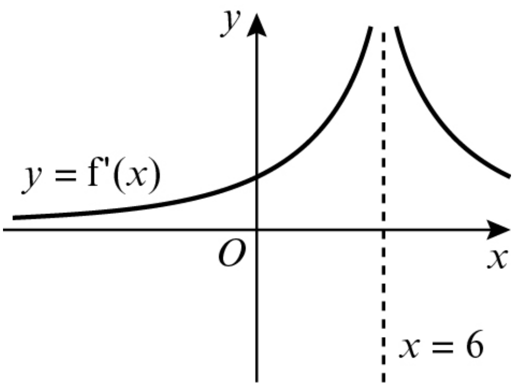
d

x	$y = f(x)$	$y = f'(x)$
$x < -2$	Negative gradient	Below x-axis
$x = -2$	Minimum	Cuts x-axis
$-2 < x < 0$	Positive gradient	Above x-axis
$x = 0$	Maximum	Cuts x-axis
$x > 4$	Negative gradient	Below x-axis with asymptote at $y = 0$



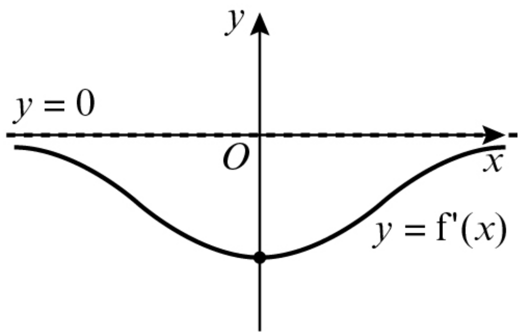
e

x	$y = f(x)$	$y = f'(x)$
$x < 6$	Positive gradient	Above x-axis with asymptote at $x = 6$
$x > 6$	Positive gradient	Above x-axis with asymptote at $x = 6$



1 f

x	$y = f(x)$	$y = f'(x)$
$x < 0$	Negative gradient	Below x-axis with asymptote at $y = 0$
$x > 0$	Negative gradient	Below x-axis with asymptote at $y = 0$



2 a $y = f(x) = (x + 1)(x - 4)^2 = x^3 - 7x^2 + 8x + 16$

When $y = 0$, $x = -1$ or $x = 4$

To find stationary points, $\frac{dy}{dx} = 0$:

$\frac{dy}{dx} = 3x^2 - 14x + 8$

$(3x - 2)(x - 4) = 0$

$x = \frac{2}{3}$ or $x = 4$

When $x = \frac{2}{3}$, $y = \left(\frac{2}{3}+1\right)\left(\frac{2}{3}-4\right)^2 = \frac{500}{27}$

When $x = 4$, $y = (4 + 1)(4 - 4)^2 = 0$

So $\left(\frac{2}{3}, \frac{500}{27}\right)$ and (4, 0) are stationary points.

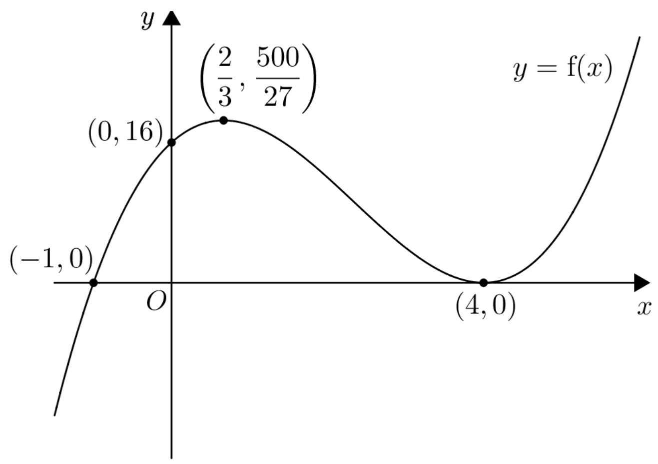
$\frac{d^2y}{dx^2} = 6x - 14$

When $x = \frac{2}{3}$, $\frac{d^2y}{dx^2} = 6\left(\frac{2}{3}\right) - 14 = -10 < 0$

So $\left(\frac{2}{3}, \frac{500}{27}\right)$ is a maximum.

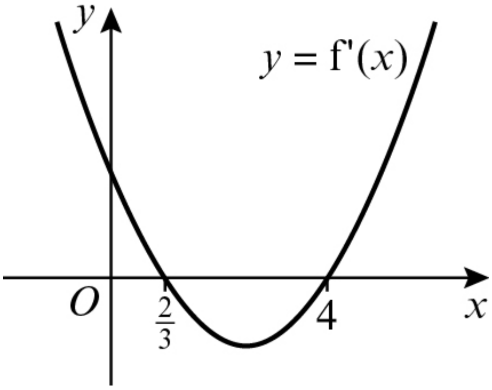
When $x = 4$, $\frac{d^2y}{dx^2} = 6(4) - 14 = 10 > 0$

So (4, 0) is a minimum.



2 b

x	$y = f(x)$	$y = f'(x)$
$x < \frac{2}{3}$	Positive gradient	Above x-axis
$x = \frac{2}{3}$	Maximum	Cuts x-axis
$\frac{2}{3} < x < 4$	Negative gradient	Below x-axis
$x = 4$	Minimum	Cuts x-axis
$x > 4$	Positive gradient	Above x-axis



c $f(x) = (x + 1)(x - 4)^2 = x^3 - 7x^2 + 8x + 16$
$f'(x) = 3x^2 - 14x + 8$
$= (3x - 2)(x - 4)$

d $f'(x) = 3x^2 - 14x + 8$
$(3x - 2)(x - 4) = 0$
$x = \dfrac{2}{3}$ or $x = 4$
When $x = 0$, $f'(x) = 8$
The points where the gradient function cuts the axes are $\left(\frac{2}{3}, 0\right)$, (4, 0) and (0, 8).

Differentiation 12K

1 $\theta = t^2 - 3t$

$\frac{d\theta}{dt} = 2t - 3$

2 $A = 2\pi r$

$\frac{dA}{dr} = 2\pi$

3 $r = \frac{12}{t} = 12t^{-1}$

$\frac{dr}{dt} = -12t^{-2} = -\frac{12}{t^2}$

When $t = 3$,

$\frac{dr}{dt} = -\frac{12}{3^2} = -\frac{12}{9} = -\frac{4}{3}$

4 $A = 4\pi r^2$

$\frac{dA}{dr} = 8\pi r$

When $r = 6$,

$\frac{dA}{dr} = 8\pi \times 6$

$= 48\pi \text{ cm}^2 \text{ per cm}$

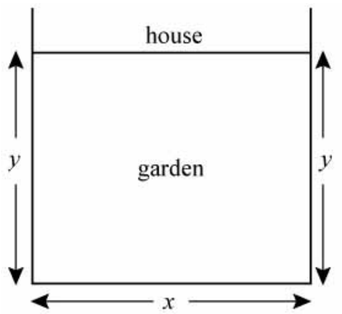
5 $s = t^2 + 8t$

$\frac{ds}{dt} = 2t + 8$

When $t = 5$,

$\frac{ds}{dt} = 2 \times 5 + 8 = 18 \text{ m s}^{-1}$

6 a



Let the width of the garden be x m.

Then $x + 2y = 80$

$x = 80 - 2y$ (1)

Area $A = xy$

$= y(80 - 2y)$

$= 80y - 2y^2$

6 b $\frac{dA}{dy} = 80 - 4y$

Putting $\frac{dA}{dy} = 0$ for maximum area:

$80 - 4y = 0$

$y = 20$

Substituting in (1): $x = 40$

So area $= 40 \text{ m} \times 20 \text{ m} = 800 \text{ m}^2$

7 a Total surface area $= 2\pi rh + 2\pi r^2$

$2\pi rh + 2\pi r^2 = 600\pi$

$rh = 300 - r^2$

Volume $= \pi r^2 h = \pi r(rh) = \pi r(300 - r^2)$

So $V = 300\pi r - \pi r^3$

b For maximum volume, $\frac{dV}{dr} = 0$

$\frac{dV}{dr} = 300\pi - 3\pi r^2$

$300\pi - 3\pi r^2 = 0$

$r^2 = 100$

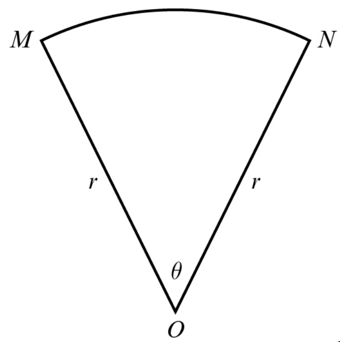
$r = 10$

Substituting $r = 10$ into V gives:

$V = 300\pi \times 10 - \pi \times 10^3 = 2000\pi$

Maximum volume $= 2000\pi \text{ cm}^3$

8 a



Let angle $MON = \theta$ radians

Then perimeter $P = 2r + r\theta$ **(1)**

and area $A = \frac{1}{2}r^2\theta$

Area $= 100 \text{ cm}^2$

$\frac{1}{2}r^2\theta = 100$

$r\theta = \frac{200}{r}$

Substituting into **(1)** gives:

$P = 2r + \frac{200}{r}$ **(2)**

8 a Since area of circle > area of sector

$$\pi r^2 > 100$$

$$r > \sqrt{\frac{100}{\pi}}$$

b For minimum perimeter, $\frac{dP}{dr} = 0$

$$\frac{dP}{dr} = 2 - \frac{200}{r^2}$$

$$2 - \frac{200}{r^2} = 0$$

$$r^2 = 100$$

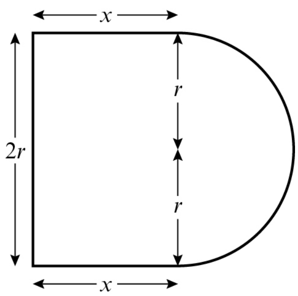
$$r = 10$$

Substituting into (2) gives:

$$P = 20 + \frac{200}{10} = 40$$

Minimum perimeter = 40 cm

9 a



Let the rectangle have dimensions $2r$ by x cm.

Perimeter of figure = $(2r + 2x + \pi r)$ cm

Perimeter = 40 cm, so

$$2r + 2x + \pi r = 40$$

$$x = \frac{40 - \pi r - 2r}{2}$$

Area = rectangle + semicircle

$$= 2rx + \frac{1}{2}\pi r^2$$

Substituting $x = \frac{40 - \pi r - 2r}{2}$:

$$A = r(40 - \pi r - 2r) + \frac{1}{2}\pi r^2$$

$$= 40r - 2r^2 - \frac{1}{2}\pi r^2$$

b For maximum area, $\frac{dA}{dr} = 0$:

$$\frac{dA}{dr} = 40 - 4r - \pi r$$

$$40 - 4r - \pi r = 0$$

9 b $r = \frac{40}{4+\pi}$

When $r = \frac{40}{4+\pi}$,

$$A = 40 \times \frac{40}{4+\pi} - 2\left(\frac{40}{4+\pi}\right)^2 - \frac{1}{2}\pi\left(\frac{40}{4+\pi}\right)^2$$

$$= \frac{1600}{4+\pi} - \left(2 + \frac{1}{2}\pi\right)\left(\frac{40}{4+\pi}\right)^2$$

$$= \frac{1600}{4+\pi} - \frac{4+\pi}{2} \times \frac{1600}{(4+\pi)^2}$$

$$= \frac{1600}{4+\pi} - \frac{800}{4+\pi}$$

$$= \frac{800}{4+\pi}$$

So maximum area $= \frac{800}{4+\pi}$ cm^2

10 a Total length of wire = $(18x + 14y)$ mm

Length = 1512 mm, so

$$18x + 14y = 1512$$

$$y = \frac{1512 - 18x}{14}$$

Total area A mm^2 is given by:

$$A = 2y \times 6x$$

Substituting $y = \frac{1512 - 18x}{14}$:

$$A = 12x\left(\frac{1512 - 18x}{14}\right)$$

$$= 1296x - \frac{108}{7}x^2$$

b For maximum area $\frac{dA}{dx} = 0$:

$$\frac{dA}{dx} = 1296 - \frac{216}{7}x$$

$$1296 - \frac{216}{7}x = 0$$

$$x = \frac{7 \times 1296}{216} = 42$$

When $x = 42$,

$$A = 1296 \times 42 - \frac{108}{7} \times 42^2$$

$$= 27\,216$$

Maximum area = 27 216 mm^2

(Check: $\frac{d^2A}{dx^2} = -\frac{216}{7} < 0 \quad \therefore$ maximum)

Differentiation, Mixed Exercise 12

1 $f(x) = 10x^2$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0}\frac{10(x+h)^2-10x^2}{h}$$
$$= \lim_{h\to 0}\frac{10x^2+20xh+10h^2-10x^2}{h}$$
$$= \lim_{h\to 0}\frac{20xh+10h^2}{h}$$
$$= \lim_{h\to 0}\frac{h(20x+10h)}{h}$$
$$= \lim_{h\to 0}(20x+10h)$$

As $h \to 0$, $20x + 10h \to 20x$
So $f'(x) = 20x$

2 a A has coordinates (1, 4).
The y-coordinate of B is
$(1 + \delta x)^3 + 3(1 + \delta x)$
$= 1^3 + 3\delta x + 3(\delta x)^2 + (\delta x)^3 + 3 + 3\delta x$
$= (\delta x)^3 + 3(\delta x)^2 + 6\delta x + 4$
Gradient of AB

$$= \frac{y_2-y_1}{x_2-x_1}$$
$$= \frac{(\delta x)^3+3(\delta x)^2+6\delta x+4-4}{\delta x}$$
$$= \frac{(\delta x)^3+3(\delta x)^2+6\delta x}{\delta x}$$
$$= (\delta x)^2+3\delta x+6$$

b As $\delta x \to 0$, $(\delta x)^2 + 3\delta x + 6 \to 6$
Therefore, the gradient of the curve at point A is 6.

3 $y = 3x^2 + 3 + \dfrac{1}{x^2} = 3x^2 + 3 + x^{-2}$

$$\frac{dy}{dx} = 6x - 2x^{-3} = 6x - \frac{2}{x^3}$$

When $x = 1$, $\dfrac{dy}{dx} = 6\times 1 - \dfrac{2}{1^3}$
$= 4$

When $x = 2$, $\dfrac{dy}{dx} = 6\times 2 - \dfrac{2}{2^3}$
$= 12 - \dfrac{2}{8}$
$= 11\dfrac{3}{4}$

When $x = 3$, $\dfrac{dy}{dx} = 6\times 3 - \dfrac{2}{3^3}$
$= 18 - \dfrac{2}{27}$
$= 17\dfrac{25}{27}$

The gradients at points A, B and C are 4, $11\frac{3}{4}$ and $17\frac{25}{27}$, respectively.

4 $y = 7x^2 - x^3$

$\dfrac{dy}{dx} = 14x - 3x^2$

$\dfrac{dy}{dx} = 16$ when

$14x - 3x^2 = 16$
$3x^2 - 14x + 16 = 0$
$(3x - 8)(x - 2) = 0$
$x = \dfrac{8}{3}$ or $x = 2$

5 $y = x^3 - 11x + 1$

$\dfrac{dy}{dx} = 3x^2 - 11$

$\dfrac{dy}{dx} = 1$ when

$3x^2 - 11 = 1$
$3x^2 = 12$
$x^2 = 4$
$x = \pm 2$

When $x = 2$, $y = 2^3 - 11(2) + 1 = -13$
When $x = -2$, $y = (-2)^3 - 11(-2) + 1 = 15$
The gradient is 1 at the points $(2, -13)$ and $(-2, 15)$.

6 a $f(x) = x + \dfrac{9}{x} = x + 9x^{-1}$

$f'(x) = 1 - 9x^{-2} = 1 - \dfrac{9}{x^2}$

6 b $f'(x) = 0$ when

$$\frac{9}{x^2} = 1$$
$$x^2 = 9$$
$$x = \pm 3$$

7

$$y = 3\sqrt{x} - \frac{4}{\sqrt{x}} = 3x^{\frac{1}{2}} - 4x^{-\frac{1}{2}}$$
$$\frac{dy}{dx} = \frac{3}{2}x^{-\frac{1}{2}} + \frac{4}{2}x^{-\frac{3}{2}}$$
$$= \frac{3}{2\sqrt{x}} + \frac{2}{\left(\sqrt{x}\right)^3}$$
$$= \tfrac{3}{2}x^{-1} + 2x^{-\frac{3}{2}}$$

8 a

$$y = 12x^{\frac{1}{2}} - x^{\frac{3}{2}}$$
$$\frac{dy}{dx} = 12\left(\frac{1}{2}\right)x^{-\frac{1}{2}} - \frac{3}{2}x^{\frac{1}{2}}$$
$$\frac{dy}{dx} = 6x^{-\frac{1}{2}} - \frac{3}{2}x^{\frac{1}{2}}$$
$$= \frac{3}{2}x^{-\frac{1}{2}}\left(4 - x\right)$$

b The gradient is zero when $\frac{dy}{dx} = 0$:

$$\frac{3}{2}x^{-\frac{1}{2}}\left(4 - x\right) = 0$$

$x = 4$

When $x = 4$, $y = 12 \times 2 - 2^3 = 16$

The gradient is zero at the point with coordinates (4, 16).

9 a

$$\left(x^{\frac{3}{2}} - 1\right)\left(x^{-\frac{1}{2}} + 1\right) = x + x^{\frac{3}{2}} - x^{-\frac{1}{2}} - 1$$

b

$$y = x + x^{\frac{3}{2}} - x^{-\frac{1}{2}} - 1$$
$$\frac{dy}{dx} = 1 + \frac{3}{2}x^{\frac{1}{2}} + \frac{1}{2}x^{-\frac{3}{2}}$$

c When $x = 4$, $\frac{dy}{dx} = 1 + \frac{3}{2} \times 2 + \frac{1}{2} \times \frac{1}{4^{\frac{3}{2}}}$

$$= 1 + 3 + \frac{1}{16}$$
$$= 4\frac{1}{16}$$

10 Let $y = 2x^3 + \sqrt{x} + \frac{x^2 + 2x}{x^2}$

$$= 2x^3 + x^{\frac{1}{2}} + \frac{x^2}{x^2} + \frac{2x}{x^2}$$
$$= 2x^3 + x^{\frac{1}{2}} + 1 + 2x^{-1}$$
$$\frac{dy}{dx} = 6x^2 + \frac{1}{2}x^{-\frac{1}{2}} - 2x^{-2}$$
$$= 6x^2 + \frac{1}{2\sqrt{x}} - \frac{2}{x^2}$$

11 The point (1, 2) lies on the curve with equation $y = ax^2 + bx + c$, so

$2 = a + b + c$ **(1)**

The point (2, 1) also lies on the curve, so

$1 = 4a + 2b + c$ **(2)**

(2) − **(1)** gives:

$-1 = 3a + b$ **(3)**

$\frac{dy}{dx} = 2ax + b$

The gradient of the curve is zero at (2, 1), so

$0 = 4a + b$ **(4)**

(4) − **(3)** gives:

$1 = a$

Substituting $a = 1$ into **(3)** gives $b = -4$

Substituting $a = 1$ and $b = -4$ into **(1)** gives $c = 5$

Therefore, $a = 1$, $b = -4$, $c = 5$

12 a $y = x^3 - 5x^2 + 5x + 2$

$$\frac{dy}{dx} = 3x^2 - 10x + 5$$

b i $\frac{dy}{dx} = 2$

$$3x^2 - 10x + 5 = 2$$
$$3x^2 - 10x + 3 = 0$$
$$(3x - 1)(x - 3) = 0$$
$$x = \frac{1}{3} \text{ or } 3$$

$x = 3$ is the coordinate at P,

so $x = \frac{1}{3}$ at Q.

12 b ii $x = 3 \quad y = 27 - 45 + 15 + 2 = -1$

So equation of the tangent is

$y + 1 = 2(x - 3)$

$y = 2x - 7$

iii When $x = 0$, $y = -7$

and when $y = 0$, $x = \frac{7}{2}$

So points R and S are $(0, -7)$ and $(\frac{7}{2}, 0)$.

Length of $RS = \sqrt{(-7)^2 + \left(\frac{7}{2}\right)^2}$

$= 7\sqrt{1+\frac{1}{4}} = \frac{7}{2}\sqrt{5}$

13 $y = \dfrac{8}{x} - x + 3x^2 = 8x^{-1} - x + 3x^2$

$\dfrac{dy}{dx} = -8x^{-2} - 1 + 6x = -\dfrac{8}{x^2} - 1 + 6x$

When $x = 2$, $y = \dfrac{8}{2} - 2 + 3 \times 2^2 = 14$

$\dfrac{dy}{dx} = -\dfrac{8}{4} - 1 + 12 = 9$

The equation of the tangent through the point (2, 14) with gradient 9 is

$y - 14 = 9(x - 2)$

$y = 9x - 18 + 14$

$y = 9x - 4$

The normal at (2, 14) has gradient $-\dfrac{1}{9}$.

So its equation is

$y - 14 = -\dfrac{1}{9}(x - 2)$

$9y + x = 128$

14 a $2y = 3x^3 - 7x^2 + 4x$

$y = \dfrac{3}{2}x^3 - \dfrac{7}{2}x^2 + 2x$

$\dfrac{dy}{dx} = \dfrac{9}{2}x^2 - 7x + 2$

At (0, 0), $x = 0$, gradient of curve is $0 - 0 + 2 = 2$.

Gradient of normal at (0, 0) is $-\dfrac{1}{2}$.

The equation of the normal at (0, 0) is

$y = -\dfrac{1}{2}x$.

At (1, 0), $x = 1$, gradient of curve is

$\dfrac{9}{2} - 7 + 2 = -\dfrac{1}{2}$.

Gradient of normal at (1, 0) is 2.

14 a The equation of the normal at (1, 0) is $y = 2(x - 1)$.

The normals meet when $y = 2x - 2$ and $y = -\dfrac{1}{2}x$:

$2x - 2 = -\dfrac{1}{2}x$

$4x - 4 = -x$

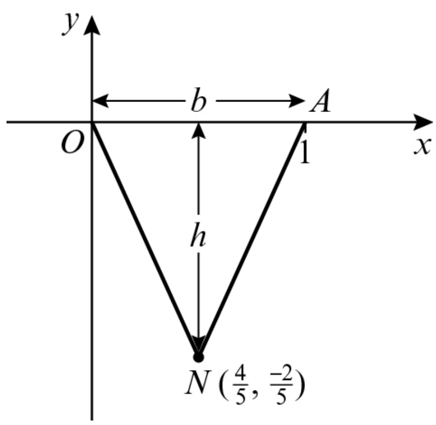
$5x = 4$

$x = \dfrac{4}{5}$

$y = 2\left(\dfrac{4}{5}\right) - 2 = -\dfrac{2}{5} \quad \left(\text{check in } y = -\dfrac{1}{2}x\right)$

N has coordinates $\left(\frac{4}{5}, -\frac{2}{5}\right)$.

b



Area of $\triangle OAN = \dfrac{1}{2}$ base × height

Base (b) = 1

Height (h) = $\dfrac{2}{5}$

Area = $\dfrac{1}{2} \times 1 \times \dfrac{2}{5} = \dfrac{1}{5}$

15 $y = x^3 - 2x^2 - 4x - 1$

When $x = 0$, $y = -1$ so the point P is $(0, -1)$

For the gradient of line L:

$\dfrac{dy}{dx} = 3x^2 - 4x - 4$

At point P, when $x = 0$, $\dfrac{dy}{dx} = -4$

The y-intercept of line L is -1.

Equation of L is $y = -4x - 1$.

Point Q is where the curve and line intersect:

$x^3 - 2x^2 - 4x - 1 = -4x - 1$

$x^3 - 2x^2 = 0$

15 $x^2(x-2)=0$
$x = 0$ or 2
$x = 0$ at point P, so $x = 2$ at point Q.
When $x = 2$, $y = -9$ substituting into the original equation
Using Pythagoras' theorem:

$$\text{distance } PQ = \sqrt{(2-0)^2+(-9-(-1))^2} = \sqrt{68} = \sqrt{4\times 17} = 2\sqrt{17}$$

16 a $y = x^{\frac{3}{2}} + \dfrac{48}{x} \quad (x>0)$

$\dfrac{dy}{dx} = \dfrac{3}{2}x^{\frac{1}{2}} - \dfrac{48}{x^2}$

Putting $\dfrac{dy}{dx} = 0$:

$\dfrac{3}{2}x^{\frac{1}{2}} = \dfrac{48}{x^2}$

$x^{\frac{5}{2}} = 32$
$x = 4$

Substituting $x = 4$ into $y = x^{\frac{3}{2}} + \dfrac{48}{x}$ gives:

$y = 8 + 12 = 20$

So $x = 4$ and $y = 20$ when $\dfrac{dy}{dx} = 0$.

b $\dfrac{d^2y}{dx^2} = \dfrac{3}{4}x^{-\frac{1}{2}} + \dfrac{96}{x^3}$

When $x = 4$, $\dfrac{d^2y}{dx^2} = \dfrac{3}{8} + \dfrac{96}{64} = \dfrac{15}{8} > 0$

$\therefore$ minimum

17 $y = x^3 - 5x^2 + 7x - 14$

$\dfrac{dy}{dx} = 3x^2 - 10x + 7$

Putting $3x^2 - 10x + 7 = 0$
$(3x-7)(x-1) = 0$

So $x = \dfrac{7}{3}$ or $x = 1$

When $x = \dfrac{7}{3}$,

$$y = \left(\frac{7}{3}\right)^3 - 5\left(\frac{7}{3}\right)^2 + 7\left(\frac{7}{3}\right) - 14 = -\frac{329}{27}$$

$y = -12\dfrac{5}{27}$

When $x = 1$,
$y = 1^3 - 5(1)^2 + 7(1) - 14$
$= -11$

So $\left(\frac{7}{3}, -12\frac{5}{27}\right)$ and $(1, -11)$ are stationary points.

18 a $f'(x) = x^2 - 2 + \dfrac{1}{x^2} \quad (x>0)$

$f''(x) = 2x - \dfrac{2}{x^3}$

When $x = 4$, $f''(x) = 8 - \dfrac{2}{64} = 7\dfrac{31}{32}$

b For an increasing function, $f'(x) \geq 0$

$x^2 - 2 + \dfrac{1}{x^2} \geq 0$

$\left(x - \dfrac{1}{x}\right)^2 \geq 0$

This is true for all x, except $x = 1$ (where $f'(1) = 0$).
So the function is an increasing function.

19 $y = x^3 - 6x^2 + 9x$

$\dfrac{dy}{dx} = 3x^2 - 12x + 9$

Putting $3x^2 - 12x + 9 = 0$
$3(x^2 - 4x + 3) = 0$
$3(x-1)(x-3) = 0$
So $x = 1$ or $x = 3$
So there are stationary points when $x = 1$ and $x = 3$.

$\dfrac{d^2y}{dx^2} = 6x - 12$

When $x = 1$, $\dfrac{d^2y}{dx^2} = 6 - 12 = -6 < 0$, so maximum point

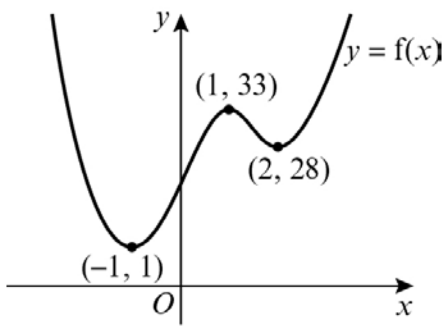
When $x = 3$, $\dfrac{d^2y}{dx^2} = 18 - 12 = 6 > 0$, so minimum point

When $x = 1$, $y = 1 - 6 + 9 = 4$

So $(1, 4)$ is a maximum point.

20 a $f(x) = 3x^4 - 8x^3 - 6x^2 + 24x + 20$
$f'(x) = 12x^3 - 24x^2 - 12x + 24$
$= 12(x^3 - 2x^2 - x + 2)$
$= 12(x - 1)(x^2 - x - 2)$
$= 12(x - 1)(x - 2)(x + 1)$
So $x = 1$, $x = 2$ or $x = -1$
$f(1) = 3 - 8 - 6 + 24 + 20$
$= 33$
$f(2) = 3(2)^4 - 8(2)^3 - 6(2)^2 + 24(2) + 20$
$= 28$
$f(-1) = 3 + 8 - 6 - 24 + 20$
$= 1$
So (1, 33), (2, 28) and (−1, 1) are stationary points.
$f''(x) = 36x^2 - 48x - 12$
$f''(1) = 36 - 48 - 12 = -24 < 0$, so maximum
$f''(2) = 36(2)^2 - 48(2) - 12 = 36 > 0$, so minimum
$f''(-1)$, $y = 36 + 48 - 12 = 72 > 0$, so minimum
So (1, 33) is a maximum point and (2, 28) and (−1, 1) are minimum points.

b



21 a $f(x) = 200 - \frac{250}{x} - x^2$

$f'(x) = \frac{250}{x^2} - 2x$

b At the maximum point, B, $f'(x) = 0$

$\frac{250}{x^2} - 2x = 0$

$\frac{250}{x^2} = 2x$

$250 = 2x^3$
$x^3 = 125$
$x = 5$

When $x = 5$, $y = f(5) = 200 - \frac{250}{5} - 5^2$
$= 125$

21 b The coordinates of B are (5, 125).

22 a P has coordinates m, $\left(x,\ 5 - \frac{1}{2}x^2\right)$.

$OP^2 = (x - 0)^2 + \left(5 - \frac{1}{2}x^2 - 0\right)^2$

$= x^2 + 25 - 5x^2 + \frac{1}{4}x^4$

$= \frac{1}{4}x^4 - 4x^2 + 25$

b Given $f(x) = \frac{1}{4}x^4 - 4x^2 + 25$
$f'(x) = x^3 - 8x$
When $f'(x) = 0$,
$x^3 - 8x = 0$
$x(x^2 - 8) = 0$
$x = 0$ or $x^2 = 8$
$x = 0$ or $x = \pm 2\sqrt{2}$

c $f''(x) = 3x^2 - 8$
When $x = 0$, $f''(x) = -8 < 0$, so maximum
When $x^2 = 8$, $f''(x) = 3 \times 8 - 8 = 16 > 0$, so minimum
Substituting $x^2 = 8$ into $f(x)$:

$OP^2 = \frac{1}{4} \times 8^2 - 4 \times 8 + 25 = 9$

So $OP = 3$ when $x = \pm 2\sqrt{2}$

23 a $y = 3 + 5x + x^2 - x^3$
Let $y = 0$, then
$3 + 5x + x^2 - x^3 = 0$
$(3 - x)(1 + 2x + x^2) = 0$
$(3 - x)(1 + x)^2 = 0$
$x = 3$ or $x = -1$ when $y = 0$
The curve touches the x-axis at $x = -1$ (A) and cuts the axis at $x = 3$ (C).
C has coordinates (3, 0)

b $\frac{dy}{dx} = 5 + 2x - 3x^2$

Putting $\frac{dy}{dx} = 0$

$5 + 2x - 3x^2 = 0$
$(5 - 3x)(1 + x) = 0$

So $x = \frac{5}{3}$ or $x = -1$

When $x = \frac{5}{3}$,

23 b $y = 3+5\left(\frac{5}{3}\right)+\left(\frac{5}{3}\right)^2-\left(\frac{5}{3}\right)^3 = 9\frac{13}{27}$

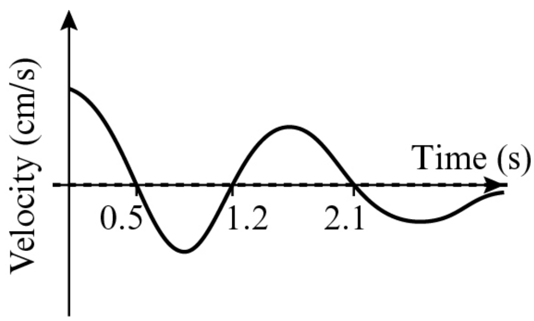
So B is $\left(\frac{5}{3}, 9\frac{13}{27}\right)$.

When $x = -1$, $y = 0$

So A is $(-1, 0)$.

24

x	$y = f(x)$	$y = f'(x)$
$0 < x < 0.5$	Positive gradient	Above x-axis
$x = 0.5$	Maximum	Cuts x-axis
$0.5 < x < 1.2$	Negative gradient	Below x-axis
$x = 1.2$	Minimum	Cuts x-axis
$1.2 < x < 2.1$	Positive gradient	Above x-axis
$x = 2.1$	Maximum	Cuts x-axis
$x > 2.1$	Negative gradient	Below x-axis with asymptote at $y = 0$



25 $V = \pi(40r - r^2 - r^3)$

$\frac{dV}{dr} = 40\pi - 2\pi r - 3\pi r^2$

Putting $\frac{dV}{dr} = 0$

$\pi(40 - 2r - 3r^2) = 0$

$(4 + r)(10 - 3r) = 0$

$r = \frac{10}{3}$ or $r = -4$

As r is positive, $r = \frac{10}{3}$

Substituting into the given expression for V:

$V = \pi\left(40\times\frac{10}{3}-\frac{100}{9}-\frac{1000}{27}\right) = \frac{2300}{27}\pi$

26 $A = 2\pi x^2 + \frac{2000}{x} = 2\pi x^2 + 2000x^{-1}$

$\frac{dA}{dx} = 4\pi x - 2000x^{-2} = 4\pi x - \frac{2000}{x^2}$

Putting $\frac{dA}{dx} = 0$

$4\pi x = \frac{2000}{x^2}$

$x^3 = \frac{2000}{4\pi} = \frac{500}{\pi}$

27 a The total length of wire is

$\left(2y + x + \frac{\pi x}{2}\right)$ m

As total length is 2 m,

$2y + x\left(1+\frac{\pi}{2}\right) = 2$

$y = 1 - \frac{1}{2}x\left(1+\frac{\pi}{2}\right)$

b Area, $R = xy + \frac{1}{2}\pi\left(\frac{x}{2}\right)^2$

Substituting $y = 1-\frac{1}{2}x\left(1+\frac{\pi}{2}\right)$ gives:

$R = x\left(1-\frac{1}{2}x-\frac{\pi}{4}x\right)+\frac{\pi}{8}x^2$

$= \frac{x}{8}(8 - 4x - 2\pi x + \pi x)$

$= \frac{x}{8}(8 - 4x - \pi x)$

c For maximum R, $\frac{dR}{dx} = 0$

$R = x - \frac{1}{2}x^2 - \frac{\pi}{8}x^2$

$\frac{dR}{dx} = 1 - x - \frac{\pi}{4}x$

Putting $\frac{dR}{dx} = 0$

$x = \frac{1}{1+\frac{\pi}{4}}$

$= \frac{4}{4+\pi}$

27 c Substituting $x = \dfrac{4}{4+\pi}$ into R:

$$R = \frac{1}{2(4+\pi)}\left(8 - \frac{16}{4+\pi} - \frac{4\pi}{4+\pi}\right)$$
$$R = \frac{1}{2(4+\pi)} \times \frac{32 + 8\pi - 16 - 4\pi}{4+\pi}$$
$$= \frac{1}{2(4+\pi)} \times \frac{16+4\pi}{4+\pi}$$
$$= \frac{4(4+\pi)}{2(4+\pi)^2}$$
$$= \frac{2}{4+\pi}$$

28 a Let the height of the tin be h cm.
The area of the curved surface of the tin = $2\pi xh$ cm^2
The area of the base of the tin = πx^2 cm^2
The area of the curved surface of the lid = $2\pi x$ cm^2
The area of the top of the lid = πx^2 cm^2
Total area of sheet metal is 80π cm^2.
So $2\pi x^2 + 2\pi x + 2\pi xh = 80\pi$

$$h = \frac{40 - x - x^2}{x}$$

The volume, V, of the tin is given by
$V = \pi x^2 h$
$$= \frac{\pi x^2(40 - x - x^2)}{x}$$
$$= \pi(40x - x^2 - x^3)$$

b $\dfrac{dV}{dx} = \pi(40 - 2x - 3x^2)$

Putting $\dfrac{dV}{dx} = 0$
$40 - 2x - 3x^2 = 0$
$(10 - 3x)(4 + x) = 0$
So $x = \dfrac{10}{3}$ or $x = -4$

But x is positive, so $x = \dfrac{10}{3}$

c $\dfrac{d^2V}{dx^2} = \pi(-2 - 6x)$

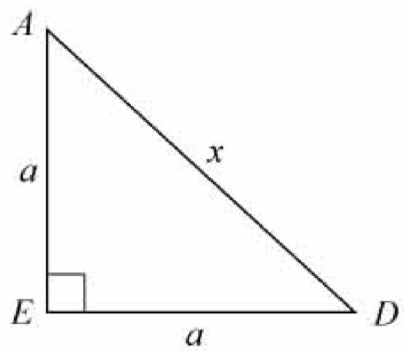
When $x = \dfrac{10}{3}$, $\dfrac{d^2V}{dx^2} = \pi(-2 - 20) < 0$
So V is a maximum.

28 d $V = \pi\left(40 \times \dfrac{10}{3} - \left(\dfrac{10}{3}\right)^2 - \left(\dfrac{10}{3}\right)^3\right)$
$$= \pi\left(\frac{400}{3} - \frac{100}{9} - \frac{1000}{27}\right)$$
$$= \frac{2300}{27}\pi$$

e Lid has surface area $\pi x^2 + 2\pi x$
When $x = \dfrac{10}{3}$,
this is $\pi\left(\dfrac{100}{9} + \dfrac{20}{3}\right) = \dfrac{160}{9}\pi$

Percentage of total surface area =
$$\frac{\frac{160}{9}\pi}{80\pi} \times 100 = \frac{200}{9} = 22.2...\%$$

29 a Let the equal sides of ΔADE be a metres.



Using Pythagoras' theorem,
$a^2 + a^2 = x^2$
$2a^2 = x^2$
$a^2 = \dfrac{x^2}{2}$

Area of $\Delta ADE = \dfrac{1}{2} \times$ base $\times$ height
$= \dfrac{1}{2} \times a \times a$
$= \dfrac{x^2}{4}$ m^2

b Area of two triangular ends
$= 2 \times \dfrac{x^2}{4} = \dfrac{x^2}{2}$
Let the length $AB = CD = y$ metres

29 b Area of two rectangular sides

$= 2 \times ay = 2ay = 2\sqrt{\frac{x^2}{2}}\,y$

So $S = \frac{x^2}{2} + 2\sqrt{\frac{x^2}{2}}\,y = \frac{x^2}{2} + xy\sqrt{2}$

But capacity of storage tank $= \frac{1}{4}x^2 \times y$

So $\frac{1}{4}x^2y = 4000$

$y = \frac{16\,000}{x^2}$

Substituting for y in equation for S gives:

$S = \frac{x^2}{2} + \frac{16\,000\sqrt{2}}{x}$

c $\frac{dS}{dx} = x - \frac{16\,000\sqrt{2}}{x^2}$

Putting $\frac{dS}{dx} = 0$

$x = \frac{16\,000\sqrt{2}}{x^2}$

$x^3 = 16\,000\sqrt{2}$

$x = 20\sqrt{2} = 28.28$ (4 s.f.)

When $x = 20\sqrt{2}$,

$S = 400 + 800 = 1200$

d $\frac{d^2S}{dx^2} = 1 + \frac{32\,000\sqrt{2}}{x^3}$

When $x = 20\sqrt{2}$, $\frac{d^2S}{dx^2} = 3 > 0$, so value is a minimum.

Challenge

a $(x+h)^7 = x^7 + \binom{7}{1}x^6h + \binom{7}{2}x^5h^2 + \binom{7}{3}x^4h^3 + \ldots$

$= x^7 + 7x^6h + 21x^5h^2 + 35x^4h^3 + \ldots$

b $$\frac{\mathrm{d}(x^7)}{\mathrm{d}x} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{(x+h)^7 - x^7}{h}$$

$$= \lim_{h\to 0}\frac{x^7 + 7x^6h + 21x^5h^2 + 35x^4h^3 - x^7}{h}$$

$$= \lim_{h\to 0}\frac{7x^6h + 21x^5h^2 + 35x^4h^3}{h}$$

$$= \lim_{h\to 0}\frac{h(7x^6 + 21x^5h + 35x^4h^2)}{h}$$

$$= \lim_{h\to 0}(7x^6 + 21x^5h + 35x^4h^2)$$

As $h \to 0$, $7x^6 + 21x^5h + 35x^4h^2 \to 7x^6$, so $\frac{\mathrm{d}(x^7)}{\mathrm{d}x} = 7x^6$

Integration 13A

1 a $\frac{\mathrm{d}y}{\mathrm{d}x} = x^5$

$y = \frac{x^6}{6} + c$

b $\frac{\mathrm{d}y}{\mathrm{d}x} = 10x^4$

$y = \frac{10x^5}{5} + c$

$y = 2x^5 + c$

c $\frac{\mathrm{d}y}{\mathrm{d}x} = -x^{-2}$

$y = -\frac{x^{-1}}{-1} + c$

$y = x^{-1} + c$ or

$y = \frac{1}{x} + c$

d $\frac{\mathrm{d}y}{\mathrm{d}x} = -4x^{-3}$

$y = \frac{-4x^{-2}}{-2} + c$

$y = 2x^{-2} + c$ or

$y = \frac{2}{x^2} + c$

e $\frac{\mathrm{d}y}{\mathrm{d}x} = x^{\frac{2}{3}}$

$y = \frac{x^{\frac{5}{3}}}{\frac{5}{3}} + c$

$y = \frac{3}{5}x^{\frac{5}{3}} + c$

f $\frac{\mathrm{d}y}{\mathrm{d}x} = 4x^{\frac{1}{2}}$

$y = 4\frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c$

$y = \frac{8}{3}x^{\frac{3}{2}} + c$

g $\frac{\mathrm{d}y}{\mathrm{d}x} = -2x^6$

$y = -\frac{x^7}{7} + c$

$y = -\frac{2}{7}x^7 + c$

h $\frac{\mathrm{d}y}{\mathrm{d}x} = x^{-\frac{1}{2}}$

$y = \frac{x^{+\frac{1}{2}}}{\frac{1}{2}} + c$

$y = 2x^{\frac{1}{2}} + c$ or

$y = 2\sqrt{x} + c$

i $\frac{\mathrm{d}y}{\mathrm{d}x} = 5x^{-\frac{3}{2}}$

$y = \frac{5x^{-\frac{1}{2}}}{-\frac{1}{2}} + c$

$y = -10x^{-\frac{1}{2}} + c$ or

$y = -\frac{10}{\sqrt{x}} + c$

j $\frac{\mathrm{d}y}{\mathrm{d}x} = 6x^{\frac{1}{3}}$

$y = \frac{6x^{\frac{4}{3}}}{\frac{4}{3}} + c$

$y = \frac{18}{4}x^{\frac{4}{3}} + c$

$y = \frac{9}{2}x^{\frac{4}{3}} + c$

k $\frac{\mathrm{d}y}{\mathrm{d}x} = 36x^{11}$

$y = \frac{36x^{12}}{12} + c$

$y = \frac{18}{6}x^{12} + c$

$y = 3x^{12} + c$

1 l $\frac{dy}{dx} = -14x^{-8}$

$y = \frac{-14x^{-7}}{-7} + c$

$y = 2x^{-7} + c$ or

$y = \frac{2}{x^7} + c$

m $\frac{dy}{dx} = -3x^{-\frac{2}{3}}$

$y = \frac{-3x^{\frac{1}{3}}}{\frac{1}{3}} + c$

$y = -9x^{\frac{1}{3}} + c$

n $\frac{dy}{dx} = -5$

$= -5x^0$

$y = \frac{-5x^1}{1} + c$

$y = -5x + c$

o $\frac{dy}{dx} = 6x$

$y = \frac{6x^2}{2} + c$

$y = 3x^2 + c$

p $\frac{dy}{dx} = 2x^{-0.4}$

$y = \frac{2x^{0.6}}{0.6} + c$

$y = \frac{20}{6}x^{0.6} + c$

$y = \frac{10}{3}x^{0.6} + c$

2 a $\frac{dy}{dx} = x^3 - \frac{3}{2}x^{-\frac{1}{2}} - 6x^{-2}$

$y = \frac{x^4}{4} - \frac{3}{2}\frac{x^{+\frac{1}{2}}}{\frac{1}{2}} - 6\frac{x^{-1}}{-1} + c$

$y = \frac{1}{4}x^4 - 3x^{\frac{1}{2}} + 6x^{-1} + c$

2 b $\frac{dy}{dx} = 4x^3 + x^{-\frac{2}{3}} - x^{-2}$

$y = \frac{4x^4}{4} + \frac{x^{\frac{1}{3}}}{\frac{1}{3}} - \frac{x^{-1}}{-1} + c$

$y = x^4 + 3x^{\frac{1}{3}} + x^{-1} + c$

c $\frac{dy}{dx} = 4 - 12x^{-4} + 2x^{-\frac{1}{2}}$

$y = 4x - \frac{12x^{-3}}{-3} + \frac{2x^{\frac{1}{2}}}{\frac{1}{2}} + c$

$y = 4x + 4x^{-3} + 4x^{\frac{1}{2}} + c$

d $\frac{dy}{dx} = 5x^{\frac{2}{3}} - 10x^4 + x^{-3}$

$y = \frac{5x^{\frac{5}{3}}}{\frac{5}{3}} - \frac{10x^5}{5} + \frac{x^{-2}}{-2} + c$

$y = 3x^{\frac{5}{3}} - 2x^5 - \frac{1}{2}x^{-2} + c$

e $\frac{dy}{dx} = -\frac{4}{3}x^{-\frac{4}{3}} - 3 + 8x$

$y = -\frac{4}{3}\frac{x^{-\frac{1}{3}}}{-\frac{1}{3}} - 3x + \frac{8x^2}{2} + c$

$y = 4x^{-\frac{1}{3}} - 3x + 4x^2 + c$

f $\frac{dy}{dx} = 5x^4 - x^{-\frac{3}{2}} - 12x^{-5}$

$y = \frac{5x^5}{5} - \frac{x^{-\frac{1}{2}}}{-\frac{1}{2}} - \frac{12x^{-4}}{-4} + c$

$y = x^5 + 2x^{-\frac{1}{2}} + 3x^{-4} + c$

3 a $f'(x) = 12x + \frac{3}{2}x^{-\frac{3}{2}} + 5$

$f(x) = \frac{12x^2}{2} + \frac{3}{2}\frac{x^{-\frac{1}{2}}}{-\frac{1}{2}} + 5x + c$

$f(x) = 6x^2 - 3x^{-\frac{1}{2}} + 5x + c$

3 b $f'(x) = 6x^5 + 6x^{-7} - \frac{1}{6}x^{-\frac{7}{6}}$

$f(x) = \frac{6x^6}{6} + \frac{6x^{-6}}{-6} - \frac{1}{6}\frac{x^{-\frac{1}{6}}}{-\frac{1}{6}} + c$

$f(x) = x^6 - x^{-6} + x^{-\frac{1}{6}} + c$

c $f'(x) = \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}}$

$f(x) = \frac{1}{2}\frac{x^{\frac{1}{2}}}{\frac{1}{2}} - \frac{1}{2}\frac{x^{-\frac{1}{2}}}{-\frac{1}{2}} + c$

$f(x) = x^{\frac{1}{2}} + x^{-\frac{1}{2}} + c$

d $f'(x) = 10x^4 + 8x^{-3}$

$f(x) = \frac{10x^5}{5} + \frac{8x^{-2}}{-2} + c$

$f(x) = 2x^5 - 4x^{-2} + c$

e $f'(x) = 2x^{-\frac{1}{3}} + 4x^{-\frac{5}{3}}$

$f(x) = \frac{2x^{\frac{2}{3}}}{\frac{2}{3}} + \frac{4x^{-\frac{2}{3}}}{-\frac{2}{3}} + c$

$f(x) = 3x^{\frac{2}{3}} - 6x^{-\frac{2}{3}} + c$

f $f'(x) = 9x^2 + 4x^{-3} + \frac{1}{4}x^{-\frac{1}{2}}$

$f(x) = \frac{9x^3}{3} + \frac{4x^{-2}}{-2} + \frac{1}{4}\frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c$

$f(x) = 3x^3 - 2x^{-2} + \frac{1}{2}x^{\frac{1}{2}} + c$

4 $\frac{dy}{dx} = (2x + 3)^2$

$= 4x^2 + 12x + 9$

$y = \frac{4x^3}{3} + 6x^2 + 9x + c$

5 $f'(x) = 3x^{-2} + 6x^{\frac{1}{2}} + x - 4$

$f(x) = \frac{3x^{-1}}{-1} + \frac{6x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{x^2}{2} - 4x + c$

$= -3x^{-1} + 4x^{\frac{3}{2}} + \frac{1}{2}x^2 - 4x + c$

Challenge

$\frac{dy}{dx} = (2\sqrt{x} - x^2)\left(\frac{3+x}{x^5}\right)$

$= (2x^{\frac{1}{2}} - x^2)\left(3x^{-5} + x^{-4}\right)$

$= 6x^{-\frac{9}{2}} + 2x^{-\frac{7}{2}} - 3x^{-3} - x^{-2}$

$y = \frac{6x^{-\frac{7}{2}}}{-\frac{7}{2}} + \frac{2x^{-\frac{5}{2}}}{-\frac{5}{2}} - \frac{3x^{-2}}{-2} - \frac{x^{-1}}{-1} + c$

$= -\frac{12}{7x^{\frac{7}{2}}} - \frac{4}{5x^{\frac{5}{2}}} + \frac{3}{2x^2} + \frac{1}{x} + c$

Integration 13B

1 a $\int x^3 \mathrm{d}x = \frac{x^4}{4} + c$

b $\int x^7 \mathrm{d}x = \frac{x^8}{8} + c$

c $\int 3x^{-4} \mathrm{d}x = \frac{3x^{-3}}{-3} + c = -x^{-3} + c$

d $\int 5x^2 \mathrm{d}x = \frac{5x^3}{3} + c$

2 a $\int (x^4 + 2x^3)\, \mathrm{d}x = \frac{5x^5}{5} + \frac{2x^4}{4} + c$
$= x^5 + \frac{x^4}{2} + c$

b $\int (2x^3 - x^2 + 5x)\, \mathrm{d}x = \frac{2x^4}{4} - \frac{x^3}{3} + \frac{5x^2}{2} + c$
$= \frac{x^4}{2} - \frac{x^3}{3} + \frac{5x^2}{2} + c$

c $\int (5x^{\frac{3}{2}} - 3x^2)\, \mathrm{d}x = \frac{5x^{\frac{5}{2}}}{\frac{5}{2}} - \frac{3x^3}{3} + c$
$= 2x^{\frac{5}{2}} - x^3 + c$

3 a $\int (4x^{-2} + 3x^{-\frac{1}{2}})\, \mathrm{d}x = \frac{4x^{-1}}{-1} + \frac{3x^{\frac{1}{2}}}{\frac{1}{2}} + c$
$= -4x^{-1} + 6x^{\frac{1}{2}} + c$

b $\int (6x^{-1} - x^{-\frac{1}{2}})\, \mathrm{d}x = \frac{6x^{-1}}{-1} - \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c$
$= -6x^{-1} - 2x^{\frac{1}{2}} + c$

c $\int (2x^{-\frac{3}{2}} + x^2 - x^{-\frac{1}{2}})\, \mathrm{d}x$
$= \frac{2x^{-\frac{1}{2}}}{-\frac{1}{2}} + \frac{x^3}{3} - \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c$
$= -4x^{-\frac{1}{2}} + \frac{x^3}{3} - 2x^{\frac{1}{2}} + c$

4 a $\int (4x^3 - 3x^{-4} + r)\, \mathrm{d}x = \frac{4x^4}{4} - \frac{3x^{-3}}{-3} + rx + c$
$= x^4 + x^{-3} + rx + c$

b $\int (x + x^{-\frac{1}{2}} + x^{-\frac{3}{2}})\, \mathrm{d}x = \frac{x^2}{2} + \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + \frac{x^{-\frac{1}{2}}}{-\frac{1}{2}} + c$
$= \frac{x^2}{2} + 2x^{\frac{1}{2}} - 2x^{-\frac{1}{2}} + c$

c $\int (px^4 + 2q + 3x^{-2})\, \mathrm{d}x$
$= \frac{px^5}{5} + \frac{4qx^2}{2} + \frac{3x^{-1}}{-1} + c$
$= \frac{px^5}{5} + 2qx^2 - 3x^{-1} + c$

5 a $\int (3t^2 - t^{-2})\, \mathrm{d}t = \frac{3t^3}{3} - \frac{t^{-1}}{-1} + c$
$= t^3 + t^{-1} + c$

b $\int (2t^2 - 3t^{-\frac{3}{2}} + 1)\, \mathrm{d}t = \frac{2t^3}{3} - \frac{3t^{-\frac{1}{2}}}{-\frac{1}{2}} + t + c$
$= \frac{2t^3}{3} + 6t^{-\frac{1}{2}} + t + c$

c $\int (pt^3 + q^2 + pr^3)\, \mathrm{d}t$
$= \frac{pt^4}{4} + q^2 t + pr^3 t + c$

6 a $\int \frac{(2x^2+3)}{x^2}\, \mathrm{d}x = \int \left(\frac{2x^2}{x^2} + \frac{3}{x^2}\right) \mathrm{d}x$
$= \int \left(2 + 3x^{-2}\right) \mathrm{d}x$
$= 2x + \frac{3x^{-1}}{-1} + c$
$= 2x - \frac{3}{x} + c$

b $\int (2x+3)^2 \mathrm{d}x = \int \left(4x^2 + 12x + 9\right) \mathrm{d}x$
$= \frac{4x^3}{3} + 6x^2 + 9x + c$

6 c $\int(2x+3)\sqrt{x}\,dx=\int(2x+3)x^{\frac{1}{2}}\,dx$

$=\int\left(2x^{\frac{3}{2}}+3x^{\frac{1}{2}}\right)dx$

$=\frac{2x^{\frac{5}{2}}}{\frac{5}{2}}+\frac{3x^{\frac{3}{2}}}{\frac{3}{2}}+c$

$=\frac{4x^{\frac{5}{2}}}{5}+2x^{\frac{3}{2}}+c$

7 a $\int\left(x+\frac{1}{x}\right)^2dx=\int\left(x^2+2+\frac{1}{x^2}\right)dx$

$=\int\left(x^2+2+x^{-2}\right)dx$

$=\frac{x^3}{3}+2x+\frac{x^{-1}}{-1}+c$

$=\frac{x^3}{3}+2x-\frac{1}{x}+c$

b $\int\left(\sqrt{x}+2\right)^2dx=\int\left(x+4\sqrt{x}+4\right)dx$

$=\int\left(x+4x^{\frac{1}{2}}+4\right)dx$

$=\frac{x^2}{2}+\frac{4x^{\frac{3}{2}}}{\frac{3}{2}}+4x+c$

$=\frac{x^2}{2}+\frac{8x^{\frac{3}{2}}}{3}+4x+c$

c $\int\left(\frac{1}{\sqrt{x}}+2\sqrt{x}\right)dx=\int(x^{-\frac{1}{2}}+2x^{\frac{1}{2}})\,dx$

$=\frac{x^{\frac{1}{2}}}{\frac{1}{2}}+\frac{2x^{\frac{3}{2}}}{\frac{3}{2}}+c$

$=2x^{\frac{1}{2}}+\frac{4x^{\frac{3}{2}}}{3}+c$

8 a $\int\left(x^{\frac{2}{3}}+\frac{4}{x^3}\right)dx=\int\left(x^{\frac{2}{3}}+4x^{-3}\right)dx$

$=\frac{x^{\frac{5}{3}}}{\frac{5}{3}}+\frac{4x^{-2}}{-2}+c$

$=\frac{3x^{\frac{5}{3}}}{5}-\frac{2}{x^2}+c$

8 b $\int\left(\frac{2+x}{x^3}+3\right)dx=\int\left(2x^{-3}+x^{-2}+3\right)dx$

$=\frac{x^{-2}}{-2}+\frac{x^{-1}}{-1}+3x+c$

$=-\frac{1}{x^2}-\frac{1}{x}+3x+c$

c $\int\left(x^2+3\right)\left(x-1\right)dx=\int\left(x^3-x^2+3x-3\right)dx$

$=\frac{x^4}{4}-\frac{x^3}{3}+\frac{3x^2}{2}-3x+c$

d $\int\frac{(2x+1)^2}{\sqrt{x}}dx=\int\left(\frac{4x^2+4x+1}{x^{\frac{1}{2}}}\right)dx$

$=\int(4x^{\frac{3}{2}}+4x^{\frac{1}{2}}+x^{\frac{-1}{2}})\,dx$

$=\frac{8x^{\frac{5}{2}}}{5}+\frac{8x^{\frac{3}{2}}}{3}+2x^{\frac{1}{2}}+c$

e $\int\left(3+\frac{\sqrt{x}+6x^3}{x}\right)dx=\int(3+x^{\frac{-1}{2}}+6x^2)\,dx$

$=3x+2x^{\frac{1}{2}}+2x^3+c$

f $\int\sqrt{x}\left(\sqrt{x}+3\right)^2dx=\int x^{\frac{1}{2}}\left(x+6x^{\frac{1}{2}}+9\right)dx$

$=\int(x^{\frac{3}{2}}+6x+9x^{\frac{1}{2}})\,dx$

$=\frac{2x^{\frac{5}{2}}}{5}+3x^2+6x^{\frac{3}{2}}+c$

9 a $\int\left(\frac{A}{x^2}-3\right)dx=\int(Ax^{-2}-3)\,dx$

$=\frac{Ax^{-1}}{-1}-3x+c$

$=-\frac{A}{x}-3x+c$

b $\int\left(\sqrt{Px}+\frac{2}{x^3}\right)dx=\int\left(\sqrt{P}x^{\frac{1}{2}}+2x^{-3}\right)dx$

$=\frac{\sqrt{P}x^{\frac{3}{2}}}{\frac{3}{2}}+\frac{2x^{-2}}{-2}+c$

$=\frac{2\sqrt{P}x^{\frac{3}{2}}}{3}-\frac{1}{x^2}+c$

9 c $\int\left(\frac{p}{x^2}+q\sqrt{x}+r\right)\mathrm{d}x$

$=\frac{px^{-1}}{-1}+\frac{qx^{\frac{3}{2}}}{\frac{3}{2}}+rx+c$

$=-\frac{p}{x}+\frac{2qx^{\frac{3}{2}}}{3}+rx+c$

10 $\int\left(\frac{6}{x^2}+4\sqrt{x}-3x+2\right)\mathrm{d}x$

$=\int(6x^{-2}+4x^{\frac{1}{2}}-3x+2)\ \mathrm{d}x$

$=\frac{6x^{-1}}{-1}+\frac{4x^{\frac{3}{2}}}{\frac{3}{2}}-\frac{3x^2}{2}+2x+c$

$=-\frac{6}{x}+\frac{8x^{\frac{3}{2}}}{3}-\frac{3x^2}{2}+2x+c$

11 $\int\left(8x^3+6x-\frac{3}{\sqrt{x}}\right)\mathrm{d}x$

$=\int(8x^3+6x-3x^{-\frac{1}{2}})\ \mathrm{d}x$

$=\frac{8x^4}{4}+\frac{6x^2}{2}-\frac{3x^{\frac{1}{2}}}{\frac{1}{2}}+c$

$=2x^4+3x^2-6\sqrt{x}+c$

12 a $(2+5\sqrt{x})^2=(2+5\sqrt{x})(2+5\sqrt{x})$

$=4+10\sqrt{x}+10\sqrt{x}+25x$

$=4+20\sqrt{x}+25x$

So $k=20$

b $\int(4+20\sqrt{x}+25x)\ \mathrm{d}x$

$=\int(4+20x^{\frac{1}{2}}+25x)\ \mathrm{d}x$

$=4x+\frac{20x^{\frac{3}{2}}}{\frac{3}{2}}+\frac{25x^2}{2}+c$

$=4x+\frac{40x^{\frac{3}{2}}}{3}+\frac{25x^2}{2}+c$

13 $\int\left(3x^5-\frac{4}{\sqrt{x}}\right)\mathrm{d}x=\int(3x^5-4x^{-\frac{1}{2}})\ \mathrm{d}x$

$=\frac{3x^6}{6}-\frac{4x^{\frac{1}{2}}}{\frac{1}{2}}+c$

$=\frac{x^6}{2}-8\sqrt{x}+c$

14 $\int\left(\frac{p}{2x^2}+pq\right)\mathrm{d}x=\int\left(\frac{px^{-2}}{2}+pq\right)\mathrm{d}x$

$=\frac{px^{-1}}{2\times-1}+pqx+c$

$=-\frac{p}{2x}+pqx+c$

$=\frac{2}{x}+10x+c$

$-\frac{p}{2}=2$ and $pq=10$

$p=-4$ and $q=-2.5$

15 a Using the binomial expansion:

$\mathrm{f}(x)=2^{10}+\binom{10}{1}2^9(-x)^1+\binom{10}{2}2^8(-x)^2$

$=1024-5120x+11520x^2$

b $\int(1024-5120x+11520x^2)\mathrm{d}x$

$=1024x-\frac{5120x^2}{2}+\frac{11520x^3}{3}+c$

$=1024x-2560x^2+3840x^3+c$

So $A=1024$, $B=-2560$ and $C=3840$

Integration 13C

1 a $\frac{dy}{dx} = 3x^2 + 2x$

$\Rightarrow y = \frac{3}{3}x^3 + \frac{2}{2}x^2 + c$

So $y = x^3 + x^2 + c$

$x = 2,\ y = 10 \Rightarrow 10 = 8 + 4 + c$

So $c = -2$

So the equation is $y = x^3 + x^2 - 2$

b $\frac{dy}{dx} = 4x^3 + \frac{2}{x^3} + 3$

$\Rightarrow y = \frac{4}{4}x^4 + \frac{2}{-2}x^{-2} + 3x + c$

So $y = x^4 - x^{-2} + 3x + c$

$x = 1,\ y = 4 \Rightarrow 4 = 1 - 1 + 3 + c$

So $c = 1$

So the equation is $y = x^4 - x^{-2} + 3x + 1$

or $y = x^4 - \frac{1}{x^2} + 3x + 1$

c $\frac{dy}{dx} = \sqrt{x} + \frac{1}{4}x^2$

$\Rightarrow y = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{1}{4}\frac{x^3}{3} + c$

So $y = \frac{2}{3}x^{\frac{3}{2}} + \frac{1}{12}x^3 + c$

$x = 4, y = 11 \Rightarrow 11 = \frac{2}{3}\times 2^3 + \frac{1}{12}\times 4^3 + c$

So $c = 11 - \frac{16}{3} - \frac{16}{3} = \frac{1}{3}$

So the equation is $y = \frac{2}{3}x^{\frac{3}{2}} + \frac{1}{12}x^3 + \frac{1}{3}$

d $\frac{dy}{dx} = \frac{3}{\sqrt{x}} - x$

$\Rightarrow y = 3\frac{x^{\frac{1}{2}}}{\frac{1}{2}} - \frac{1}{2}x^2 + c$

So $y = 6\sqrt{x} - \frac{1}{2}x^2 + c$

$x = 4, y = 0 \Rightarrow 0 = 6\times 2 - \frac{1}{2}\times 16 + c$

So $c = -4$

So the equation is $y = 6\sqrt{x} - \frac{1}{2}x^2 - 4$

e $\frac{dy}{dx} = (x+2)^2$

$= x^2 + 4x + 4$

$\Rightarrow y = \frac{1}{3}x^3 + 2x^2 + 4x + c$

$x = 1, y = 7 \Rightarrow 7 = \frac{1}{3} + 2 + 4 + c$

So $c = \frac{2}{3}$

So the equation is $y = \frac{1}{3}x^3 + 2x^2 + 4x + \frac{2}{3}$

f $\frac{dy}{dx} = \frac{x^2 + 3}{\sqrt{x}} = x^{\frac{3}{2}} + 3x^{-\frac{1}{2}}$

$\Rightarrow y = \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + 3\frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c$

So $y = \frac{2}{5}x^{\frac{5}{2}} + 6x^{\frac{1}{2}} + c$

$x = 0,\ y = 1 \Rightarrow 1 = \frac{2}{5}\times 0 + 6\times 0 + c$

So $c = 1$

So the equation is $y = \frac{2}{5}x^{\frac{5}{2}} + 6x^{\frac{1}{2}} + 1$

2 $f'(x) = 2x^3 - \frac{1}{x^2}$

$= 2x^3 - x^{-2}$

So $f(x) = \frac{2}{4}x^4 - \frac{x^{-1}}{-1} + c = \frac{1}{2}x^4 + \frac{1}{x} + c$

But $f(1) = 2$

So $2 = \frac{1}{2} + 1 + c$

$\Rightarrow c = \frac{1}{2}$

So $f(x) = \frac{1}{2}x^4 + \frac{1}{x} + \frac{1}{2}$

3 $\frac{dy}{dx} = \frac{\sqrt{x} + 3}{x^2}$

$= x^{-\frac{3}{2}} + 3x^{-2}$

$\Rightarrow y = \frac{x^{-\frac{1}{2}}}{-\frac{1}{2}} + 3\frac{x^{-1}}{-1} + c$

So $y = -2x^{-\frac{1}{2}} - 3x^{-1} + c$

$= -\frac{2}{\sqrt{x}} - \frac{3}{x} + c$

3 $x = 9, y = 0 \Rightarrow 0 = -\frac{2}{3} - \frac{3}{9} + c$

So $c = \frac{2}{3} + \frac{1}{3} = 1$

So the equation is $y = 1 - \dfrac{2}{\sqrt{x}} - \dfrac{3}{x}$

4 $y = \int (9x^2 + 4x - 3)\,dx$

$= \dfrac{9x^3}{3} + \dfrac{4x^2}{2} - 3x + c$

$= 3x^3 + 2x^2 - 3x + c$

When $x = -1$ and $y = 0$,

$0 = 3(-1)^3 + 2(-1)^2 - 3(-1) + c$

$-3 + 2 + 3 + c = 0$

$c = -2$

$f(x) = 3x^3 + 2x^2 - 3x - 2$

5 $y = \int (3x^{-\frac{1}{2}} - 2x\sqrt{x})\,dx$

$= \int (3x^{-\frac{1}{2}} - 2x^{\frac{3}{2}})\,dx$

$= \dfrac{3x^{\frac{1}{2}}}{\frac{1}{2}} - \dfrac{2x^{\frac{5}{2}}}{\frac{5}{2}} + c$

$= 6x^{\frac{1}{2}} - \frac{4}{5}x^{\frac{5}{2}} + c$

When $x = 4$ and $y = 10$,

$10 = 6(4)^{\frac{1}{2}} - \frac{4}{5}(4)^{\frac{5}{2}} + c$

$12 - \frac{128}{5} + c = 10$

$c = \frac{118}{5}$

$y = 6x^{\frac{1}{2}} - \frac{4}{5}x^{\frac{5}{2}} + \frac{118}{5}$

6 a $\dfrac{6x + 5x^{\frac{3}{2}}}{\sqrt{x}} = \dfrac{6x + 5x^{\frac{3}{2}}}{x^{\frac{1}{2}}}$

$= x^{-\frac{1}{2}}(6x + 5x^{\frac{3}{2}})$

$= 6x^{\frac{1}{2}} + 5x$

$p = \frac{1}{2}$ and $q = 1$

b $y = \int (6x^{\frac{1}{2}} + 5x)\,dx$

$= \dfrac{6x^{\frac{3}{2}}}{\frac{3}{2}} + \dfrac{5x^2}{2} + c$

$= 4x^{\frac{3}{2}} + \dfrac{5x^2}{2} + c$

6 b When $x = 9$ and $y = 100$,

$100 = 4(9)^{\frac{3}{2}} + \dfrac{5(9)^2}{2} + c$

$108 + \frac{405}{2} + c = 100$

$c = -\frac{421}{2}$

$y = 4x^{\frac{3}{2}} + \frac{5}{2}x^2 - \frac{421}{2}$

7 a $f(t) = \int (10 - 5t)\,dt$

$= 10t - \dfrac{5t^2}{2} + c$

When $x = 0$ and $y = 0$,

$f(0) = 10(0) - \dfrac{5(0)^2}{2} + c = 0$

$c = 0$

$f(t) = 10t - \frac{5}{2}t^2$

b $f(3) = 10(3) - \dfrac{5(3)^2}{2}$

$= 7\frac{1}{2}$

8 a $f(t) = \int (-9.8t)\,dt$

$= -\dfrac{9.8t^2}{2} + c$

$= -4.9t^2 + c$

When $x = 0$ and $y = 35$,

$f(0) = -4.9(0)^2 + c = 35$

$c = 35$

$f(t) = -4.9t^2 + 35$

b $f(1.5) = -4.9(1.5)^2 + 35 = 23.975$

The height of the arrow is 23.975 m.

c $f(0) = 35$

The height of the castle is 35 m.

d The arrow will hit the ground when the height is 0.

$-4.9t^2 + 35 = 0$

$t = \sqrt{\dfrac{-35}{-4.9}} = 2.67$ or -2.67

The time must be positive, so 2.67 seconds.

e The assumption is that the ground is flat.

Challenge

1 a $f_2'(x) = f_1(x) = x^2$

So $f_2(x) = \frac{1}{3}x^3 + c$

The curve passes through $(0, 0)$.

so $f_2(0) = 0 \Rightarrow c = 0$

So $f_2(x) = \frac{1}{3}x^3$

$f_3'(x) = \frac{1}{3}x^3$

$f_3(x) = \frac{1}{12}x^4 + c$

But $c = 0$ since $f_3(0) = 0$.

So $f_3(x) = \frac{1}{12}x^4$

b $f_2(x) = \frac{1}{3}x^3$, $f_3(x) = \dfrac{x^4}{3 \times 4}$

So the power of x is $n+1$ for $f_n(x)$.

The denominator is $3 \times 4 \times \ldots$ up to $n+1$.

$$f_n(x) = \frac{x^{n+1}}{3 \times 4 \times 5 \times \ldots \times (n+1)}$$

2 $f_2'(x) = f_1(x) = 1$

$\Rightarrow f_2(x) = x + c$

But $f_2(0) = 1 \Rightarrow 1 = 0 + c \Rightarrow c = 1$

So $f_2(x) = x + 1$

$f_3'(x) = f_2(x) = x + 1$

$\Rightarrow f_3(x) = \frac{1}{2}x^2 + x + c$

But $f_3(0) = 1 \Rightarrow 1 = 0 + c \Rightarrow c = 1$

So $f_3(x) = \frac{1}{2}x^2 + x + 1$

$f_4'(x) = f_3(x) = \frac{1}{2}x^2 + x + 1$

$\Rightarrow f_4(x) = \frac{1}{6}x^3 + \frac{1}{2}x^2 + x + c$

But $f_4(0) = 1 \Rightarrow 1 = 0 + c \Rightarrow c = 1$

So $f_4(x) = \frac{1}{6}x^3 + \frac{1}{2}x^2 + x + 1$

Integration 13D

1 a $\int_2^5 x^3 \mathrm{d}x = \left[\frac{x^4}{4}\right]_2^5$

$= \left(\frac{5^4}{4}\right) - \left(\frac{2^4}{4}\right)$

$= \frac{609}{4}$

$= 152\frac{1}{4}$

b $\int_1^3 x^4 \mathrm{d}x = \left[\frac{x^5}{5}\right]_1^3$

$= \left(\frac{3^5}{5}\right) - \left(\frac{1^5}{5}\right)$

$= \frac{242}{5}$

$= 48\frac{2}{5}$

c $\int_0^4 \sqrt{x} \mathrm{d}x = \int_0^4 x^{\frac{1}{2}} \mathrm{d}x$

$= \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^4$

$= \left[\frac{2x^{\frac{3}{2}}}{3}\right]_0^4$

$= \left(\frac{2(4)^{\frac{3}{2}}}{3}\right) - \left(\frac{2(0)^{\frac{3}{2}}}{3}\right)$

$= \frac{16}{3}$

$= 5\frac{1}{3}$

d $\int_1^3 \frac{3}{x^2} \mathrm{d}x = \int_1^3 3x^{-2} \mathrm{d}x$

$= \left[\frac{3x^{-1}}{-1}\right]_1^3$

$= \left[-\frac{3}{x}\right]_1^3$

$= \left(-\frac{3}{3}\right) - \left(-\frac{3}{1}\right)$

$= 2$

2 a $\int_1^2 \left(\frac{2}{x^3} + 3x\right) \mathrm{d}x = \int_1^2 \left(2x^{-3} + 3x\right) \mathrm{d}x$

$= \left(\frac{2x^{-2}}{-2} + \frac{3x^2}{2}\right)_1^2$

$= \left(-x^{-2} + \frac{3}{2}x^2\right)_1^2$

$= \left(-\frac{1}{4} + \frac{3}{2} \times 4\right) - \left(-1 + \frac{3}{2}\right)$

$= \left(-\frac{1}{4} + 6\right) - \frac{1}{2}$

$= 5\frac{1}{4}$

b $\int_0^2 \left(2x^3 - 4x + 5\right) \mathrm{d}x = \left(\frac{2x^4}{4} - \frac{4x^2}{2} + 5x\right)_0^2$

$= \left(\frac{x^4}{2} - 2x^2 + 5x\right)_0^2$

$= \left(\frac{16}{2} - 2 \times 4 + 10\right) - (0)$

$= 8 - 8 + 10$

$= 10$

c $\int_4^9 \left(\sqrt{x} - \frac{6}{x^2}\right) \mathrm{d}x = \int_4^9 \left(x^{\frac{1}{2}} - 6x^{-2}\right) \mathrm{d}x$

$= \left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{6x^{-1}}{-1}\right)_4^9 \mathrm{d}x$

$= \left(\frac{2}{3}x^{\frac{3}{2}} + 6x^{-1}\right)_4^9$

$= \left(\frac{2}{3} \times 3^3 + \frac{2}{3}\right) - \left(\frac{2}{3} \times 2^3 + \frac{3}{2}\right)$

$= 18 + \frac{2}{3} - \frac{16}{3} - \frac{3}{2}$

$= 16\frac{1}{2} - \frac{14}{3}$

$= 11\frac{5}{6}$

2 d $\int_1^8 \left(x^{-\frac{1}{3}}+2x-1\right) dx = \left(\dfrac{x^{\frac{2}{3}}}{\frac{2}{3}}+\dfrac{2x^2}{2}-x\right)_1^8$

$$= \left(\tfrac{3}{2}x^{\frac{2}{3}}+x^2-x\right)_1^8$$
$$= \left(\tfrac{3}{2}\times 2^2+64-8\right)-\left(\tfrac{3}{2}+1-1\right)$$
$$= 62-\tfrac{3}{2}$$
$$= 60\tfrac{1}{2}$$

3 a $\int_1^3 \left(\dfrac{x^3+2x^2}{x}\right) dx$

$$= \int_1^3 \left(x^2+2x\right) dx$$
$$= \left(\frac{x^3}{3}+x^2\right)_1^3$$
$$= \left(\tfrac{27}{3}+9\right)-\left(\tfrac{1}{3}+1\right)$$
$$= 18-\tfrac{4}{3}$$
$$= 16\tfrac{2}{3}$$

b $\int_3^6 \left(x-\dfrac{3}{x}\right)^2 dx = \int_3^6 \left(x^2-6+\dfrac{9}{x^2}\right) dx$

$$= \int_3^6 \left(x^2-6+9x^{-2}\right) dx$$
$$= \left(\frac{x^3}{3}-6x+\frac{9x^{-1}}{-1}\right)_3^6$$
$$= \left(\frac{x^3}{3}-6x-9x^{-1}\right)_3^6$$
$$= \left(\tfrac{216}{3}-36-\tfrac{9}{6}\right)-\left(\tfrac{27}{3}-18-\tfrac{9}{3}\right)$$
$$= 72-36-\tfrac{3}{2}-9+18+3$$
$$= 48-\tfrac{3}{2}$$
$$= 46\tfrac{1}{2}$$

c $\int_0^1 x^2\left(\sqrt{x}+\dfrac{1}{x}\right) dx = \int_0^1 \left(x^{\frac{5}{2}}+x\right) dx$

$$= \left(\frac{x^{\frac{7}{2}}}{\frac{7}{2}}+\frac{x^2}{2}\right)_0^1$$
$$= \left(\tfrac{2}{7}+\tfrac{1}{2}\right)-(0)$$
$$= \tfrac{11}{14}$$

3 d $\int_1^4 \left(\dfrac{2+\sqrt{x}}{x^2}\right) dx = \int_1^4 \left(\dfrac{2}{x^2}+\dfrac{1}{x^{\frac{3}{2}}}\right) dx$

$$= \int_1^4 \left(2x^{-2}+x^{-\frac{3}{2}}\right) dx$$
$$= \left(\frac{2x^{-1}}{-1}+\frac{x^{-\frac{1}{2}}}{-\frac{1}{2}}\right)_1^4$$
$$= \left(-2x^{-1}-2x^{-\frac{1}{2}}\right)_1^4$$
$$= \left(-\tfrac{2}{4}-\tfrac{2}{2}\right)-\left(-2-2\right)$$
$$= -1\tfrac{1}{2}+4$$
$$= 2\tfrac{1}{2}$$

4 $\int_1^4 (6\sqrt{x}-A)\,dx = A^2$

$$\int_1^4 (6x^{\frac{1}{2}}-A)\,dx = A^2$$
$$\left[\frac{6x^{\frac{3}{2}}}{\frac{3}{2}}-Ax\right]_1^4 = A^2$$
$$\left[4x^{\frac{3}{2}}-Ax\right]_1^4 = A^2$$
$$\left(4(4)^{\frac{3}{2}}-A(4)\right)-\left(4(1)^{\frac{3}{2}}-A(1)\right) = A^2$$
$$(32-4A)-(4-A) = A^2$$
$$28-3A = A^2$$
$$A^2+3A-28 = 0$$
$$(A+7)(A-4) = 0$$
$$A = -7 \text{ or } A = 4$$

5 $\int_1^9 (2x-3\sqrt{x})\,dx = \int_1^9 (2x-3x^{\frac{1}{2}})\,dx$

$$= \left[\frac{2x^2}{2}-\frac{3x^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^9$$
$$= \left[x^2-2x^{\frac{3}{2}}\right]_1^9$$

5 $\int_1^9 (2x - 3\sqrt{x})\ \mathrm{d}x = \left(9^2 - 2(9)^{\frac{3}{2}}\right) - \left(1^2 - 2(1)^{\frac{3}{2}}\right)$

$$= (81 - 54) - (1 - 2)$$
$$= 28$$

6 $\int_4^{12} (\frac{2}{\sqrt{x}})\,\mathrm{d}x = \int_4^{12} (2x^{-\frac{1}{2}})\,\mathrm{d}x$

$$= \left[\frac{2x^{\frac{1}{2}}}{\frac{1}{2}}\right]_4^{12}$$
$$= \left[4x^{\frac{1}{2}}\right]_4^{12}$$
$$= \left(4(12)^{\frac{1}{2}}\right) - \left(4(4)^{\frac{1}{2}}\right)$$
$$= 4\sqrt{12} - 8$$
$$= 4\sqrt{4 \times 3} - 8$$
$$= -8 + 8\sqrt{3}$$

7 $\int_1^k \frac{1}{\sqrt{x}}\,\mathrm{d}x = 3$

$$\int_1^k x^{-\frac{1}{2}}\,\mathrm{d}x = 3$$
$$\left[\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right]_1^k = 3$$
$$\left[2\sqrt{x}\right]_1^k = 3$$
$$2\sqrt{k} - 2\sqrt{1} = 3$$
$$2\sqrt{k} = 5$$
$$\sqrt{k} = \tfrac{5}{2}$$
$$k = \tfrac{25}{4}$$

8 $s = \int_0^{10} (20 + 5t)\mathrm{d}t$

$$= \left[20t + \frac{5t^2}{2}\right]_0^{10}$$
$$= \left(20(10) + \frac{5(10)^2}{2}\right) - \left(20(0) + \frac{5(0)^2}{2}\right)$$
$$= 450 \text{ m}$$

Challenge

$$\int_k^{3k} \frac{3x+2}{8}\ \mathrm{d}x = 7$$
$$\int_k^{3k} \left(\frac{3x}{8} + \frac{1}{4}\right)\ \mathrm{d}x = 7$$
$$\left[\frac{1}{2}\frac{3x^2}{8} + \frac{x}{4}\right]_k^{3k} = 7$$
$$\left[\frac{3x^2}{16} + \frac{x}{4}\right]_k^{3k} = 7$$
$$\left(\frac{3(3k)^2}{16} + \frac{(3k)}{4}\right) - \left(\frac{3k^2}{16} + \frac{k}{4}\right) = 7$$
$$\left(\frac{27k^2}{16} + \frac{3k}{4}\right) - \left(\frac{3k^2}{16} + \frac{k}{4}\right) = 7$$
$$\frac{24k^2}{16} + \frac{k}{2} = 7$$
$$24k^2 + 8k - 112 = 0$$
$$3k^2 + k - 14 = 0$$
$$(3k + 7)(k - 2) = 0$$
$$k = -\tfrac{7}{3} \text{ or } k = 2$$

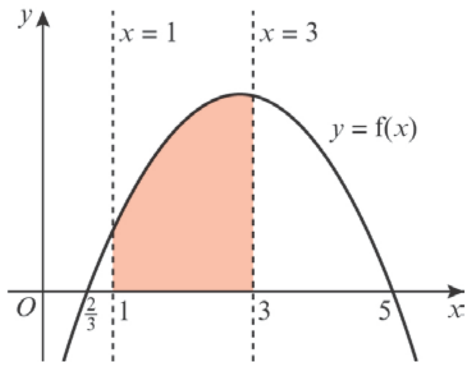
As $k > 0$, $k = 2$

Integration 13E

1 a $-3x^2 + 17x - 10 = 0$

$(-3x + 2)(x - 5) = 0$

$x = \frac{2}{3}$ or $x = 5$



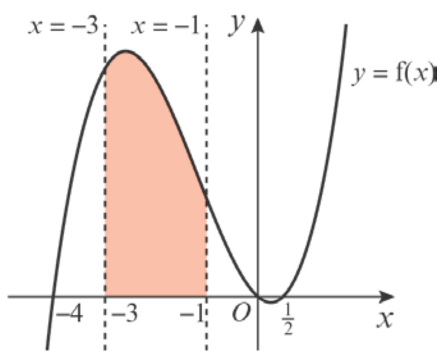
$$\int_1^3 (-3x^2 + 17x - 10)\,dx$$
$$= \left[\frac{-3x^3}{3} + \frac{17x^2}{2} - 10x\right]_1^3$$
$$= \left[-x^3 + \frac{17x^2}{2} - 10x\right]_1^3$$
$$= \left(-(3)^3 + \frac{17(3)^2}{2} - 10(3)\right) - \left(-(1)^3 + \frac{17(1)^2}{2} - 10(1)\right)$$
$$= \left(-27 + \frac{153}{2} - 30\right) - \left(-1 + \frac{17}{2} - 10\right)$$
$$= 22$$

b $2x^3 + 7x^2 - 4x = 0$

$x(2x^2 + 7x - 4) = 0$

$x(2x - 1)(x + 4) = 0$

$x = 0$, $x = \frac{1}{2}$ or $x = -4$

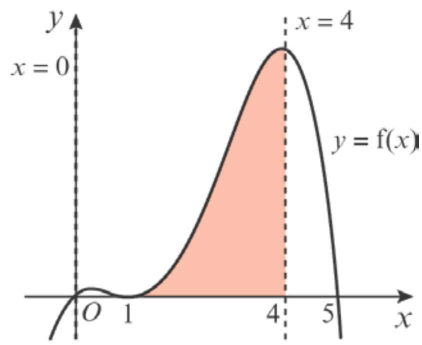


$$\int_{-3}^{-1} (2x^3 + 7x^2 - 4x)\,dx$$
$$= \left[\frac{2x^4}{4} + \frac{7x^3}{3} - \frac{4x^2}{2}\right]_{-3}^{-1}$$
$$= \left(\frac{(-1)^4}{2} + \frac{7(-1)^3}{3} - 2(-1)^2\right) - \left(\frac{(-3)^4}{2} + \frac{7(-3)^3}{3} - 2(-3)^2\right)$$
$$= \left(\frac{1}{2} - \frac{7}{3} - 2\right) - \left(\frac{81}{2} - \frac{189}{3} - 18\right)$$
$$= 36\tfrac{2}{3}$$

c $-x^4 + 7x^3 - 11x^2 + 5x = 0$

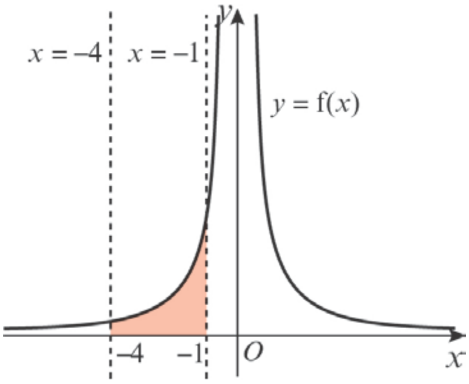
$-x(x - 1)^2(x - 5) = 0$

$x = 0$, $x = 1$ or $x = 5$



$$\int_0^4 (-x^4 + 7x^3 - 11x^2 + 5x)\,dx$$
$$= \left[-\frac{x^5}{5} + \frac{7x^4}{4} - \frac{11x^3}{3} + \frac{5x^2}{2}\right]_0^4$$
$$= \left(-\frac{4^5}{5} + \frac{7(4)^4}{4} - \frac{11(4)^3}{3} + \frac{5(4)^2}{2}\right) - \left(-\frac{0^5}{5} + \frac{7(0)^4}{4} - \frac{11(0)^3}{3} + \frac{5(0)^2}{2}\right)$$
$$= \left(-\frac{1024}{5} + 448 - \frac{704}{3} + 40\right)$$
$$= 48\tfrac{8}{15}$$

1 d



$$\int_{-4}^{-1}\left(\frac{8}{x^2}\right)\,dx = \int_{-4}^{-1}(8x^{-2})\,dx$$
$$= \left[\frac{8x^{-1}}{-1}\right]_{-4}^{-1}$$
$$= \left[-\frac{8}{x}\right]_{-4}^{-1}$$
$$= \left(-\frac{8}{(-1)}\right) - \left(-\frac{8}{(-4)}\right)$$
$$= (8) - (2)$$
$$= 6$$

2 $$A = \int_{-2}^{0} x\left(x^2 - 4\right)\,dx = \int_{-2}^{0}\left(x^3 - 4x\right)\,dx$$
$$= \left(\frac{x^4}{4} - \frac{4x^2}{2}\right)_{-2}^{0}$$
$$= \left(\frac{x^4}{4} - 2x^2\right)_{-2}^{0}$$
$$= (0) - \left(\frac{16}{4} - 2\times 4\right)$$
$$= -4 + 8$$
$$= 4$$

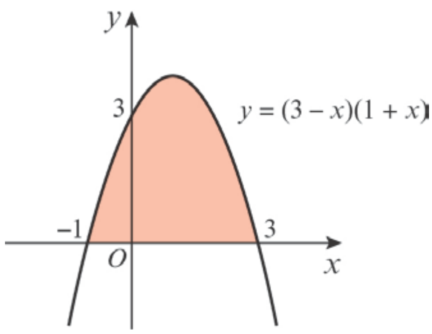
3 $$A = \int_{1}^{3}\left(3x + \frac{6}{x^2} - 5\right)\,dx$$
$$= \int_{1}^{3}\left(3x + 6x^{-2} - 5\right)\,dx$$
$$= \left(\frac{3x^2}{2} + \frac{6x^{-1}}{-1} - 5x\right)_{1}^{3}$$
$$= \left(\frac{3}{2}x^2 - 6x^{-1} - 5x\right)_{1}^{3}$$

3 $$A = \left(\frac{3}{2}\times 9 - \frac{6}{3} - 15\right) - \left(\frac{3}{2} - 6 - 5\right)$$
$$= \frac{27}{2} - 17 - \frac{3}{2} + 11$$
$$= \frac{24}{2} - 6$$
$$= 6$$

4 $y = (3 - x)(1 + x)$ is $\cap$ shaped

$y = 0 \Rightarrow x = 3, -1$

$x = 0 \Rightarrow y = 3$

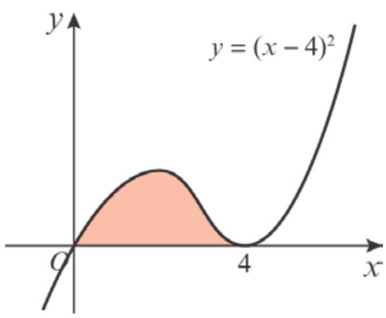


$$A = \int_{-1}^{3}(3 - x)(1 + x)\,dx$$
$$= \int_{-1}^{3}\left(3 + 2x - x^2\right)\,dx$$
$$= \left(3x + x^2 - \frac{x^3}{3}\right)_{-1}^{3}$$
$$= \left(9 + 9 - \frac{27}{3}\right) - \left(-3 + 1 + \frac{1}{3}\right)$$
$$= 9 + 1\tfrac{2}{3}$$
$$= 10\tfrac{2}{3}$$

5 $y = x(x - 4)^2$

$y = 0 \Rightarrow x = 0, 4\ (\text{twice})$

There is a turning point at $(4, 0)$.



5 Area $= \int_0^4 x(x-4)^2 \, dx$

$$= \int_0^4 x(x^2 - 8x + 16) \, dx$$

$$= \int_0^4 (x^3 - 8x^2 + 16x) \, dx$$

$$= \left(\frac{x^4}{4} - \frac{8x^3}{3} + 8x^2\right)_0^4$$

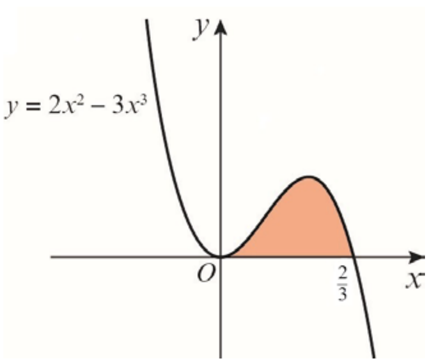
$$= \left(64 - \frac{8}{3} \times 64 + 128\right) - (0)$$

$$= \tfrac{64}{3} \text{ or } 21\tfrac{1}{3}$$

6 $2x^2 - 3x^3 = 0$

$x^2(2 - 3x) = 0$

$x = 0$ or $x = \frac{2}{3}$



$$\int_0^{\frac{2}{3}} (2x^2 - 3x^3) \, dx = \left[\frac{2x^3}{3} - \frac{3x^4}{4}\right]_0^{\frac{2}{3}}$$

$$= \left(\frac{2(\frac{2}{3})^3}{3} - \frac{3(\frac{2}{3})^4}{4}\right) - \left(\frac{2(0)^3}{3} - \frac{3 \times 0^4}{4}\right)$$

$$= \tfrac{16}{81} - \tfrac{12}{81}$$

$$= \tfrac{4}{81}$$

7

$$\int_0^k (3x^2 - 2x + 2) \, dx = 8$$

$$\left[\frac{3x^3}{3} - \frac{2x^2}{2} + 2x\right]_0^k = 8$$

$$\left[x^3 - x^2 + 2x\right]_0^k = 8$$

$$(k^3 - k^2 + 2k) - (0^3 - 0^2 + 2(0)) = 8$$

$$k^3 - k^2 + 2k - 8 = 0$$

7 Using the factor theorem, $k = 2$ as

$2^3 - 2^2 + 2(2) - 8 = 0$

Therefore, $k = 2$

8 a $-x^2 + 2x + 3 = 0$

$(-x + 3)(x + 1) = 0$

$x = 3$ or $x = -1$

$A(-1, 0)$ and $B(3, 0)$

b $\int_{-1}^3 (-x^2 + 2x + 3) \, dx$

$$= \left[-\frac{x^3}{3} + \frac{2x^2}{2} + 3x\right]_{-1}^3$$

$$= \left[-\frac{x^3}{3} + x^2 + 3x\right]_{-1}^3$$

$$= \left(-\frac{3^3}{3} + 3^2 + 3(3)\right) - \left(-\frac{(-1)^3}{3} + (-1)^2 + 3(-1)\right)$$

$$= (-9 + 9 + 9) - \left(\frac{1}{3} + 1 - 3\right)$$

$$= 10\tfrac{2}{3}$$

9 $\int_0^2 x^2(2 - x) \, dx = \int_0^2 2x^2 - x^3 \, dx$

$$= \left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^2$$

$$= \left(\frac{2(2)^3}{3} - \frac{2^4}{4}\right) - \left(\frac{2(0)^3}{3} - \frac{0^4}{4}\right)$$

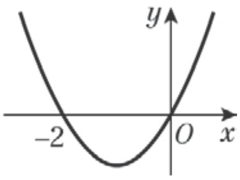
$$= \left(\frac{16}{3} - \frac{16}{4}\right)$$

$$= 1\tfrac{1}{3}$$

Integration 13F

1 a $y = x(x+2)$ is $\bigvee$ shaped

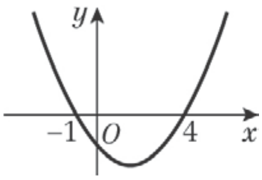
$y = 0 \Rightarrow x = 0, -2$



$$\text{Area} = \int_{-2}^{0} x(x+2)\ \mathrm{d}x$$
$$= -\int_{-2}^{0} \left(x^2 + 2x\right)\ \mathrm{d}x$$
$$= -\left(\frac{x^3}{3} + x^2\right)_{-2}^{0}$$
$$= \left\{(0) - \left(-\frac{8}{3} + 4\right)\right\}$$
$$= -\left(-\frac{4}{3}\right)$$
$$= \tfrac{4}{3} \text{ or } 1\tfrac{1}{3}$$

b $y = (x+1)(x-4)$ is $\bigvee$ shaped

$y = 0 \Rightarrow x = -1, 4$



$$\int_{-1}^{4} (x+1)(x-4)\ \mathrm{d}x = \int_{-1}^{4} \left(x^2 - 3x - 4\right)\ \mathrm{d}x$$
$$= \left(\frac{x^3}{3} - \frac{3x^2}{2} - 4x\right)_{-1}^{4}$$
$$= \left(\frac{64}{3} - \frac{3}{2} \times 16 - 16\right)$$
$$- \left(-\frac{1}{3} - \frac{3}{2} + 4\right)$$
$$= \frac{64}{3} - 40 + \frac{11}{6} - 4$$
$$= -20\tfrac{5}{6}$$

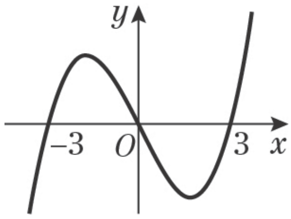
So area $= 20\tfrac{5}{6}$

c $y = (x+3)x(x-3)$

$y = 0 \Rightarrow x = -3, 0, 3$

$x \to \infty, y \to \infty$

$x \to -\infty, y \to -\infty$



$$\int y\mathrm{d}x = \int \left(x^3 - 9x\right)\ \mathrm{d}x$$
$$= \left(\frac{x^4}{4} - \frac{9}{2}x^2\right)$$
$$\int_{-3}^{0} y\ \mathrm{d}x = (0) - \left(\frac{81}{4} - \frac{9}{2} \times 9\right)$$
$$= +\frac{81}{4}$$
$$\int_{0}^{3} y\ \mathrm{d}x = \left(\frac{81}{4} - \frac{9}{2} \times 9\right) - (0)$$
$$= -\frac{81}{4}$$

So area $= \dfrac{81}{4} + \dfrac{81}{4}$

$$= \frac{81}{2} \text{ or } 40\tfrac{1}{2}$$

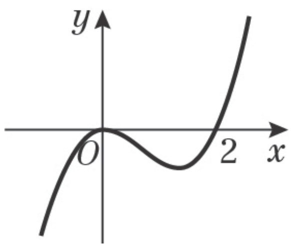
d $y = x^2(x-2)$

$y = 0 \Rightarrow x = 0\ (\text{twice}), 2$

There is a turning point at (0, 0).

$x \to \infty, y \to \infty$

$x \to -\infty, y \to -\infty$



1 d Area $= -\int_0^2 x^2(x-2)\,dx$

$= -\int_0^2 (x^3 - 2x^2)\,dx$

$= -\left(\frac{x^4}{4} - \frac{2}{3}x^3\right)_0^2$

$= -\left\{\left(\frac{16}{4} - \frac{2}{3}\times 8\right) - (0)\right\}$

$= -\left(4 - \frac{16}{3}\right)$

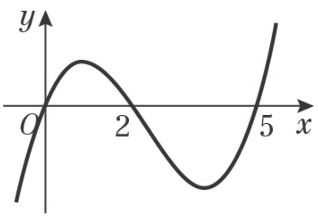
$= \frac{4}{3}$ or $1\frac{1}{3}$

e $y = x(x-2)(x-5)$

$y = 0 \Rightarrow x = 0, 2, 5$

$x \to \infty, y \to \infty$

$x \to -\infty, y \to -\infty$



$\int y\,dx = \int x(x^2 - 7x + 10)\,dx$

$= \int (x^3 - 7x^2 + 10x)\,dx$

$= \left(\frac{x^4}{4} - \frac{7}{3}x^3 + 5x^2\right)$

$\int_0^2 y\,dx = \left(\frac{16}{4} - \frac{7}{3}\times 8 + 20\right) - (0)$

$= 24 - \frac{56}{3}$

$= 5\frac{1}{3}$

$\int_2^5 y\,dx = \left(\frac{625}{4} - \frac{7}{3}\times 125 + 125\right) - \left(5\frac{1}{3}\right)$

$= -15\frac{3}{4}$

So area $= 5\frac{1}{3} + 15\frac{3}{4}$

$= 21\frac{1}{12}$

2 a $x(x + 3)(2 - x) = 0$

$x = 0$, $x = -3$ or $x = 2$

$A(-3, 0)$, $B(2, 0)$

b $\int_0^2 x(x+3)(2-x)\,dx - \int_{-3}^0 x(x+3)(2-x)\,dx$

$= \int_0^2 (-x^3 - x^2 + 6x)\,dx$

$- \int_{-3}^0 (-x^3 - x^2 + 6x)\,dx$

$\int_0^2 (-x^3 - x^2 + 6x)\,dx$

$= \left[-\frac{x^4}{4} - \frac{x^3}{3} + \frac{6x^2}{2}\right]_0^2$

$= \left[-\frac{x^4}{4} - \frac{x^3}{3} + 3x^2\right]_0^2$

$= \left(-\frac{2^4}{4} - \frac{2^3}{3} + 3(2)^2\right) - \left(-\frac{0^4}{4} - \frac{0^3}{3} + 3(0)^2\right)$

$= \left(-4 - \frac{8}{3} + 12\right)$

$= 5\frac{1}{3}$

$\int_{-3}^0 (-x^3 - x^2 + 6x)\,dx$

$= \left[-\frac{x^4}{4} - \frac{x^3}{3} + 3x^2\right]_{-3}^0$

$= \left(-\frac{0^4}{4} - \frac{0^3}{3} + 3(0)^2\right)$

$- \left(-\frac{(-3)^4}{4} - \frac{(-3)^3}{3} + 3(-3)^2\right)$

$= -\left(-\frac{81}{4} + 9 + 27\right)$

$= -15\frac{3}{4}$

Total area $= 5\frac{1}{3} + 15\frac{3}{4}$

$= 21\frac{1}{12}$

3 a $f(-3) = -(-3)^3 + 4(-3)^2 + 11(-3) - 30$

$= 27 + 36 - 33 - 30 = 0$

3 b

$$\begin{array}{r} -x^2+7x-10 \\ x+3\overline{\big)-x^3+4x^2+11x-30} \\ \underline{-x^3-3x^2} \\ 7x^2+11x \\ \underline{7x^2+21x} \\ -10x-30 \\ \underline{-10x-30} \\ 0 \end{array}$$

$f(x) = (x + 3)(-x^2 + 7x - 10)$

c $f(x) = (x + 3)(-x + 2)(x - 5)$

d $x = -3$, $x = 2$ or $x = 5$
$(-3, 0)$, $(2, 0)$ and $(5, 0)$

e Total area is:

$$\int_2^5(-x^3+4x^2+11x-30)\,dx$$
$$-\int_{-3}^2(-x^3+4x^2+11x-30)\,dx$$

$$\int_2^5(-x^3+4x^2+11x-30)\,dx$$
$$=\left[-\frac{x^4}{4}+\frac{4x^3}{3}+\frac{11x^2}{2}-30x\right]_2^5$$
$$=\left(-\frac{5^4}{4}+\frac{4(5)^3}{3}+\frac{11(5)^2}{2}-30(5)\right)$$
$$-\left(-\frac{2^4}{4}+\frac{4(2)^3}{3}+\frac{11(2)^2}{2}-30(2)\right)$$
$$=\left(-\frac{625}{4}+\frac{500}{3}+\frac{275}{2}-150\right)$$
$$-\left(-4+\frac{32}{3}+22-60\right)$$
$$=29\tfrac{1}{4}$$

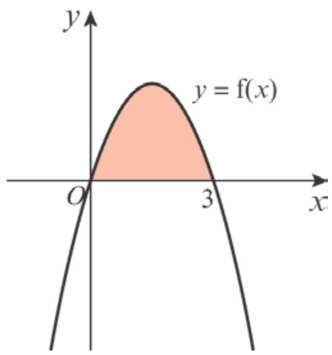
$$\int_{-3}^2(-x^3+4x^2+11x-30)\,dx$$
$$=\left[-\frac{x^4}{4}+\frac{4x^3}{3}+\frac{11x^2}{2}-30x\right]_{-3}^2$$
$$=\left(-\frac{2^4}{4}+\frac{4(2)^3}{3}+\frac{11(2)^2}{2}-30(2)\right)$$
$$-\left(-\frac{(-3)^4}{4}+\frac{4(-3)^3}{3}+\frac{11(-3)^2}{2}-30(-3)\right)$$

3 e $$\int_{-3}^2(-x^3+4x^2+11x-30)\,dx$$
$$=\left(-4+\frac{32}{3}+22-60\right)$$
$$-\left(-\frac{81}{4}-\frac{108}{3}+\frac{99}{2}+90\right)$$
$$=-114\tfrac{7}{12}$$

Total area = $29\tfrac{1}{4} + 114\tfrac{7}{12}$
$= 143\tfrac{5}{6}$

Challenge

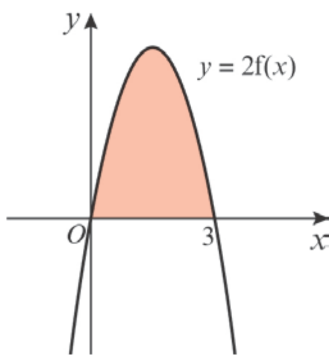
1 a $x(3 - x) = 0$
$x = 0$ or $x = 3$



$f(x) = 3x - x^2$

Area $= \int_0^3(3x-x^2)\,dx$
$$=\left[\frac{3x^2}{2}-\frac{x^3}{3}\right]_0^3$$
$$=\left(\frac{3(3)^2}{2}-\frac{3^3}{3}\right)-\left(\frac{3(0)^2}{2}-\frac{0^3}{3}\right)$$
$$=\left(\tfrac{27}{2}-9\right)$$
$$=4\tfrac{1}{2}$$

b



$f(x) = 6x - 2x^2$

1 b Area $= \int_0^3 (6x - 2x^2)\,dx$

$$= \left[\frac{6x^2}{2} - \frac{2x^3}{3}\right]_0^3$$

$$= \left[3x^2 - \frac{2x^3}{3}\right]_0^3$$

$$= \left(3(3)^2 - \frac{2(3)^3}{3}\right) - \left(3(0)^2 - \frac{2(0)^3}{3}\right)$$

$$= (27 - 18)$$

$$= 9$$

c $f(x) = a(3x - x^2)$
Area $= a \times$ area of $f(x)$
$= a \times 4\frac{1}{2}$
$= \dfrac{9a}{2}$

d $y = f(x + a)$ is a translation of $y = f(x)$ by $\begin{pmatrix} -a \\ 0 \end{pmatrix}$.
Therefore, the area of $y = f(x + a)$ is equal to the area of $y = f(x)$.
The area of $y = f(x + a)$ is $4\frac{1}{2}$

e $f(ax) = 3ax - a^2x^2$

Area $= \int_0^{\frac{3}{a}} (3ax - a^2x^2)\,dx$

$$= \left[\frac{3ax^2}{2} - \frac{a^2x^3}{3}\right]_0^{\frac{3}{a}}$$

$$= \left(\frac{3a\left(\frac{3}{a}\right)^2}{2} - \frac{a^2\left(\frac{3}{a}\right)^3}{3}\right)$$

$$-\left(\frac{3(0)^2}{2} - \frac{0^3}{3}\right)$$

$$= \left(\frac{27}{2a} - \frac{9}{a}\right)$$

$$= \frac{9}{2a}$$

2 a When $y = 0$, $x = -2$ or $x = 0$ or $x = 1$
$B(1, 0)$
$y = x(x^2 + x - 2)$
$= x^3 + x^2 - 2x$

2 a Area under the curve $= \int_0^1 (x^3 + x^2 - 2x)\,dx$

$$= \left[\frac{x^4}{4} + \frac{x^3}{3} - x^2\right]_0^1$$

$$= \left(\frac{1^4}{4} + \frac{1^3}{3} - 1^2\right)$$

$$-\left(\frac{0^4}{4} + \frac{0^3}{3} - 0^2\right)$$

$$= \left(-\frac{5}{12}\right) - 0$$

$$= -\frac{5}{12}$$

So area $= \dfrac{5}{12}$

Area above the curve from $x = 0$ to $x = x$

$$\int_x^0 (x^3 + x^2 - 2x)\,dx = \frac{5}{12}$$

$$\left[\frac{x^4}{4} + \frac{x^3}{3} - x^2\right]_x^0 = \frac{5}{12}$$

$$\left(\frac{0^4}{4} + \frac{0^3}{3} - 0^2\right) - \left(\frac{x^4}{4} + \frac{x^3}{3} - x^2\right) = \frac{5}{12}$$

$$-\left(\frac{x^4}{4} + \frac{x^3}{3} - x^2\right) = \frac{5}{12}$$

$3x^4 + 4x^3 - 12x^2 + 5 = 0$

Using the factor theorem when $x = 1$
$3(1)^4 + 4(1)^3 - 12(1)^2 + 5 = 0$

So $(x - 1)$ is a factor

$$\begin{array}{r}
3x^3 + 7x^2 - 5x - 5 \\
x - 1 \overline{\big) 3x^4 + 4x^3 - 12x^2 + 0x + 5} \\
\underline{3x^4 - 3x^3} \qquad\qquad\qquad\quad \\
7x^3 - 12x^2 \qquad\qquad \\
\underline{7x^3 - 7x^2} \qquad\qquad \\
-5x^2 + 0x \qquad \\
\underline{-5x^2 + 5x} \qquad \\
-5x + 5 \\
\underline{-5x + 5} \\
0
\end{array}$$

2 a $3x^4 + 4x^3 - 12x^2 + 5$
$= (x - 1)(3x^3 + 7x^2 - 5x - 5)$

Using the factor theorem when $x = 1$
$3(1)^3 + 7(1)^2 - 5(1) - 5 = 0$

So $(x - 1)$ is a factor

$$\begin{array}{r@{}l} & 3x^2 + 10x + 5 \\ x-1 \big) & 3x^3 + 7x^2 - 5x - 5 \\ & \underline{3x^3 - 3x^2} \\ & \quad 10x^2 - 5x \\ & \quad \underline{10x^2 - 10x} \\ & \qquad 5x - 5 \\ & \qquad \underline{5x - 5} \\ & \qquad\quad 0 \end{array}$$

So $3x^4 + 4x^3 - 12x^2 + 5$
$= (x - 1)^2(3x^2 + 10x + 5)$

2 b When $(x - 1) = 0$, $x = 1$
When $3x^2 + 10x + 5 = 0$

$$x = \frac{-10 \pm \sqrt{10^2 - 4(3)(5)}}{2\times 3}$$
$$= \frac{-10 \pm \sqrt{40}}{6}$$
$$= \frac{-10 \pm \sqrt{4\times 10}}{6}$$
$$= \frac{-10 \pm 2\sqrt{10}}{6}$$
$$= \frac{-5 \pm \sqrt{10}}{3}$$

As $y = x^3 + x^2 - 2x$ going back to the original curve

When $x = \dfrac{-5+\sqrt{10}}{3}$

$$y = \left(\frac{-5+\sqrt{10}}{3}\right)^3 + \left(\frac{-5+\sqrt{10}}{3}\right)^2 - 2\left(\frac{-5+\sqrt{10}}{3}\right)$$
$$= \left(\frac{-275+85\sqrt{10}}{27}\right) + \left(\frac{35-10\sqrt{10}}{9}\right) - \left(\frac{-10+2\sqrt{10}}{3}\right)$$
$$= \frac{-80+37\sqrt{10}}{27}$$

$$A\left(\frac{-5+\sqrt{10}}{3}, \frac{-80+37\sqrt{10}}{27}\right)$$

The roots at 1 correspond to point B.

The root $\dfrac{-5-\sqrt{10}}{3}$ gives a point on the curve to the left of -2 below the x-axis, so cannot be A.

Integration 13G

1 a A, B are given by $6 = x^2 + 2$

$$x^2 = 4$$
$$x = \pm 2$$

So A is (−2, 6) and B is (2, 6).

b $\text{Area} = \int_{-2}^{2} \left(6 - \left(x^2 + 2\right)\right) \, dx$

$$= \int_{-2}^{2} \left(4 - x^2\right) dx$$
$$= \left(4x - \frac{x^3}{3}\right)_{-2}^{2}$$
$$= \left(8 - \tfrac{8}{3}\right) - \left(-8 + \tfrac{8}{3}\right)$$
$$= 16 - 2 \times \tfrac{8}{3}$$
$$= 10\tfrac{2}{3}$$

2 a A, B are given by $3 = 4x - x^2$

$$x^2 - 4x + 3 = 0$$
$$(x-3)(x-1) = 0$$
$$x = 1, 3$$

So A is (1, 3) and B is (3, 3).

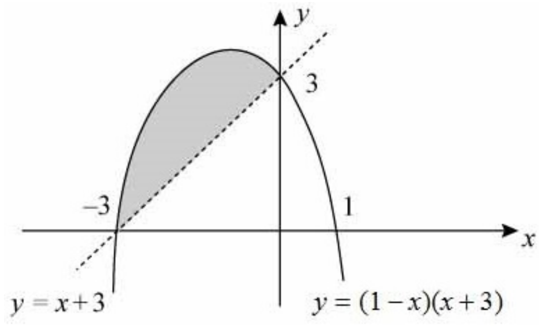
b $\text{Area} = \int_{1}^{3} \left(\left(4x - x^2\right) - 3\right) dx$

$$= \int_{1}^{3} \left(4x - x^2 - 3\right) dx$$
$$= \left(2x^2 - \frac{x^3}{3} - 3x\right)_{1}^{3}$$
$$= (18 - 9 - 9) - \left(2 - \tfrac{1}{3} - 3\right)$$
$$= 1\tfrac{1}{3}$$

3 $\text{Area} = \int_{-1}^{1} (\text{curve} - \text{line}) \, dx$

$$= \int_{-1}^{1} \left(9 - 3x - 5x^2 - x^3 - (4 - 4x)\right) dx$$
$$= \int_{-1}^{1} \left(5 + x - 5x^2 - x^3\right) dx$$
$$= \left(5x + \frac{x^2}{2} - \frac{5}{3}x^3 - \frac{x^4}{4}\right)_{-1}^{1}$$
$$= \left(5 + \tfrac{1}{2} - \tfrac{5}{3} - \tfrac{1}{4}\right) - \left(-5 + \tfrac{1}{2} + \tfrac{5}{3} - \tfrac{1}{4}\right)$$
$$= 10 - \tfrac{10}{3}$$
$$= \tfrac{20}{3} \text{ or } 6\tfrac{2}{3}$$

4 $y = (1-x)(x+3)$ is $\cap$ shaped and crosses the x-axis at (1, 0) and (−3, 0).
$y = x + 3$ is a straight line passing through (−3, 0) and (0, 3)



Intersections occur when

$$x + 3 = (1-x)(x+3)$$

$$0 = (x+3)(1 - x - 1)$$
$$0 = -x(x+3)$$
$$x = -3 \text{ or } x = 0$$

$$\text{Area} = \int_{-3}^{0} \left((1-x)(x+3) - (x+3)\right) dx$$
$$= \int_{-3}^{0} \left(-x^2 - 3x\right) dx$$
$$= \left(-\frac{x^3}{3} - \frac{3x^2}{2}\right)_{-3}^{0}$$
$$= (0) - \left(\tfrac{27}{3} - \tfrac{27}{2}\right)$$
$$= \tfrac{27}{6}$$
$$= \tfrac{9}{2} \text{ or } 4\tfrac{1}{2}$$

5 a A is given by $x(4+x) = 12$

$$x^2 + 4x - 12 = 0$$
$$(x+6)(x-2) = 0$$
$$x = 2 \text{ or } x = -6$$

So A is (2, 12)

b R is found by $\int_0^2 x(4+x) \, dx$ away from a rectangle of area 12 × 2 = 24.

So area of $R = 24 - \int_0^2 \left(x^2 + 4x\right) dx$

$$= 24 - \left(\frac{x^3}{3} + 2x^2\right)_0^2$$

5 b Area of $R = 24 - \left\{\left(\frac{8}{3}+8\right)-(0)\right\}$

$= 24 - \frac{32}{3}$

$= \frac{40}{3}$ or $13\frac{1}{3}$

6 a Intersections occur when $7 - x = x^2 + 1$

$0 = x^2 + x - 6$

$0 = (x+3)(x-2)$

$x = 2$ or -3

Area of R_1, is given by

$\int_{-3}^{2}\left(7 - x - \left(x^2+1\right)\right)\,dx$

$= \int_{-3}^{2}\left(6 - x - x^2\right)\,dx$

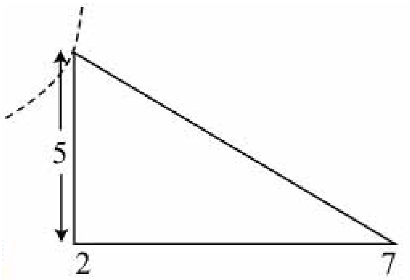
$= \left(6x - \frac{x^2}{2} - \frac{x^3}{3}\right)_{-3}^{2}$

$= \left(12 - \frac{4}{2} - \frac{8}{3}\right) - \left(-18 - \frac{9}{2} + \frac{27}{3}\right)$

$= 20\frac{5}{6}$

b Area of R_2, is given by

$\int_{0}^{2}\left(x^2+1\right)\,dx$ + area of the triangle.



Area of $R_2 = \left(\frac{x^3}{3} + x\right)_0^2 + \frac{1}{2}\times5\times5$

$= \left(\frac{8}{3}+2\right) - (0) + \frac{25}{2}$

$= 17\frac{1}{6}$

7 a When $x = 1$, $y = 1 - \frac{2}{1} + 1$

$= 0$

So $(1, 0)$ lies on C.

b When $x = 8$, $y = 8^{\frac{2}{3}} - \dfrac{2}{8^{\frac{1}{3}}} + 1$

$= 2^2 - \frac{2}{2} + 1$

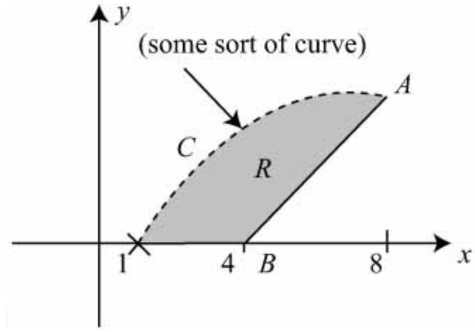
$= 4$

So $(8, 4)$ lies on C.

7 c A is the point (8, 4) and B is the point (4, 0).

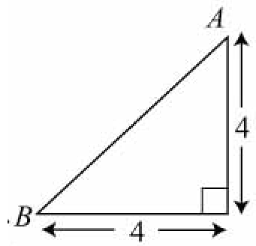
Gradient of line through AB is $\dfrac{4-0}{8-4} = 1$.

So the equation is $y - 0 = x - 4$ or $y = x - 4$



d Area of R is given by

$\int_{1}^{8}(\text{curve})\,dx$ – area of the triangle.



Area $R = \int_{1}^{8}\left(x^{\frac{2}{3}} - \dfrac{2}{x^{\frac{1}{3}}} + 1\right)dx - \frac{1}{2}\times4\times4$

$= \left(\frac{3}{5}x^{\frac{5}{3}} - \dfrac{2x^{\frac{2}{3}}}{\frac{2}{3}} + x\right)_1^8 - 8$

$= \left(\frac{3}{5}\times32 - 3\times4 + 8\right)$

$\quad - \left(\frac{3}{5} - 3 + 1\right) - 8$

$= \frac{76}{5} + \frac{7}{5} - 8$

$= \frac{43}{5}$

$= 8\frac{3}{5}$

8 Area $= \int_{\frac{1}{2}}^{2}\left(\text{line } AB - \left(\frac{2}{x^2} + x\right)\right)dx$

Substitute $\frac{1}{2}$ and 2 for x into the equation to find

A is $\left(\frac{1}{2}, 8\frac{1}{2}\right)$ and B is $\left(2, 2\frac{1}{2}\right)$.

The gradient of $AB = \dfrac{6}{-1\frac{1}{2}} = -4$

8 So the equation is $y-2\frac{1}{2}=-4(x-2)$

$$y=10\tfrac{1}{2}-4x$$

$$\text{Area}=\int_{\frac{1}{2}}^{2}\left(10\tfrac{1}{2}-5x-2x^{-2}\right)\,dx$$
$$=\left(\frac{21}{2}x-\frac{5}{2}x^2-\frac{2x^{-1}}{-1}\right)_{\frac{1}{2}}^{2}$$
$$=\left(\frac{21}{2}x-\frac{5}{2}x^2+\frac{2}{x}\right)_{\frac{1}{2}}^{2}$$
$$=(21-10+1)-\left(\tfrac{21}{4}-\tfrac{5}{8}+4\right)$$
$$=12-8\tfrac{5}{8}$$
$$=3\tfrac{3}{8}\text{ or }3.375$$
$$=3.38\ (3\text{ s.f.})$$

9 a On the line, when $x=4$, $y=4-\frac{1}{2}\times 4$

$=2$

On the curve, when $x=4$,

$$y=3\times\sqrt{4}-\sqrt{64}+4$$
$$=6-8+4$$
$$=2$$

So the point (4, 2) lies on the line and the curve.

b $$\text{Area}=\int_0^4\left(3x^{\frac{1}{2}}-x^{\frac{3}{2}}+4-\left(4-\tfrac{1}{2}x\right)\right)\,dx$$
$$=\int_0^4\left(3x^{\frac{1}{2}}-x^{\frac{3}{2}}+\tfrac{1}{2}x\right)\,dx$$
$$=\left(\frac{3x^{\frac{3}{2}}}{\frac{3}{2}}-\frac{x^{\frac{5}{2}}}{\frac{5}{2}}+\frac{x^2}{4}\right)_0^4$$
$$=\left(2x^{\frac{3}{2}}-\tfrac{2}{5}x^{\frac{5}{2}}+\tfrac{1}{4}x^2\right)_0^4$$
$$=\left(2\times 8-\tfrac{2}{5}\times 32+4\right)-(0)$$
$$=20-\tfrac{64}{5}$$
$$=\tfrac{36}{5}\text{ or }7.2$$

10 a $y=x^2(x+4)$

$y=0\Rightarrow x=0\,(\text{twice}),-4$

10 a Area of R_1 is

$$\int_{-4}^{0}\left(x^3+4x^2\right)\,dx=\left(\frac{x^4}{4}+\frac{4}{3}x^3\right)_{-4}^{0}$$
$$=(0)-\left(\frac{4^4}{4}-\frac{4^4}{3}\right)$$
$$=\frac{4^4}{12}$$
$$=\frac{4^3}{3}$$
$$=\tfrac{64}{3}\text{ or }21\tfrac{1}{3}$$

b Area of R_2 is $\int_0^2\left(x^3+4x^2\right)\,dx+$ area of the triangle.

$$\text{Area of }R_2=\left(\frac{x^4}{4}+\frac{4}{3}x^3\right)_0^2+12(b-2)$$
$$=\left(\tfrac{16}{4}+\tfrac{32}{3}\right)-(0)+12(b-2)$$
$$=14\tfrac{2}{3}+12b-24$$
$$=-9\tfrac{1}{3}+12b$$

Area of R_2 = area of R_1

$\Rightarrow -9\frac{1}{3}+12b=21\frac{1}{3}$

So $12b=30\frac{2}{3}$

$\Rightarrow b=2\frac{5}{9}$ or $2.56\ (3\text{ s.f.})$

11 a The intersections occur when

$$10-x=2x^2-5x+4$$
$$0=2x^2-4x-6$$
$$0=2(x+1)(x-3)$$
$$x=-1\text{ or }x=3$$

When $x=-1$, $y=11$, A is (–1, 11).
When $x=3$, $y=7$, B is (3, 7).

b $$\text{Area}=\int_{-1}^{3}[(10-x)-\left(2x^2-5x+4\right)]\,dx$$
$$=\int_{-1}^{3}(10-x-2x^2+5x-4)\,dx$$
$$=\int_{-1}^{3}(6+4x-2x^2)\,dx$$
$$=\left[6x+2x^2-\tfrac{2}{3}x^3\right]_{-1}^{3}$$
$$=(18+18-18)-\left(-6+2+\tfrac{2}{3}\right)$$
$$=18+3\tfrac{1}{3}$$
$$=21\tfrac{1}{3}$$

Integration, Mixed Exercise 13

1 a $\int (x+1)(2x-5)\,dx = \int (2x^2-3x-5)\,dx$

$= 2\frac{x^3}{3} - 3\frac{x^2}{2} - 5x + c$

$= \frac{2}{3}x^3 - \frac{3}{2}x^2 - 5x + c$

b $\int \left(x^{\frac{1}{3}} + x^{-\frac{1}{3}} \right) dx = \frac{x^{\frac{4}{3}}}{\frac{4}{3}} + \frac{x^{\frac{2}{3}}}{\frac{2}{3}} + c$

$= \frac{3}{4}x^{\frac{4}{3}} + \frac{3}{2}x^{\frac{2}{3}} + c$

2 $f'(x) = x^2 - 3x - \frac{2}{x^2} = x^2 - 3x - 2x^{-2}$

So $f(x) = \frac{x^3}{3} - 3\frac{x^2}{2} - 2\frac{x^{-1}}{-1} + c$

So $f(x) = \frac{1}{3}x^3 - \frac{3}{2}x^2 + \frac{2}{x} + c$

But $f(1) = 1 \Rightarrow \frac{1}{3} - \frac{3}{2} + 2 + c = 1$

So $c = \frac{1}{6}$

So the equation is $y = \frac{1}{3}x^3 - \frac{3}{2}x^2 + \frac{2}{x} + \frac{1}{6}$

3 a $\int (8x^3 - 6x^2 + 5)\,dx = 8\frac{x^4}{4} - 6\frac{x^3}{3} + 5x + c$

$= 2x^4 - 2x^3 + 5x + c$

b $\int (5x+2)x^{\frac{1}{2}}\,dx = \int \left(5x^{\frac{3}{2}} + 2x^{\frac{1}{2}} \right) dx$

$= 5\frac{x^{\frac{5}{2}}}{\frac{5}{2}} + 2\frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c$

$= 2x^{\frac{5}{2}} + \frac{4}{3}x^{\frac{3}{2}} + c$

4 $y = \frac{(x+1)(2x-3)}{\sqrt{x}}$

$y = (2x^2 - x - 3)x^{-\frac{1}{2}}$

$y = 2x^{\frac{3}{2}} - x^{\frac{1}{2}} - 3x^{-\frac{1}{2}}$

$\int y\,dx = \int \left(2x^{\frac{3}{2}} - x^{\frac{1}{2}} - 3x^{-\frac{1}{2}} \right) dx$

$= 2\frac{x^{\frac{5}{2}}}{\frac{5}{2}} - \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - 3\frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= \frac{4}{5}x^{\frac{5}{2}} - \frac{2}{3}x^{\frac{3}{2}} - 6x^{\frac{1}{2}} + c$

5 $\frac{dx}{dt} = (t+1)^2 = t^2 + 2t + 1$

$\Rightarrow x = \frac{t^3}{3} + 2\frac{t^2}{2} + t + c$

But $x = 0$ when $t = 2$.

So $0 = \frac{8}{3} + 4 + 2 + c$

$\Rightarrow c = -\frac{26}{3}$

So $x = \frac{1}{3}t^3 + t^2 + t - \frac{26}{3}$

When $t = 3$, $x = \frac{27}{3} + 9 + 3 - \frac{26}{3}$

So $x = \frac{37}{3}$ or $12\frac{1}{3}$

6 a $y^{\frac{1}{2}} = x^{\frac{1}{3}} + 3$

So $y = \left(x^{\frac{1}{3}} + 3 \right)^2$

So $y = \left(x^{\frac{1}{3}} \right)^2 + 6x^{\frac{1}{3}} + 9$

So $y = x^{\frac{2}{3}} + 6x^{\frac{1}{3}} + 9$

$(A = 6,\ B = 9)$

b $\int y\,dx = \int \left(x^{\frac{2}{3}} + 6x^{\frac{1}{3}} + 9 \right) dx$

$= \frac{x^{\frac{5}{3}}}{\frac{5}{3}} + 6\frac{x^{\frac{4}{3}}}{\frac{4}{3}} + 9x + c$

$= \frac{3}{5}x^{\frac{5}{3}} + \frac{9}{2}x^{\frac{4}{3}} + 9x + c$

7 a $y^{\frac{1}{2}} = 3x^{\frac{1}{4}} - 4x^{-\frac{1}{4}}$

$y = (3x^{\frac{1}{4}} - 4x^{-\frac{1}{4}})^2$

$= 9x^{\frac{1}{2}} - 24 + 16x^{-\frac{1}{2}}$

$\frac{dy}{dx} = \frac{9}{2}x^{-\frac{1}{2}} - 8x^{-\frac{3}{2}}$

b $\int \left(9x^{\frac{1}{2}} - 24 + 16x^{-\frac{1}{2}} \right) dx$

$= \frac{9x^{\frac{3}{2}}}{\frac{3}{2}} - 24x + \frac{16x^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= 6x^{\frac{3}{2}} - 24x + 32x^{\frac{1}{2}} + c$

8 $\int \left(\frac{a}{3x^3} - ab \right) dx = \int \left(\frac{a}{3}x^{-3} - ab \right) dx$

$= \frac{a}{3} \times \frac{x^{-2}}{-2} - abx + c$

8 $\int\left(\frac{a}{3x^3}-ab\right)\,dx=-\frac{a}{6x^2}-abx+c$

$=-\frac{2}{3x^2}+14x+c$

Equating coefficients $-\frac{a}{6}=-\frac{2}{3}$ and $-ab=14$

$a=4,\ b=-3.5$

9 $f'(t)=-9.8t$

$f(t)=-\frac{9.8t^2}{2}+c$
$=-4.9t^2+c$

$f(0)=-4.9(0)^2+c$
$=70$
$c=70$

$f(t)=-4.9t^2+70$

$f(3)=-4.9(3)^2+70$
$=25.9$

The height of the rock above the ground after 3 seconds is 25.9 m.

10 a $f(t)=\int(5+2t)\,dt$

$=5t+\frac{2t^2}{2}+c$

$=5t+t^2+c$

As $f(0)=0,\ 5(0)+0^2+c=0$
$c=0$

$f(t)=5t+t^2$

b When $f(t)=100,\ 5t+t^2=100$
$t^2+5t-100=0$

Using the formula

$t=\frac{-5\pm\sqrt{5^2-4(1)(-100)}}{2(1)}$

$t=\frac{-5\pm\sqrt{425}}{2}$

$t=7.8$ or $t=-12.8$
As $t>0$, $t=7.8$ seconds

11 a $2=5+2x-x^2$
$\Rightarrow x^2-2x-3=0$
$\Rightarrow(x-3)(x+1)=0$
$\Rightarrow x=-1(A),3(B)$

b Area of $R=\int_{-1}^{3}\left(5+2x-x^2-2\right)dx$
$=\int_{-1}^{3}\left(3+2x-x^2\right)dx$
$=\left(3x+x^2-\frac{1}{3}x^3\right)_{-1}^{3}$
$=\left(9+9-\frac{27}{3}\right)-\left(-3+1+\frac{1}{3}\right)$
$=9+2-\frac{1}{3}$
$=10\frac{2}{3}$

12 a $\left(x^{\frac{1}{2}}-4\right)\left(x^{-\frac{1}{2}}-1\right)$
$=1-4x^{-\frac{1}{2}}-x^{\frac{1}{2}}+4=5-4x^{-\frac{1}{2}}-x^{\frac{1}{2}}$

$\int\left(x^{\frac{1}{2}}-4\right)\left(x^{-\frac{1}{2}}-1\right)\,dx=5x-\frac{4x^{\frac{1}{2}}}{\frac{1}{2}}-\frac{x^{\frac{3}{2}}}{\frac{3}{2}}+c$
$=5x-8x^{\frac{1}{2}}-\frac{2}{3}x^{\frac{3}{2}}+c$

b $\int_{1}^{4}\left(x^{\frac{1}{2}}-4\right)\left(x^{-\frac{1}{2}}-1\right)\,dx=\left(5x-8x^{\frac{1}{2}}-\frac{2}{3}x^{\frac{3}{2}}\right)_{1}^{4}$
$=\left(20-8\times2-\frac{2}{3}\times2^3\right)-\left(5-8-\frac{2}{3}\right)$
$=4-\frac{16}{3}+3+\frac{2}{3}$
$=7-\frac{14}{3}$
$=\frac{7}{3}$ or $2\frac{1}{3}$

13 a $(x-3)^2=x^2-6x+9$
So $x(x-3)^2=x^3-6x^2+9x$
$y=0\Rightarrow x=0$ or 3 (twice)
So A is the point $(3,0)$.

b $\frac{dy}{dx}=0\Rightarrow0=3x^2-12x+9$
$\Rightarrow0=3\left(x^2-4x+3\right)$
$\Rightarrow0=3(x-3)(x-1)$
$\Rightarrow0=1$ or 3
$x=3$ at A, the minimum, so B is $(1,4)$
(Found by substituting $x=1$ into original equation.)

13 c Area of $R = \int_0^3 \left(x^3 - 6x^2 + 9x\right)\,\mathrm{d}x$

$= \left(\tfrac{1}{4}x^4 - 2x^3 + \tfrac{9}{2}x^2\right)_0^3$

$= \left(\tfrac{81}{4} - 2\times 27 + \tfrac{9}{2}\times 9\right) - (0)$

$= 6\tfrac{3}{4}$

14 a $y = 3x^{\frac{1}{2}} - 4x^{-\frac{1}{2}}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = \tfrac{3}{2}x^{-\frac{1}{2}} + \tfrac{1}{2}\times 4x^{-\frac{3}{2}}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = \tfrac{3}{2}x^{-\frac{1}{2}} + 2x^{-\frac{3}{2}}$

b $\int y\,\mathrm{d}x = \int\left(3x^{\frac{1}{2}} - 4x^{-\frac{1}{2}}\right)\mathrm{d}x$

$= \frac{3x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{4x^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= 2x^{\frac{3}{2}} - 8x^{\frac{1}{2}} + c$

c $\int_1^3 y\,\mathrm{d}x = \left(2x^{\frac{3}{2}} - 8x^{\frac{1}{2}}\right)_1^3$

$= \left(2\times 3\sqrt{3} - 8\sqrt{3}\right) - (2 - 8)$

$= -2\sqrt{3} + 6$

$= 6 - 2\sqrt{3}$

So $A = 6$ and $B = -2$

15 a $y = 12x^{\frac{1}{2}} - x^{\frac{3}{2}}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = 6x^{-\frac{1}{2}} - \tfrac{3}{2}x^{\frac{1}{2}}$

$= \tfrac{3}{2}x^{-\frac{1}{2}}(4 - x)$

b $\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow x = 4, y = 12\times 2 - 2^3 = 16$

So B is the point $(4, 16)$.

c Area $= \int_0^{12}\left(12x^{\frac{1}{2}} - x^{\frac{3}{2}}\right)\mathrm{d}x$

$= \left(\frac{12x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{5}{2}}}{\frac{5}{2}}\right)_0^{12}$

$= \left(8x^{\frac{3}{2}} - \tfrac{2}{5}x^{\frac{5}{2}}\right)_0^{12}$

$= \left(8\times\sqrt{12^3} - \tfrac{2}{5}\sqrt{12^5}\right) - (0)$

$= 133\ (3\text{ s.f.})$

16 a $x(8 - x) = 12$

$\Rightarrow 8x - x^2 = 12$

$\Rightarrow 0 = x^2 - 8x + 12$

$\Rightarrow 0 = (x - 6)(x - 2)$

$\Rightarrow x = 2$ or $x = 6$

M is on the same line as L.

So M is the point $(6, 12)$.

b Area $= \int_6^8\left(8x - x^2\right)\mathrm{d}x$

$= \left(4x^2 - \frac{x^3}{3}\right)_6^8$

$= \left(4\times 64 - \tfrac{512}{3}\right) - \left(4\times 36 - \tfrac{216}{3}\right)$

$= 256 - 170\tfrac{2}{3} - 144 + 72$

$= 13\tfrac{1}{3}$

17 a A is the point $(1, 0)$, B is the point $(5, 0)$.

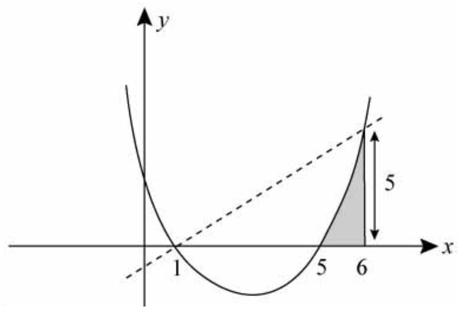
$x - 1 = (x - 1)(x - 5)$

$\Rightarrow 0 = (x - 1)(x - 5 - 1)$

$\Rightarrow 0 = (x - 1)(x - 6)$

$\Rightarrow x = 1, x = 6$

So C is the point $(6, 5)$.

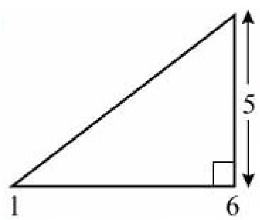


b Drop a perpendicular from C to the x-axis to a point D.

The area of the shaded region is

Area of triangle $ABD - \int_5^6 (x - 1)(x - 5)\,\mathrm{d}x$

$=$ Area of $ABD - \int_5^6\left(x^2 - 6x + 5\right)\mathrm{d}x$



17 b Area $=\left(\frac{1}{2}\times5\times5\right)-\int_5^6\left(x^2-6x+5\right)\,dx$

$=12\frac{1}{2}-\left[\frac{1}{3}x^3-3x^2+5x\right]_5^6$

$=12\frac{1}{2}-\left[\left(72-108+30\right)-\left(41\frac{2}{3}-75+25\right)\right]$

$=12\frac{1}{2}-\left[\left(-6\right)-\left(-8\frac{1}{3}\right)\right]$

$=12\frac{1}{2}-2\frac{1}{3}$

$=10\frac{1}{6}$

18 a For the point A, which lies on the line and the curve

$4q+25=p+40-16$

$\Rightarrow\ 4q=p-1\qquad$ **(1)**

For the point B, which lies on the line and the curve

$8q+25=p+80-64$

$\Rightarrow\ 8q=p-9\qquad$ **(2)**

Subtracting **(2)** − **(1)**

$\Rightarrow 4q=-8$

$\Rightarrow q=-2$

Substituting into **(1)**

$\Rightarrow p=1+4q$

$\Rightarrow p=-7$

b At A, $y=4q+25=17$

So C is given by

$17=-7+10x-x^2$

$x^2-10x+24=0$

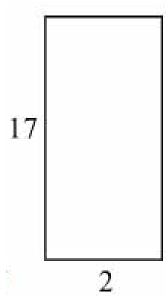
$(x-6)(x-4)=0$

$x=4,\ x=6$

So C is the point $(6,17)$

c The required area is

$\int_4^6(-7+10x-x^2)\,dx-\text{area of rectangle}$



18 c Area $=\left(-7x+5x^2-\frac{1}{3}x^3\right)_4^6-34$

$=\left(-42+180-72\right)-\left(-28+80-\frac{64}{3}\right)-34$

$=\frac{4}{3}$ or $1\frac{1}{3}$

19 $\int\left(\frac{9}{x^2}-8\sqrt{x}+4x-5\right)\,dx$

$=\int(9x^{-2}-8x^{\frac{1}{2}}+4x-5)\,dx$

$=\frac{9x^{-1}}{-1}-\frac{8x^{\frac{3}{2}}}{\frac{3}{2}}+\frac{4x^2}{2}-5x+c$

$=-\frac{9}{x}-\frac{16x^{\frac{3}{2}}}{3}+2x^2-5x+c$

20 $A^2=\int_4^9\left(\frac{3}{\sqrt{x}}-A\right)\,dx$

$=\int_4^9\left(3x^{-\frac{1}{2}}-A\right)\,dx$

$=\left[\frac{3x^{\frac{1}{2}}}{\frac{1}{2}}-Ax\right]_4^9$

$=\left[6x^{\frac{1}{2}}-Ax\right]_4^9$

$=\left(6(9)^{\frac{1}{2}}-A(9)\right)-\left(6(4)^{\frac{1}{2}}-A(4)\right)$

$=\left(18-9A\right)-\left(12-4A\right)$

$0=(A+6)(A-1)$

$A=-6$ or $A=1$

21 a $f'(x)=\frac{(2-x^2)^3}{x^2}$

$=\frac{(2-x^2)(2-x^2)(2-x^2)}{x^2}$

$=\frac{(4-4x^2+x^4)(2-x^2)}{x^2}$

$=x^{-2}(8-12x^2+6x^4-x^6)$

$=8x^{-2}-12+6x^2-x^4$

So $A=6$ and $B=-1$

b $f''(x)=-16x^{-3}+12x-4x^3$

c $f(x)=\int(8x^{-2}-12+6x^2-x^4)\,dx$

$=\frac{8x^{-1}}{-1}-12x+\frac{6x^3}{3}-\frac{x^5}{5}+c$

$=-\frac{8}{x}-12x+2x^3-\frac{x^5}{5}+c$

21 c When $x = -2$ and $y = 9$

$$-\frac{8}{-2} - 12(-2) + 2(-2)^3 - \frac{(-2)^5}{5} + c = 9$$

$$4 + 24 - 16 + \tfrac{32}{5} + c = 9$$

$$c = -\tfrac{47}{5}$$

$$f(x) = -\frac{8}{x} - 12x + 2x^3 - \frac{x^5}{5} - \frac{47}{5}$$

22 a $y = 3 - 5x - 2x^2$

When $y = 0$, $3 - 5x - 2x^2 = 0$

$(3 + x)(1 - 2x) = 0$

$x = -3$ or $x = \frac{1}{2}$

The points are $A(-3, 0)$ and $B(\frac{1}{2}, 0)$.

b $\int_{-3}^{\frac{1}{2}} (3 - 5x - 2x^2)\,dx$

$$= \left[3x - \frac{5x^2}{2} - \frac{2x^3}{3}\right]_{-3}^{\frac{1}{2}}$$

$$= \left(3\left(\tfrac{1}{2}\right) - \frac{5\left(\frac{1}{2}\right)^2}{2} - \frac{2\left(\frac{1}{2}\right)^3}{3}\right) - \left(3(-3) - \frac{5(-3)^2}{2} - \frac{2(-3)^3}{3}\right)$$

$$= \left(\tfrac{3}{2} - \tfrac{5}{8} - \tfrac{1}{12}\right) - \left(-9 - \tfrac{45}{2} + \tfrac{54}{3}\right)$$

$$= 14\tfrac{7}{24}$$

23 a $(x - 4)(2x + 3) = 0$

$x = 4$ or $x = -\frac{3}{2}$

The points are $A(-\frac{3}{2}, 0)$ and $B(4, 0)$.

b $R = \int_{-\frac{3}{2}}^{4} (x - 4)(2x + 3)\,dx$

$$= \int_{-\frac{3}{2}}^{4} (2x^2 - 5x - 12)\,dx$$

$$= \left[\frac{2x^3}{3} - \frac{5x^2}{2} - 12x\right]_{-\frac{3}{2}}^{4}$$

$$= \left(\frac{2(4)^3}{3} - \frac{5(4)^2}{2} - 12(4)\right) - \left(\frac{2(-\frac{3}{2})^3}{3} - \frac{5(-\frac{3}{2})^2}{2} - 12(-\tfrac{3}{2})\right)$$

$$= \left(\tfrac{128}{3} - 40 - 48\right) - \left(-\tfrac{9}{4} - \tfrac{45}{8} + 18\right)$$

$$= -55\tfrac{11}{24}$$

Area $= 55\frac{11}{24}$

24 a $x(x - 3)(x + 2) = 0$

$x = 0$, $x = 3$ or $x = -2$

The points are $A(-2, 0)$ and $B(3, 0)$.

b $\int_{-2}^{0} x(x - 3)(x + 2)\,dx - \int_{0}^{3} x(x - 3)(x + 2)\,dx$

$$= \int_{-2}^{0} (x^3 - x^2 - 6x)\,dx - \int_{0}^{3} (x^3 - x^2 - 6x)\,dx$$

$$\int_{-2}^{0} (x^3 - x^2 - 6x)\,dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - \frac{6x^2}{2}\right]_{-2}^{0}$$

$$= \left[\frac{x^4}{4} - \frac{x^3}{3} - 3x^2\right]_{-2}^{0}$$

$$= \left(\frac{0^4}{4} - \frac{0^3}{3} - 3(0)^2\right) - \left(\frac{(-2)^4}{4} - \frac{(-2)^3}{3} - 3(-2)^2\right)$$

$$= 0 - \left(4 + \tfrac{8}{3} - 12\right)$$

$$= 5\tfrac{1}{3}$$

$$\int_{0}^{3} (x^3 - x^2 - 6x)\,dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - 3x^2\right]_{0}^{3}$$

$$= \left(\frac{3^4}{4} - \frac{3^3}{3} - 3(3)^2\right) - \left(\frac{0^4}{4} - \frac{0^3}{3} - 3(0)^2\right)$$

$$= \left(\tfrac{81}{4} - 9 - 27\right)$$

$$= -15\tfrac{3}{4}$$

Total area is $5\frac{1}{3} - (-15\frac{3}{4}) = 21\frac{1}{12}$

Challenge

To find the points of intersection:

$x^2 - 5x + 7 = \frac{1}{2}x^2 - \frac{5}{2}x + 7$
$2x^2 - 10x + 14 = x^2 - 5x + 14$
$x^2 - 5x = 0$
$x(x - 5) = 0$
$x = 0$ or $x = 5$

Area R =
(area under the curve $y = \frac{1}{2}x^2 - \frac{5}{2}x + 7$)
– (area under the curve $y = x^2 - 5x + 7$)

Area under the curve: $y = \frac{1}{2}x^2 - \frac{5}{2}x + 7$:

$$\begin{aligned}
\text{Area under the curve} &= \int_0^5 \left(\tfrac{1}{2}x^2 - \tfrac{5}{2}x + 7\right) \mathrm{d}x \\
&= \left[\frac{1}{3}\times\frac{x^3}{2} - \frac{1}{2}\times\frac{5x^2}{2} + 7x\right]_0^5 \\
&= \left[\frac{x^3}{6} - \frac{5x^2}{4} + 7x\right]_0^5 \\
&= \left(\frac{5^3}{6} - \frac{5(5)^2}{4} + 7(5)\right) \\
&\quad - \left(\frac{0^3}{6} - \frac{5(0)^2}{4} + 7(0)\right) \\
&= \left(\tfrac{125}{6} - \tfrac{125}{4} + 35\right) \\
&\; 24\tfrac{7}{12}
\end{aligned}$$

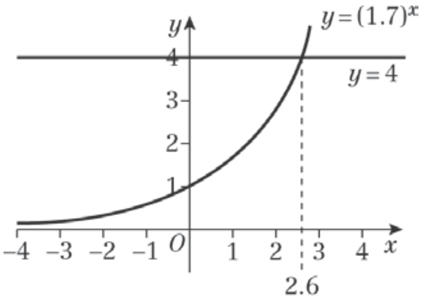
Area under the curve: $y = x^2 - 5x + 7$:

$$\begin{aligned}
\text{Area under the curve} &= \int_0^5 (x^2 - 5x + 7)\, \mathrm{d}x \\
&= \left[\frac{x^3}{3} - \frac{5x^2}{2} + 7x\right]_0^5 \\
&= \left(\frac{5^3}{3} - \frac{5(5)^2}{2} + 7(5)\right) \\
&\quad - \left(\frac{0^3}{3} - \frac{5(0)^2}{2} + 7(0)\right) \\
&= \left(\tfrac{125}{3} - \tfrac{125}{2} + 35\right) \\
&= 14\tfrac{1}{6}
\end{aligned}$$

$$\begin{aligned}
R &= 24\tfrac{7}{12} - 14\tfrac{1}{6} \\
&= 10\tfrac{5}{12}
\end{aligned}$$

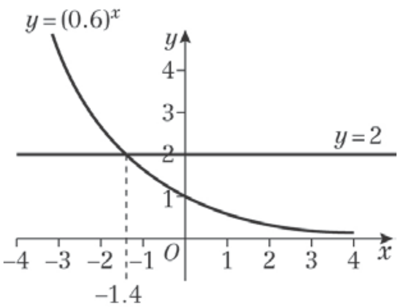
Exponentials and logarithms 14A

1 a



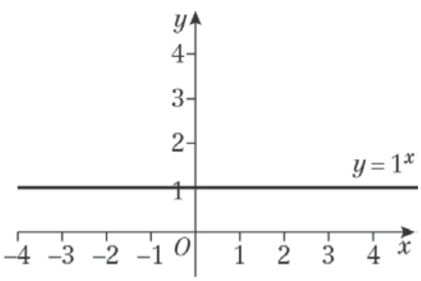
b Where $y = 4$, $x \approx 2.6$

2 a



b Where $y = 2$, $x \approx -1.4$

3

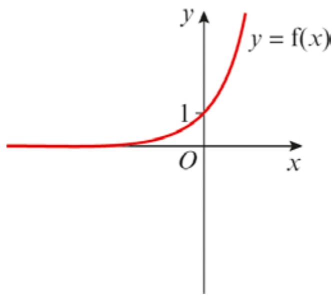


4 a True because, when $x = 0$, $a^0 = 1$ when a is positive

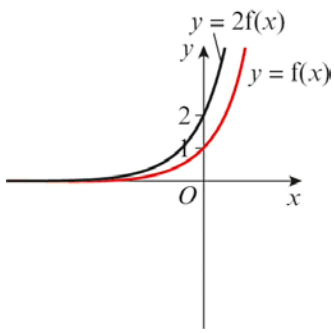
b False. For example, when $a = \frac{1}{2}$, the function $f(x) = a^x$ is not an increasing function.

c True because, when a is positive, $a^x > 0$ for all values of x.

5 a The graph crosses the y-axis when $x = 0$.
$y = 3^0$
So $y = 1$
The graph crosses the y-axis at (0, 1).
Asymptote is at $y = 0$.

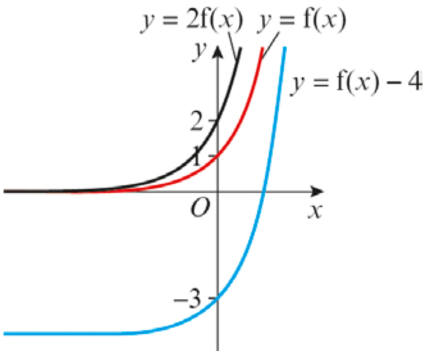


b The graph is a vertical stretch by scale factor 2.
The graph crosses the y-axis when $x = 0$.
$y = 2 \times 3^0$
So $y = 2$
The graph crosses the y-axis at (0, 2).
Asymptote is at $y = 0$.

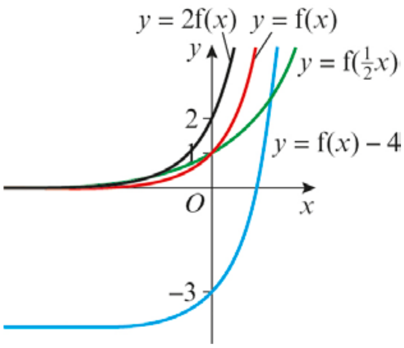


c The graph is a translation by the vector $\begin{pmatrix} 0 \\ -4 \end{pmatrix}$.
The graph crosses the y-axis when $x = 0$.
$y = 3^0 - 4$
So $y = -3$
The graph crosses the y-axis at (0, −3).
Asymptote is at $y = -4$.

5 c



d The graph is a horizontal stretch by scale factor 2.
The graph crosses the y-axis when $x = 0$.
$y = 3^{\frac{1}{2} \times 0}$
So $y = 1$
The graph crosses the y-axis at (0, 1).
Asymptote is at $y = 0$.

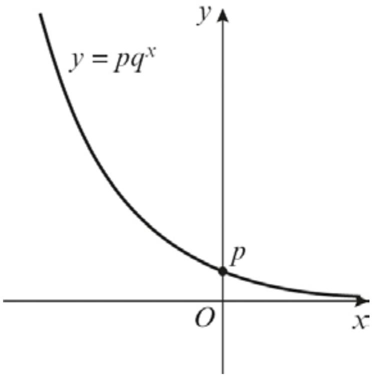


6 Substitute the coordinates into $y = ka^x$.
$6 = ka^1$ (equation 1)
$48 = ka^4$ (equation 2)
Solve simultaneously: divide equation 2 by equation 1,

$$48 \div 6 = \frac{ka^4}{ka}$$

$a^3 = 8$
$a = 2, k = 3$

7 a As x increases, y decreases



b Substitute the coordinates into $y = pq^x$.
$150 = pq^{-3}$ (equation 1)
$0.048 = pq^2$ (equation 2)
Solve simultaneously, divide equation 2 by equation 1.

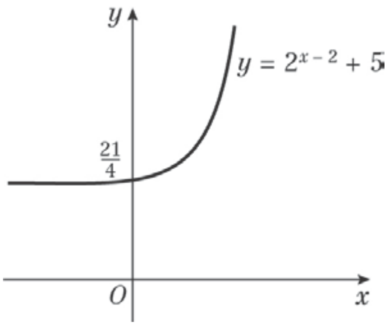
$$0.048 \div 150 = \frac{pq^2}{pq^{-3}}$$

$q^5 = 0.00032$
$q = 0.2$
$p = 0.048 \div 0.2^2 = 1.2$
$p = 1.2, q = 0.2$

Challenge

To draw the graph, note that it is a translation of the graph $y = 2^x$ by the vector $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$.

The graph crosses the y-axis when $x = 0$, so $y = 2^{0-2} + 5$
$y = 5.25$
The graph crosses the y-axis at (0, 5.25).
Asymptote is at $y = 5$.



Exponentials and logarithms 14B

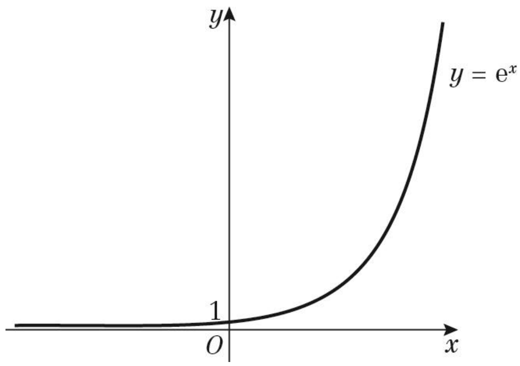
1 a 2.71828

b 54.59815

c 0.00005

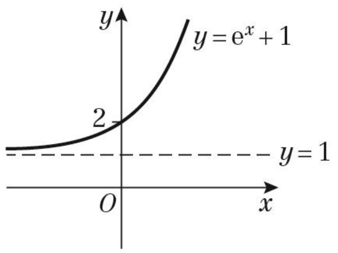
d 1.22140

2 a



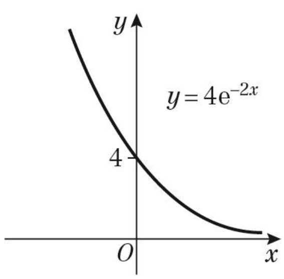
e = 2.71828…
$e^3 = 20.08553…$

3 a $y = e^x + 1$



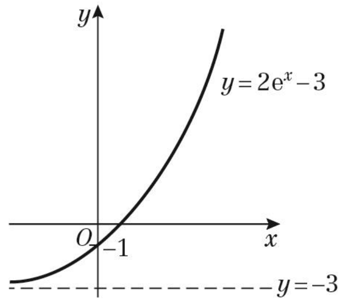
This is the normal $y = e^x$ ‘moved up’ (translated) 1 unit

b $y = 4e^{-2x}$



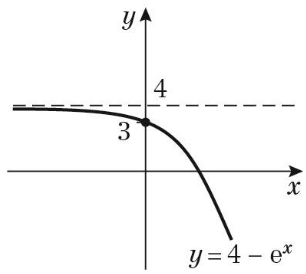
$x = 0 \Rightarrow y = 4$
As $x \to -\infty, y \to \infty$
As $x \to \infty, y \to 0$
This is an exponential decay type of graph.

c $y = 2e^x - 3$



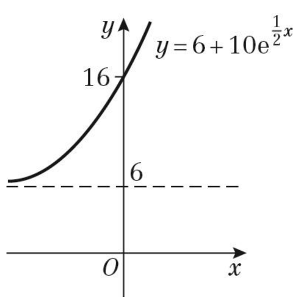
$x = 0 \Rightarrow y = 2\times1-3 = -1$
As $x \to \infty,\ y \to \infty$
As $x \to -\infty,\ y \to 2\times0-3 = -3$

d $y = 4 - e^x$



$x = 0 \Rightarrow y = 4-1 = 3$
As $x \to \infty,\ y \to 4-\infty$, i.e. $y \to -\infty$
As $x \to -\infty,\ y \to 4-0 = 4$

e $y = 6 + 10e^{\frac{1}{2}x}$

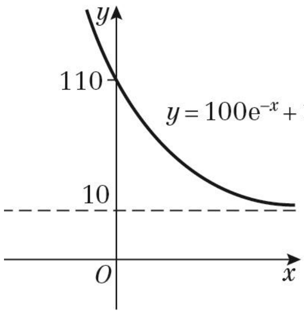


3 e $x=0 \Rightarrow y=6+10\times1=16$

As $x \to \infty, y \to \infty$

As $x \to -\infty, y \to 6+10\times0=6$

f $y=100\mathrm{e}^{-x}+10$



$x=0 \Rightarrow y=100\times1+10=110$

As $x \to \infty,\ y \to 100\times0+10=10$

As $x \to -\infty,\ y \to \infty$

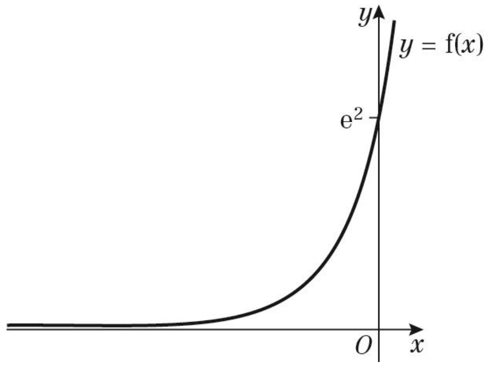
4 a The graph is increasing so b is positive.
The line $y = 5$ is an asymptote, so $C = 5$.
When $x = 0$, $6 = A\mathrm{e}^{b\times0} + C = A + 5$, so $A = 1$.

b The graph is decreasing so b is negative.
The line $y = 0$ is an asymptote, so $C = 0$.
When $x = 0$, $4 = A\mathrm{e}^{b\times0} + C = A + 0$, so $A = 4$.

c The graph is increasing so b is positive.
The line $y = 2$ is an asymptote, so $C = 2$.
When $x = 0$, $8 = A\mathrm{e}^{b\times0} + C = A + 2$, so $A = 6$.

5 $\mathrm{f}(x) = \mathrm{e}^{3x+2}$
$= \mathrm{e}^{3x}\times\mathrm{e}^{2}$
$= \mathrm{e}^{2}\mathrm{e}^{3x}$
$A = \mathrm{e}^2$ and $b = 3$

5



6 a $y = \mathrm{e}^{6x}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = 6\mathrm{e}^{6x}$

b $y = \mathrm{e}^{-\frac{1}{3}x}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{3}\mathrm{e}^{-\frac{1}{3}}$

c $y = 7\mathrm{e}^{2x}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = 2\times7\mathrm{e}^{2x} = 14\mathrm{e}^{2x}$

d $y = 5\mathrm{e}^{0.4x}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = 0.4\times5\mathrm{e}^{0.4x} = 2\mathrm{e}^{0.4x}$

e $y = \mathrm{e}^{3x} + 2\mathrm{e}^{x}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = 3\mathrm{e}^{3x} + 2\mathrm{e}^{x}$

f $y = \mathrm{e}^{x}(\mathrm{e}^{x}+1) = \mathrm{e}^{2x} + \mathrm{e}^{x}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = 2\mathrm{e}^{2x} + \mathrm{e}^{x}$

7 a $y = \mathrm{e}^{3x}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = 3\mathrm{e}^{3x}$

When $x = 2$,

$\frac{\mathrm{d}y}{\mathrm{d}x} = 3\mathrm{e}^{3\times2} = 3\mathrm{e}^{6}$

7 b When $x = 0$,

$$\frac{dy}{dx} = 3e^{3 \times 0} = 3$$

c When $x = -0.5$,

$$\frac{dy}{dx} = 3e^{3 \times -0.5} = 3e^{-1.5}$$

8 $f(x) = e^{0.2x}$
$f'(x) = 0.2e^{0.2x}$
The gradient of the tangent when $x = 5$
is $f'(5) = 0.2e^{0.2 \times 5} = 0.2e$
$f(5) = e^{0.2 \times 5} = e$
The equation of the tangent in the form
$y = mx + c$
is $e = 0.2e \times 5 + c$
$e = e + c$
so $c = 0$
Therefore the tangent to the curve at the point $(5, c)$ is in the form $y = mx$.
Thus it so goes through the origin.

Exponentials and logarithms 14C

1 $V = 20\,000\,\mathrm{e}^{-\frac{1}{12}}$

a The new value is when $t = 0$

$$\Rightarrow V = 20\,000 \times \mathrm{e}^{-\frac{0}{12}}$$
$$= 20\,000 \times 1$$
$$= 20\,000$$

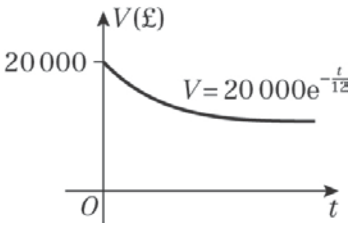
New value = £20 000

b Value after 4 years is given when $t = 4$

$$\Rightarrow V = 20\,000 \times \mathrm{e}^{-\frac{4}{12}}$$
$$= 20\,000 \times \mathrm{e}^{-\frac{1}{3}}$$
$$= 14\,330.63$$

Value after 4 years is £14 331 (to nearest £).

c



2 $P = 20 + 10\,\mathrm{e}^{\frac{t}{50}}$

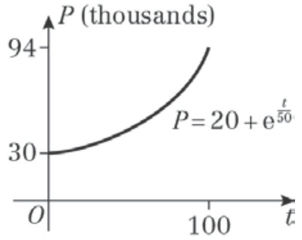
a The year 2000 corresponds to $t = 0$.

Substitute $t = 0$ into $P = 20 + 10\,\mathrm{e}^{\frac{t}{50}}$

$P = 20 + 10 \times \mathrm{e}^0 = 20 + 10 \times 1 = 30$
Population = 30 thousand

b $P = 20 + 10\mathrm{e}^{\frac{30}{50}}$
P = 38.221... thousand

2 c



Year 2100 is $t = 100$

$$P = 20 + 10\,\mathrm{e}^{\frac{100}{50}}$$
$$= 20 + 10\,\mathrm{e}^2$$
$$= 93.891 \text{ thousand}$$

d $P = 20 + 10\mathrm{e}^{\frac{500}{50}}$
= 220 284.6579... thousand
The model predicts the population of the country to be over 220 million. This is highly unlikely and by 2500 new factors are likely to affect population growth. Therefore, the model is not valid for predictions that far into the future.

3 $N = 300 - 100\,\mathrm{e}^{-0.5t}$

a The number first diagnosed means when $t = 0$.

Substitute $t = 0$ in $N = 300 - 100\,\mathrm{e}^{-0.5t}$

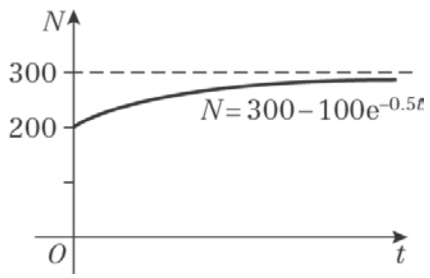
$$N = 300 - 100 \times \mathrm{e}^{-0.5 \times 0}$$
$$= 300 - 100 \times 1$$
$$= 200$$

b The long term prediction suggests $t \to \infty$.

As $t \to \infty, \mathrm{e}^{-0.5t} \to 0$

So $N \to 300 - 100 \times 0 = 300$

3 c



4 a i $R = 12e^{0.2m}$

$R = 12e^{0.2 \times 1}$

$= 14.6568...$

15 rabbits

ii $R = 12e^{0.2m}$

$R = 12e^{0.2 \times 12}$

$= 132.278...$

132 rabbits

b When $m = 0$, $R = 12$

12 is the initial number of rabbits in the population.

c $\dfrac{dR}{dm} = 0.2 \times 12e^{0.2m}$

$= 2.4e^{0.2m}$

When $m = 6$,

$\dfrac{dR}{dm} = 2.4e^{0.2\times6}$

$= 7.9682...$

≈ 8

d This model will stop giving valid results for large enough values of t as new factors are likely to affect population growth, such as the rabbits running out of food or space.

5 a $p = e^{-0.13h}$

$p = e^{-0.13 \times 4.394}$

$= 0.5648359...$

0.565 bars

5 b $\dfrac{dp}{dh} = -0.13e^{-0.13h}$

As $p = e^{-0.13h}$

$\dfrac{dp}{dh} = -0.13p$

$k = -0.13$

c The atmospheric pressure decreases exponentially as the altitude increases.

d $\dfrac{dp}{dh} = -0.13e^{-0.13h}$

When $h = 0$,

$\dfrac{dp}{dh} = -0.13e^{-0.13\times0}$

$= -0.13$

When $h = 1$,

$\dfrac{dp}{dh} = -0.13e^{-0.13\times1}$

$= -0.114152406...$

Percentage change

$= \dfrac{-0.11415206 + 0.13}{-0.13} \times 100$

$= 12.19...$

$= 12\%$

6 a Model 1: $T = 20\,000e^{-0.24t}$

When $t = 1$, $T = 20\,000e^{-0.24 \times 1}$

$= 15\,732.557...$

$= £15\,733$

Model 2: $T = 19\,000e^{-0.255t} + 1000$

When $t = 1$, $T = 19\,000e^{-0.255 \times 1} + 1000$

$= 15\,723.413...$

$= £15\,723$

b Model 1: $T = 20\,000e^{-0.24t}$

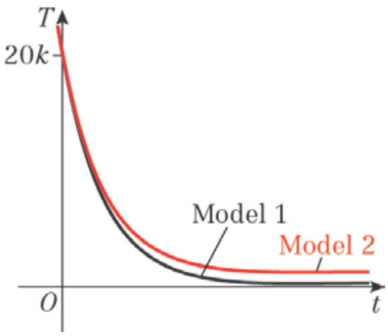
When $t = 10$, $T = 20\,000e^{-0.24 \times 10}$

$= 1814.359...$

$= £1814$

6 b Model 2: $T = 19\,000e^{-0.255t} + 1000$
When $t = 10$,
$T = 19\,000e^{-0.255 \times 10} + 1000$
$= 2483.5516...$
$= £2484$
So, Model 2 predicts a larger value.

c



d In Model 2 the tractor will always be worth at least £1000. This could be the value of the tractor as scrap metal.

Exponentials and logarithms 14D

1 a $\log_4 256 = 4$

b $\log_3\left(\frac{1}{9}\right) = -2$

c $\log_{10} 1\ 000\ 000 = 6$

d $\log_{11} 11 = 1$

e $\log_{0.2} 0.008 = 3$

2 a $2^4 = 16$

b $5^2 = 25$

c $9^{\frac{1}{2}} = 3$

d $5^{-1} = 0.2$

e $10^5 = 100\ 000$

3 a If $\log_2 8 = x$ then $2^x = 8$, so $x = 3$

b If $\log_5 25 = x$ then $5^x = 25$, so $x = 2$

c If $\log_{10} 10\ 000\ 000 = x$
then $10^x = 10\ 000\ 000$, so $x = 7$

d If $\log_{12} 12 = x$ then $12^x = 12$, so $x = 1$

e If $\log_3 729 = x$ then $3^x = 729$, so $x = 6$

f If $\log_{10}\sqrt{10} = x$
then $10^x = \sqrt{10}$, so $x = \frac{1}{2}$
(Power $\frac{1}{2}$ means 'square root'.)

g If $\log_4(0.25) = x$ then $4^x = 0.25 = \frac{1}{4}$,
so $x = -1$
(Negative power means 'reciprocal'.)

h $\log_{0.25} 16 = x$
$\Rightarrow 0.25^x = 16$
$\Rightarrow \left(\frac{1}{4}\right)^x = 16$, so $x = -2$
$$\left(\left(\frac{1}{4}\right)^{-2} = \frac{1}{\left(\frac{1}{4}\right)^2} = \frac{1}{\left(\frac{1}{16}\right)} = 16\right)$$

i $\log_a\left(a^{10}\right) = x$
$\Rightarrow a^x = a^{10}$, so $x = 10$

j $\log_{\left(\frac{2}{3}\right)}\left(\frac{9}{4}\right) = x$

$$\Rightarrow \left(\frac{2}{3}\right)^x = \frac{9}{4} = \frac{1}{\left(\frac{2}{3}\right)^2} = \frac{1}{\left(\frac{4}{9}\right)} = \frac{9}{4} \Rightarrow x = -2$$

4 a Using a power, $5^4 = x$
So $x = 625$

b Using a power, $x^2 = 81$
So $x = 9$
(The base of a logarithm cannot be negative, so $x = -9$ is not possible.)

c Using a power, $7^1 = x$
So $x = 7$

d $2^3 = x - 1$
$x = 2^3 + 1$
$= 9$

4 e $3^4 = 4x + 1$
$4x = 3^4 - 1$
$x = \frac{1}{4}(3^4 - 1)$
$= 20$

f Using a power,
$x^2 = 2x$
$x^2 - 2x = 0$
$x(x-2) = 0$
$x = 2$
(The base of a logarithm cannot be 0, so $x = 0$ is not possible)

5 a $\log_9 230 = 2.475$

b $\log_5 33 = 2.173$

c $\log_{10} 1020 = 3.009$

d $\log_e 3 = 1.099$

6 a Let $\log_2 50 = x$
$2^x = 50$
As $2^5 = 32$ and $2^6 = 64$,
$32 < 2^x < 64$
$2^5 < 2^x < 2^6$
So $5 < x < 6$

b $\log_2 50 = 5.644$

7 a i $\log_2 2 = 1$

ii $\log_3 3 = 1$

iii $\log_{17} 17 = 1$

b Let $\log_a a = x$
$a^x = a$
$x = 1$
$\therefore \log_a a = 1$

8 a i $\log_2 1 = 0$

ii $\log_3 1 = 0$

iii $\log_{17} 1 = 0$

b Let $\log_a 1 = x$
$a^x = 1$
$x = 0$
$\therefore \log_a 1 = 0$

Exponentials and logarithms 14E

1 a $\log_2(7\times 3)=\log_2 21$

b $\log_2\left(\frac{36}{4}\right)=\log_2 9$

c $3\log_5 2=\log_5 2^3=\log_5 8$

$\log_5 8+\log_5 10=\log_5(8\times 10)=\log_5 80$

d $2\log_5 8=\log_6 8^2=\log_6 64$

$4\log_6 3=\log_6 3^4=\log_6 81$

$\log_6 64-\log_6 81=\log_6\left(\frac{64}{81}\right)$

e $\log_{10} 5+\log_{10} 6=\log_{10}(5\times 6)=\log_{10} 30$

$\log_{10} 30-\log_{10}\left(\frac{1}{4}\right)=\log_{10}\left(\frac{30}{\frac{1}{4}}\right)$

$=\log_{10} 120$

2 a $\log_2\left(\frac{40}{5}\right)=\log_2 8=3 \quad (2^3=8)$

b $\log_6(4\times 9)=\log_6 36=2 \quad (6^2=36)$

c $2\log_{12} 3+4\log_{12} 2$

$=\log_{12}(3^2)+\log_{12}(2^4)$

$=\log_{12} 9+\log_{12} 16$

$=\log_{12}(9\times 16)$

$=\log_{12} 144$

$=2 \quad (12^2=144)$

d $\log_8(25\times 10)-\log_8(5^3)$

$=\log_8 250-\log_8 125$

$=\log_8\left(\frac{250}{125}\right)$

$=\log_8 2$

$=\frac{1}{3} \quad \left(8^{\frac{1}{3}}=2\right)$

2 e $\log_{10}(2^2)-\log_{10}(5\times 8)$

$=\log_{10} 4-\log_{10} 40$

$=\log_{10}\left(\frac{4}{40}\right)$

$=\log_{10}\left(\frac{1}{10}\right)$

$=-1 \quad \left(10^{-1}=\frac{1}{10}\right)$

3 a $\log_a x^3+\log_a y^4+\log_a z$

$=3\log_a x+4\log_a y+\log_a z$

b $\log_a x^5-\log_a y^2$

$=5\log_a x-2\log_a y$

c $\log_a a^2+\log_a x^2$

$=2\log_a a+2\log_a x$

$=2+2\log_a x \quad (\log_a a=1)$

d $\log_a\left(\frac{x}{\sqrt{yz}}\right)$

$=\log_a x-\log_a\sqrt{yz}$

$=\log_a x-(\log_a\sqrt{y}+\log_a z)$

$=\log_a x-\frac{1}{2}\log_a y-\log_a z$

e $\log_a\left((ax)^{\frac{1}{2}}\right)$

$=\frac{1}{2}\log_a(ax)$

$=\frac{1}{2}\log_a a+\frac{1}{2}\log_a x$

$=\frac{1}{2}+\frac{1}{2}\log_a x$

4 a $\log_2 3+\log_2 x=2$

$\log_2(3\times x)=2$

$2^2=3x$

$x=\frac{2^2}{3}$

$=\frac{4}{3}$

4 b $\log_6 12 - \log_6 x = 3$

$\log_6\left(\frac{12}{x}\right) = 3$

$6^3 = \frac{12}{x}$

$x = \frac{12}{6^3} = \frac{1}{18}$

c $2\log_5 x = 1 + \log_5 6$

$2\log_5 x - \log_5 6 = 1$

$\log_5 x^2 - \log_5 6 = 1$

$\log_5\left(\frac{x^2}{6}\right) = 1$

$5^1 = \frac{x^2}{6}$

$x^2 = 30$

$x = \sqrt{30}$

d $2\log_9(x+1) = 2\log_9(2x-3) + 1$

$2\log_9(x+1) - 2\log_9(2x-3) = 1$

$\log_9(x+1)^2 - \log_9(2x-3)^2 = 1$

$\log_9\left(\frac{x+1}{2x-3}\right)^2 = 1$

$\left(\frac{x+1}{2x-3}\right)^2 = 9^1$

$\frac{x+1}{2x-3} = 3$

$x + 1 = 6x - 9$

$x = 2$

5 a $\log_3(x+1) = 1 + 2\log_3(x-1)$

$\log_3(x+1) - 2\log_3(x-1) = 1$

$\log_3(x+1) - \log_3(x-1)^2 = 1$

$\log_3\frac{x+1}{(x-1)^2} = 1$

$\frac{x+1}{(x-1)^2} = 3^1$

$3(x-1)^2 = x + 1$

$3(x^2 - 2x + 1) = x + 1$

$3x^2 - 6x + 3 - x - 1 = 0$

$3x^2 - 7x + 2 = 0$

5 b $3x^2 - 7x + 2 = 0$

$(3x - 1)(x - 2) = 0$

$x = \frac{1}{3}$ or $x = 2$

The equation contains $\log(x - 1)$, so $x > 1$, therefore $x = 2$

6 $\log_6 a + \log_6 b = 2$

$\log_6(ab) = 2$

$6^2 = ab$

$ab = 36$

Rearrange $a + b = 13$

$13 - b = a$

Using substitution

$(13 - b)b = 36$

$13b - b^2 = 36$

$b^2 - 13b + 36 = 0$

$(b - 9)(b - 4) = 0$

$b = 9$ or 4

When $b = 9$, $a = 4$

When $b = 4$, $a = 9$

As $a > b$, $a = 9$ and $b = 4$

Challenge

$\log_a x = m$ and $\log_a y = n$

$a^m = x$ and $a^n = y$

$\frac{x}{y} = \frac{a^m}{a^n} = a^{m-n}$

$\log_a\left(\frac{x}{y}\right) = m - n = \log_a x - \log_a y$

Exponentials and logarithms 14F

1 a
$$\begin{aligned} 2^x &= 75 \\ \log 2^x &= \log 75 \\ x\log 2 &= \log 75 \\ x &= \frac{\log 75}{\log 2} \\ &= 6.23\ (3\text{ s.f.}) \end{aligned}$$

b
$$\begin{aligned} 3^x &= 10 \\ \log 3^x &= \log 10 \\ x\log 3 &= \log 10 \\ x &= \frac{\log 10}{\log 3} \\ &= 2.10\ (3\text{ s.f.}) \end{aligned}$$

c
$$\begin{aligned} 5^x &= 2 \\ \log 5^x &= \log 2 \\ x\log 5 &= \log 2 \\ x &= \frac{\log 2}{\log 5} \\ &= 0.431\ (3\text{ s.f.}) \end{aligned}$$

d
$$\begin{aligned} 4^{2x} &= 100 \\ \log 4^{2x} &= \log 100 \\ 2x\log 4 &= \log 100 \\ x &= \frac{\log 100}{2\log 4} \\ &= 1.66\ (3\text{ s.f.}) \end{aligned}$$

e
$$\begin{aligned} 9^{x+5} &= 50 \\ \log 9^{x+5} &= \log 50 \\ (x+5)\log 9 &= \log 50 \\ x\log 9 + 5\log 9 &= \log 50 \\ x\log 9 &= \log 50 - 5\log 9 \\ x &= \frac{\log 50 - 5\log 9}{\log 9} \\ &= -3.22\ (3\text{ s.f.}) \end{aligned}$$

f
$$\begin{aligned} 7^{2x-1} &= 23 \\ \log 7^{2x-1} &= \log 23 \\ (2x-1)\log 7 &= \log 23 \\ 2x\log 7 - \log 7 &= \log 23 \\ 2x\log 7 &= \log 23 + \log 7 \\ x &= \frac{\log 23 + \log 7}{2\log 7} \\ &= 1.31\ (3\text{ s.f.}) \end{aligned}$$

g
$$\begin{aligned} 11^{3x-2} &= 65 \\ \log 11^{3x-2} &= \log 65 \\ (3x-2)\log 11 &= \log 65 \\ 3x-2 &= \frac{\log 65}{\log 11} \\ &= 1.740855 \\ x &= 1.25\ (3\text{ s.f.}) \end{aligned}$$

h
$$\begin{aligned} 2^{3-2x} &= 88 \\ \log 2^{3-2x} &= \log 88 \\ (3-2x)\log 2 &= \log 88 \\ \log_2 88 &= 3-2x \\ 3-2x &= 6.45943 \\ x &= -1.73\ (3\text{ s.f.}) \end{aligned}$$

2 a Let $y = 2^x$

$$y^2 - 6y + 5 = 0$$
$$(y-1)(y-5) = 0$$

So $y = 1$ or $y = 5$

If $y = 1,\ 2^x = 1,\ x = 0$

If $y = 5,\ 2^x = 5$

$$\begin{aligned} \log 2^x &= \log 5 \\ x\log 2 &= \log 5 \\ x &= \frac{\log 5}{\log 2} \\ x &= 2.32\ (3\text{ s.f.}) \end{aligned}$$

So $x = 0$ or $x = 2.32$

2 b Let $y = 3^x$

$$y^2 - 15y + 44 = 0$$
$$(y-4)(y-11) = 0$$

So $y = 4$ or $y = 11$

If $y = 4$, $3^x = 4$

$$\log 3^x = \log 4$$
$$x \log 3 = \log 4$$
$$x = \frac{\log 4}{\log 3}$$
$$x = 1.26 \text{ (3 s.f.)}$$

If $y = 11$, $3^x = 11$

$$\log 3^x = \log 11$$
$$x \log 3 = \log 11$$
$$x = \frac{\log 11}{\log 3}$$
$$x = 2.18 \text{ (3 s.f.)}$$

So $x = 1.26$ or $x = 2.18$

c Let $y = 5^x$

$$y^2 - 6y - 7 = 0$$
$$(y+1)(y-7) = 0$$

So $y = -1$ or $y = 7$

If $y = -1$, $5^x = -1$. No Solution.

If $y = 7$, $5^x = 7$

$$\log 5^x = \log 7$$
$$x \log 5 = \log 7$$
$$x = \frac{\log 7}{\log 5}$$
$$x = 1.21 \text{ (3 s.f.)}$$

d Let $y = 3^x$

$$(3^x)^2 + (3^x \times 3) - 10 = 0$$
$$y^2 + 3y - 10 = 0$$
$$(y+5)(y-2) = 0$$

So $y = -5$ or $y = 2$

If $y = -5$, $3^x = -5$. No Solution.

If $y = 2$, $3^x = 2$

$$\log 3^x = \log 2$$
$$x \log 3 = \log 2$$

d $x = \dfrac{\log 2}{\log 3}$

$x = 0.631 \text{ (3 s.f.)}$

e Let $y = 7^x$

$$(7^x)^2 + 12 = 7^x \times 7$$
$$y^2 + 12 = 7y$$
$$y^2 - 7y + 12 = 0$$
$$(y-3)(y-4) = 0$$

So $y = 3$ or $y = 4$

If $y = 3$, $7^x = 3$

$$x \log 7 = \log 3$$
$$x = \frac{\log 3}{\log 7}$$
$$x = 0.565 \text{ (3 s.f.)}$$

If $y = 4$, $7^x = 4$

$$x \log 7 = \log 4$$
$$x = \frac{\log 4}{\log 7}$$
$$x = 0.712 \text{ (3 s.f.)}$$

So $x = 0.565$ or $x = 0.712$

f Let $y = 2^x$

Then $y^2 + 3y - 4 = 0$

So $(y+4)(y-1) = 0$

So $y = -4$ or $y = 1$

$2^x = -4$ has no solution.

Therefore $2^x = 1$

So $x = 0$ is the only solution.

g Let $y = 3^x$

Then $3y^2 - 26y - 9 = 0$

So $(3y+1)(y-9) = 0$

So $y = -\frac{1}{3}$ or $y = 9$

$3^x = -\frac{1}{3}$ has no solution.

Therefore $3^x = 9$

So $x = 2$ is the only solution.

2 h Let $y = 3^x$

Then $12y^2 + 17y - 7 = 0$

So $(3y-1)(4y+7) = 0$

So $y = \frac{1}{3}$ or $y = -\frac{7}{4}$

$3^x = -\frac{7}{4}$ has no solution.

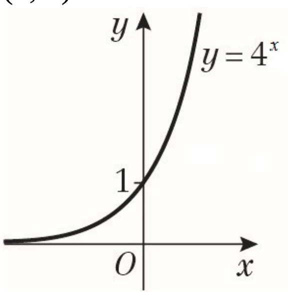
Therefore $3^x = \frac{1}{3}$

So $x = -1$ is the only solution.

3 a
$$3^{x+1} = 2000$$
$$\log_3 2000 = x + 1$$
$$x + 1 = 6.9186$$
$$x = 5.92 \text{ (3 s.f.)}$$

b
$$5^{-1} = x - 3$$
$$x - 3 = \tfrac{1}{5}$$
$$x = 3.2$$

4 a (0, 1)



b Let $y = 4^x$
$$4^{2x} - 10(4^x) + 16 = 0$$
$$y^2 - 10y + 16 = 0$$
$$(y-2)(y-8) = 0$$
$$y = 2 \text{ or } y = 8$$
Therefore, $4^x = 2$ or $4^x = 8$
$$\log_4 2 = x \text{ or } \log_4 8 = x$$
$$x = \tfrac{1}{2} \text{ or } x = \tfrac{3}{2}$$

5 a
$$5^x = 2^{x+1}$$
$$\log 5^x = \log 2^{x+1}$$
$$x\log 5 = (x+1)\log 2$$
$$x\log 5 = x\log 2 + \log 2$$
$$x\log 5 - x\log 2 = \log 2$$
$$x(\log 5 - \log 2) = \log 2$$
$$x = \frac{\log 2}{\log 5 - \log 2}$$
$$x = 0.7565 \text{ (4 d.p.)}$$

b
$$3^{x+5} = 6^x$$
$$\log 3^{x+5} = \log 6^x$$
$$(x+5)\log 3 = x\log 6$$
$$x\log 3 + 5\log 3 = x\log 6$$
$$5\log 3 = x\log 6 - x\log 3$$
$$5\log 3 = x(\log 6 - \log 3)$$
$$x = \frac{5\log 3}{\log 6 - \log 3}$$
$$x = 7.9248 \text{ (4 d.p.)}$$

c
$$7^{x+1} = 3^{x+2}$$
$$\log 7^{x+1} = \log 3^{x+2}$$
$$(x+1)\log 7 = (x+2)\log 3$$
$$x\log 7 + \log 7 = x\log 3 + 2\log 3$$
$$x\log 7 - x\log 3 = 2\log 3 - \log 7$$
$$x(\log 7 - \log 3) = 2\log 3 - \log 7$$
$$x = \frac{2\log 3 - \log 7}{\log 7 - \log 3}$$
$$x = 0.2966 \text{ (4 d.p.)}$$

Exponentials and logarithms 14G

1 a When $e^x = 6$
$\ln(e^x) = \ln 6$
$x = \ln 6$

b When $e^{2x} = 11$
$\ln(e^{2x}) = \ln 11$
$2x = \ln 11$
$x = \frac{1}{2}\ln 11$

c When $e^{-x+3} = 20$
$\ln(e^{-x+3}) = \ln 20$
$-x + 3 = \ln 20$
$x = 3 - \ln 20$

d When $3e^{4x} = 1$
$e^{4x} = \frac{1}{3}$
$\ln(e^{4x}) = \ln \frac{1}{3}$
$4x = \ln \frac{1}{3}$
$x = \frac{1}{4}\ln \frac{1}{3}$

e When $e^{2x+6} = 3$
$\ln(e^{2x+6}) = \ln 3$
$2x + 6 = \ln 3$
$x = \ln 3 - 6$
$x = \frac{1}{2}\ln 3 - 3$

f When $e^{5-x} = 19$
$\ln(e^{5-x}) = \ln 19$
$5 - x = \ln 19$
$x = 5 - \ln 19$

2 a When $\ln x = 2$
$e^{\ln x} = e^2$
$x = e^2$

b When $\ln(4x) = 1$
$e^{\ln(4x)} = e^1$
$4x = e^1$
$x = \dfrac{e}{4}$

c When $\ln(2x + 3) = 4$
$e^{\ln(2x+3)} = e^4$
$2x + 3 = e^4$
$2x = e^4 - 3$
$x = \frac{1}{2}e^4 - \frac{3}{2}$

2 d When $2\ln(6x - 2) = 5$
$\ln(6x - 2) = \frac{5}{2}$
$e^{\ln(6x-2)} = e^{\frac{5}{2}}$
$6x - 2 = e^{\frac{5}{2}}$
$6x = e^{\frac{5}{2}} + 2$
$x = \frac{1}{6}(e^{\frac{5}{2}} + 2)$

e When $\ln(18 - x) = \frac{1}{2}$
$e^{\ln(18-x)} = e^{\frac{1}{2}}$
$18 - x = e^{\frac{1}{2}}$
$x = 18 - e^{\frac{1}{2}}$

f When $\ln(x^2 - 7x + 11) = 0$
$e^{\ln(x^2-7x+11)} = e^0$
$x^2 - 7x + 11 = 1$
$x^2 - 7x + 10 = 0$
$(x - 2)(x - 5) = 0$
$x = 2$ or $x = 5$

3 a $e^{2x} - 8e^x + 12 = 0$
Let $u = e^x$
$u^2 - 8u + 12 = 0$
$(u - 2)(u - 6) = 0$
$u = 2$ or $u = 6$
$e^x = 2$ or $e^x = 6$

When $e^x = 2$
$\ln(e^x) = \ln 2$
$x = \ln 2$
When $e^x = 6$
$\ln(e^x) = \ln 6$
$x = \ln 6$

$x = \ln 2$ or $x = \ln 6$

b $e^{4x} - 3e^{2x} + 2 = 0$
Let $u = e^{2x}$
$u^2 - 3u + 2 = 0$
$(u - 1)(u - 2) = 0$
$u = 1$ or $u = 2$
$e^{2x} = 1$ or $e^{2x} = 2$

3 b When $e^{2x} = 1$
$\ln(e^{2x}) = \ln 1$
$2x = 0$
$x = 0$
When $e^{2x} = 2$
$\ln(e^{2x}) = \ln 2$
$2x = \ln 2$
$x = \frac{1}{2}\ln 2$

$x = 0$ or $x = \frac{1}{2}\ln 2$

c $(\ln x)^2 + 2\ln x - 15 = 0$
Let $u = \ln x$
$u^2 + 2u - 15 = 0$
$(u + 5)(u - 3) = 0$
$u = -5$ or $u = 3$

When $\ln x = -5$
$e^{\ln x} = e^{-5}$
$x = e^{-5}$
When $\ln x = 3$
$e^{\ln x} = e^3$
$x = e^3$

$x = e^{-5}$ or $x = e^3$

d $e^x - 5 + 4e^{-x} = 0$
Multiply each term by e^x
$e^{2x} - 5e^x + 4 = 0$
Let $u = e^x$
$u^2 - 5u + 4 = 0$
$(u - 1)(u - 4) = 0$
$u = 1$ or $u = 4$
$e^x = 1$ or $e^x = 4$

When $e^x = 1$
$\ln(e^x) = \ln 1$
$x = 0$
When $e^x = 4$
$\ln(e^x) = \ln 4$
$x = \ln 4$

$x = 0$ or $x = \ln 4$

e $3e^{2x} - 16e^x + 5 = 0$
Let $u = e^x$
$3u^2 - 16u + 5 = 0$
$(3u - 1)(u - 5) = 0$
$u = \frac{1}{3}$ or $u = 5$
$e^x = \frac{1}{3}$ or $e^x = 5$

e When $e^x = \frac{1}{3}$
$\ln(e^x) = \ln \frac{1}{3}$
$x = \ln \frac{1}{3}$
When $e^x = 5$
$\ln(e^x) = \ln 5$
$x = \ln 5$

$x = \ln \frac{1}{3}$ or $x = \ln 5$

f $(\ln x)^2 - 4\ln x - 12 = 0$
Let $u = \ln x$
$u^2 - 4u - 12 = 0$
$(u + 2)(u - 6) = 0$
$u = -2$ or $u = 6$

When $\ln x = -2$
$e^{\ln x} = e^{-2}$
$x = e^{-2}$
When $\ln x = 6$
$e^{\ln x} = e^6$
$x = e^6$

$x = e^{-2}$ or $x = e^6$

4 $e^x - 7 + 12e^{-x} = 0$
Multiply each term by e^x
$e^{2x} - 7e^x + 12 = 0$
Let $u = e^x$
$u^2 - 7u + 12 = 0$
$(u - 3)(u - 4) = 0$
$u = 3$ or $u = 4$
$e^x = 3$ or $e^x = 4$

When $e^x = 3$
$\ln(e^x) = \ln 3$
$x = \ln 3$
When $e^x = 4$
$\ln(e^x) = \ln 4$
$x = \ln 2^2$
$x = 2\ln 2$

$x = \ln 3$ or $x = 2\ln 2$

5 a When $\ln(8x - 3) = 2$
$e^{\ln(8x-3)} = e^2$
$8x - 3 = e^2$
$8x = e^2 + 3$
$x = \frac{1}{8}(e^2 + 3)$

5 b When $e^{5(x-8)} = 3$
$\ln(e^{5(x-8)}) = \ln 3$
$5(x-8) = \ln 3$
$x - 8 = \frac{1}{5}\ln 3$
$x = \frac{1}{5}\ln 3 + 8$

c $e^{10x} - 8e^{5x} + 7 = 0$
Let $u = e^{5x}$
$u^2 - 8u + 7 = 0$
$(u-1)(u-7) = 0$
$u = 1$ or $u = 7$
$e^{5x} = 1$ or $e^{5x} = 7$

When $e^{5x} = 1$
$\ln(e^{5x}) = \ln 1$
$5x = 0$
$x = 0$
When $e^{5x} = 7$
$\ln(e^{5x}) = \ln 7$
$5x = \ln 7$
$x = \frac{1}{5}\ln 7$

$x = 0$ or $x = \frac{1}{5}\ln 7$

d When $(\ln x - 1)^2 = 4$
$(\ln x)^2 - 2\ln x - 3 = 0$
Let $u = \ln x$
$u^2 - 2u - 3 = 0$
$(u+1)(u-3) = 0$
$u = -1$ or $u = 3$

When $\ln x = -1$
$e^{\ln x} = e^{-1}$
$x = e^{-1}$
When $\ln x = 3$
$e^{\ln x} = e^3$
$x = e^3$

$x = e^{-1}$ or $x = e^3$

6 When $3^x e^{4x-1} = 5$
$\ln(3^x e^{4x-1}) = \ln 5$
$\ln(3^x) + \ln(e^{4x-1}) = \ln 5$
$x\ln 3 + 4x - 1 = \ln 5$
$x\ln 3 + 4x = 1 + \ln 5$
$x(\ln 3 + 4) = 1 + \ln 5$
$$x = \frac{1+\ln 5}{4+\ln 3}$$

7 a $D = 6$ when $t = 0$ so 6 is the initial concentration of the drug in mg/l.

7 b $D = 6e^{\frac{-t}{10}}$
When $t = 2$
$D = 6e^{\frac{-2}{10}}$
$D = 4.91$ mg/l (3 s.f.)

c When $6e^{\frac{-t}{10}} = 3$
$$e^{\frac{-t}{10}} = \frac{1}{2}$$
$$\ln e^{\frac{-t}{10}} = \ln\frac{1}{2}$$
$$\frac{-t}{10} = \ln\frac{1}{2}$$
$$t = -10\ln\frac{1}{2}$$
$t = 6.931471...$
$t = 6$ hours and 55.888... minutes
$t = 6$ hours and 56 minutes

8 a A is where $x = 0$
Substitute $x = 0$ into
$y = 3 + \ln(4-x)$ to give
$y = 3 + \ln 4$
$A = (0, 3 + \ln 4)$

b B is where $y = 0$
Substitute $y = 0$ into
$y = 3 + \ln(4-x)$ to give
$0 = 3 + \ln(4-x)$
$-3 = \ln(4-x)$
$e^{-3} = 4 - x$
$x = 4 - e^{-3}$
$B = (4 - e^{-3}, 0)$

9 a When $t = 0$, $V = 27\,000$,
so $27\,000 = Ae^{k \times 0} = A$
When $t = 5$, $V = 18\,000$,
so $18\,000 = Ae^{5k}$

Substituting in $A = 27\,000$

$$18000 = 27000e^{5k}$$

$$\frac{18000}{27000} = e^{5k}$$

$$\frac{2}{3} = e^{5k}$$

$$\ln\left(\frac{2}{3}\right) = \ln\left(e^{5k}\right)$$

$$\ln\left(\frac{2}{3}\right) = 5k$$

$k = -0.08109\ldots = -0.0811$ (3 s.f.)

So $A = 27\,000$, $k = -0.0811$ (3 s.f.)

b According to the model,
when $t = 8$, $V = 14\,100$ (3 s.f.)
so model is reliable.

10 a Consider linear model in the form:

$P = mt + c$

When $t = 0$, $P = 7.6$, so $c = 7.6$

When $t = 20$, $P = 12.1$, so $12.1 = 20m + 7.6$

Solve to find m:

$$12.1 = 20m + 7.6$$

$$m = \frac{12.1 - 7.6}{20} = 0.225t$$

Linear model: $P = 0.225t + 7.6$

b Consider exponential model in the form:

$P = ab^t$

When $t = 0$, $P = 7.6$, so $a = 7.6$

When $t = 20$, $P = 12.1$, so $12.1 = 7.6b^{20}$

Solve to find b:

$$12.1 = 7.6b^{20}$$

$$\frac{12.1}{7.6} = b^{20}$$

$$\ln\left(\frac{12.1}{7.6}\right) = 20\ln b$$

$$\ln\left(\frac{12.1}{7.6}\right)^{\frac{1}{20}} = \ln b$$

$$b = 1.0235 \text{ (4 d.p.)}$$

c When $t = 50$, linear model predicts 18.85 million people, and exponential model predicts 24.3 million people. Exponential model is best supported by the given fact.

Challenge

$g(0) = Ae^{B \times 0} + C = 5$
$A + C = 5$
As $y = 2$ is an asymptote, $C = 2$
$A = 3$ and $g(6) = 3e^{B \times 6} + 2 = 10$

$$3e^{6B} = 8$$

$$e^{6B} = \tfrac{8}{3}$$

$$\ln(e^{6B}) = \ln\tfrac{8}{3}$$

$$6B = \ln\tfrac{8}{3}$$

$$B = \frac{\ln\frac{8}{3}}{6}$$

Exponentials and logarithms 14H

1 a When $S = 4 \times 7^x$

$\log S = \log (4 \times 7^x)$

$\log S = \log 4 + \log 7^x$

$\log S = \log 4 + x \log 7$

b $\log S = x \log 7 + \log 4$

Gradient = $\log 7$

Intercept = $\log 4$

2 a When $A = 6x^4$

$\log A = \log (6x^4)$

$\log A = \log 6 + \log x^4$

$\log A = \log 6 + 4 \log x$

b $\log A = 4 \log x + \log 6$

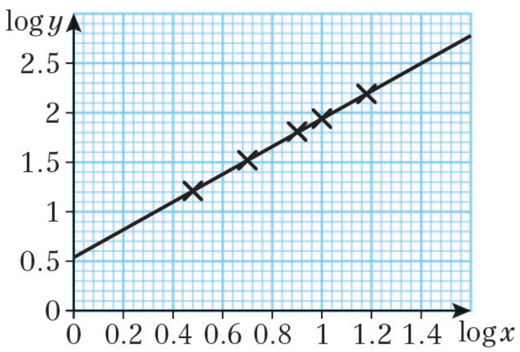
Gradient = 4

Intercept = $\log 6$

3 a

$\log x$	0.48	0.70	0.90	1	1.18
$\log y$	1.21	1.52	1.81	1.94	2.19

b



c $y = ax^n$

$\log y = \log (ax^n)$

$\log y = \log a + \log x^n$

$\log y = \log a + n \log x$

$\log y = n \log x + \log a$

Gradient = n

Intercept = $\log a$

Calculating the gradient from the table,

$$n = \frac{2.19 - 1.21}{1.18 - 0.48} = \frac{0.98}{0.7} = 1.4$$

Reading the intercept from the graph,

$\log a = 0.55$

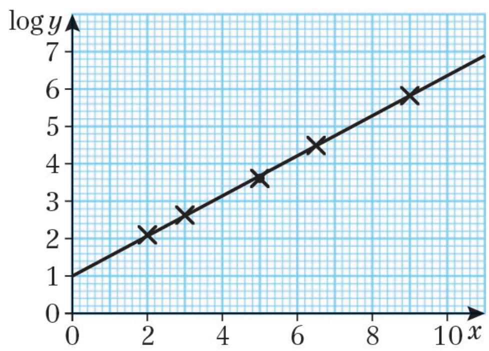
$a = 10^{0.55} = 3.548...$

$a = 3.5$, $n = 1.4$

4 a

x	2	3	5	6.5	9
$\log y$	2.10	2.63	3.61	4.49	5.82

b



c $y = ab^x$

$\log y = \log (ab^x)$

$\log y = \log a + \log b^x$

$\log y = \log a + x \log b$

$\log y = x \log b + \log a$

Gradient = $\log b$

Intercept = $\log a$

Calculating the gradient from the table and the graph,

$$\log b = \frac{5.82 - 1}{9 - 0} = \frac{4.82}{9} = 0.53555...$$

$$b = 10^{0.53555...} = 3.43...$$

Reading the intercept from the graph,

$\log a = 1$

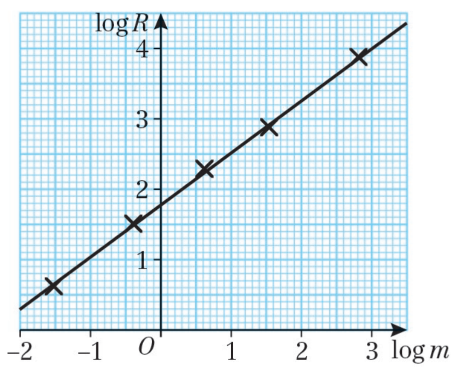
$a = 10^1 = 10$

$a = 10$, $b = 3.4$

5 a

$\log m$	−1.52	−0.39	0.62	1.54	2.81
$\log R$	0.62	1.51	2.29	2.88	3.88

b



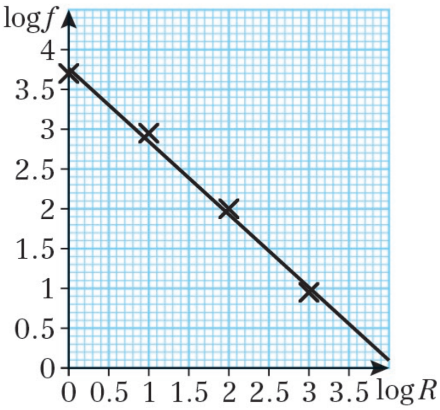
5 c $R = am^b$
$\log R = \log (am^b)$
$\log R = \log a + \log m^b$
$\log R = \log a + b \log m$
Gradient = b
Intercept = $\log a$
Calculating the gradient from the table,
$$b = \frac{3.88-0.62}{2.81-(-1.52)} = \frac{3.26}{4.33} = 0.75288...$$
Reading the intercept from the graph,
$\log a = 1.78$
$a = 10^{1.78} = 60.255...$
$a = 60,\ b = 0.75$

d $R = 60m^{0.75}$
When $m = 80$
$R = 60(80)^{0.75} = 1604.97...$
1600 kcal/day (2 s.f.)

6 a

$\log R$	0	1	2	3
$\log f$	3.69	2.94	1.96	0.95

b



c $f = AR^b$
$\log f = \log (AR^b)$
$\log f = \log A + \log R^b$
$\log f = \log A + b \log R$
$\log y = b \log R + \log A$
Gradient = b
Intercept = $\log A$
Calculating the gradient from the table,
$$b = \frac{0.95-3.69}{3-0} = \frac{-2.74}{3} = -0.91...$$
Reading the intercept from the graph,
$\log A = 3.76$
$A = 10^{3.76} = 5754.39...$
$A = 5800,\ b = -0.9$

6 d $f = 5800R^{-0.9}$ per 100 000 words
When $R = 57$
$f = 152.45...$
For 455 125 words, $4.55125 \times f = 693.85...$
690 times (2 s.f.)

7 a $N = ab^t$
$\log N = \log (ab^t)$
$\log N = \log a + \log b^t$
$\log N = \log a + t \log b$
$$\text{Gradient} = \frac{2.55-1.6}{10-0} = \frac{0.95}{10} = 0.095$$
Intercept = 1.6
$\log N = 0.095t + 1.6$

b $\log a = 1.6$
$a = 10^{1.6} = 39.8...$
$\log b = 0.095$
$b = 10^{0.095} = 1.2445...$
$a = 40,\ b = 1.2$

c a is the initial number of sick people

d $N = ab^t$
$N = 40(1.2)^{30} = 9495.052 = 9500$ (2 s.f.)
After 30 days people may start to recover, or the disease may stop spreading as quickly.

8 a $A = pw^q$
$\log A = m \log w + c$
Intercept = -0.1049
Gradient = 2
$\log A = 2 \log w - 0.1049$

b $A = pw^q$
$\log A = \log (pw^q)$
$\log A = \log p + \log w^q$
$\log A = \log p + q \log w$
Equating coefficients
$q = 2$
$\log p = -0.1049$
$p = 10^{-0.1049}$
$p = 0.785416... = 0.7854$ (4 s.f.)

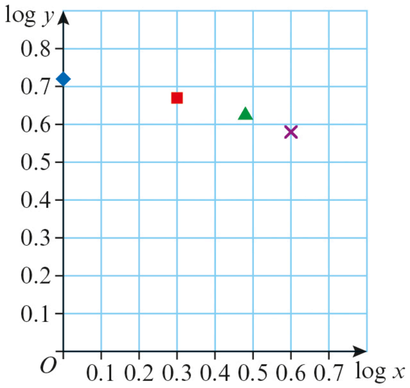
c The shapes are circles.
Multiply p by 4
$4p = 3.1416... \approx \pi$
So p is approximately $\frac{1}{4}$ of π
So $A = \frac{\pi}{4}w^2$

8 c The width is the diameter of the circle

so $A = \frac{\pi}{4}(2r)^2 = \pi r^2$

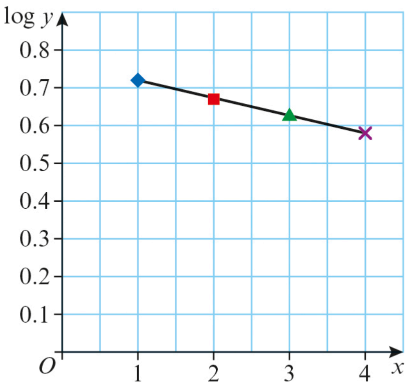
Challenge

$\log x$	0	0.30	0.48	0.60
$\log y$	0.72	0.67	0.63	0.58



The relationship between $\log x$ and $\log y$ is not linear so the relationship is perhaps $y = ax^n$

x	1	2	3	4
$\log y$	0.72	0.67	0.63	0.58



The second graph, $\log y$ against x, is a linear relationship so the relationship is of the form $y = ab^x$.

$\log y = \log (ab^x)$
$\log y = \log a + \log b^x$
$\log y = \log a + x \log b$
Intercept = 0.75
$\log a = 0.75$
$a = 10^{0.75} = 5.8$

$$\text{Gradient} = \frac{0.58 - 0.72}{4 - 1} = -\frac{0.14}{3} = -0.04666...$$

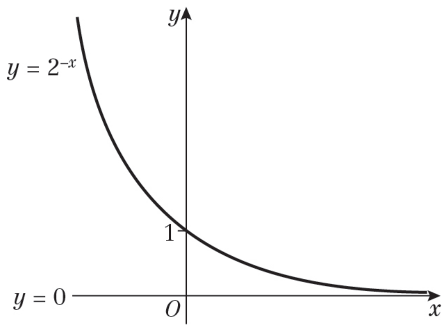
Challenge

$\log b = -0.04666...$
$b = 10^{-0.04666...}$
$= 0.90$

So the formula is $y = 5.8 \times 0.9^x$

Exponentials and logarithms, Mixed Exercise 14

1 a $y = 2^{-x} = (2^{-1})^x = (\frac{1}{2})^x$



b $y = 5e^x - 1$

The graph is a translation by the vector $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$ and a vertical stretch scale factor 5 of the graph $y = e^x$.

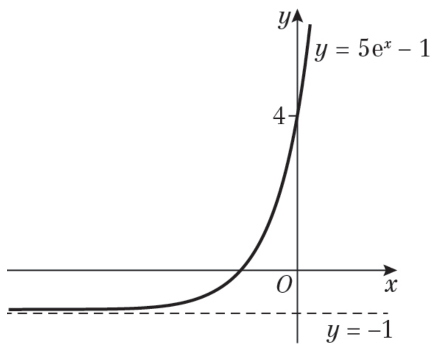
The graph crosses the y-axis when $x = 0$.

$y = 5 \times e^0 - 1$

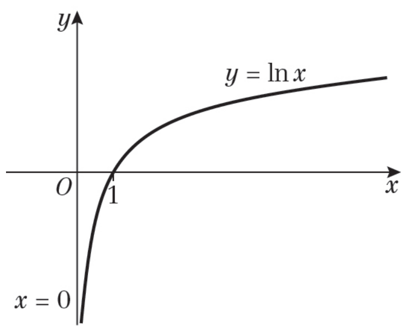
$y = 4$

The graph crosses the y-axis at (0, 4).

Asymptote is at $y = -1$.



c $y = \ln x$



2 a $\log_a(p^2 q) = \log_a(p^2) + \log_a q$

$= 2\log_a p + \log_a q$

b $\log_a(pq) = \log_a p + \log_a q$

So

$\log_a p + \log_a q = 5 \quad$ **(1)**

$2\log_a p + \log_a q = 9 \quad$ **(2)**

Subtract **(1)** from **(2)**:

$\log_a p = 4$

So $\log_a q = 1$

3 a $p = \log_q 16$

$= \log_q(2^4)$

$= 4\log_q 2$

$\log_q 2 = \dfrac{p}{4}$

b $\log_{q+1}(8q) = \log_q 8 + \log_q q$

$= \log_q(2^3) + \log_q q$

$= 3\log_q 2 + \log_q q$

$= 3 \times \dfrac{p}{4} + 1$

$= \dfrac{3p}{4} + 1$

4 a $4^x = 23$

$\log_4 23 = x$

$x = 2.26$

b $7^{(2x+1)} = 1000$

$\log_7 1000 = 2x + 1$

$2x = \log_7 1000 - 1$

$x = \frac{1}{2}\log_7 1000 - \frac{1}{2}$

$= 1.27$

c $10^x = 6^{x+2}$

$\log(10^x) = \log(6^{x+2})$

$x \log 10 = (x + 2)\log 6$

$x \log 10 - x \log 6 = 2 \log 6$

$x(\log 10 - \log 6) = 2 \log 6$

$x = \dfrac{2\log 6}{\log 10 - \log 6}$

$= 7.02$

5 a $4^x - 2^{x+1} - 15 = 0$

$2^{2x} - 2 \times 2^x - 15 = 0$

$(2^x)^2 - 2 \times 2^x - 15 = 0$

Let $u = 2^x$

$u^2 - 2u - 15 = 0$

b $(u+3)(u-5) = 0$

So $u = -3$ or $u = 5$

If $u = -3$, $2^x = -3$. No solution.

If $u = 5$, $2^x = 5$

$\log 2^x = \log 5$

$x \log 2 = \log 5$

$x = \dfrac{\log 5}{\log 2}$

$= 2.32\ (2\text{ d.p.})$

6 $\log_2(x+10) - \log_2(x-5) = 4$

$\log_2\left(\dfrac{x+10}{x-5}\right) = 4$

$\dfrac{x+10}{x-5} = 2^4$

$16x - 80 = x + 10$

$15x = 90$

$x = 6$

7 a $y = e^{-x}$

$\dfrac{dy}{dx} = -e^{-x}$

b $y = e^{11x}$

$\dfrac{dy}{dx} = 11e^{11x}$

c $y = 6e^{5x}$

$\dfrac{dy}{dx} = 5 \times 6e^{5x} = 30e^{5x}$

8 a $\ln(2x-5) = 8 \quad$ (inverse of ln)

$2x - 5 = e^8 \quad (+5)$

$2x = e^8 + 5 \quad (\div 2)$

$x = \dfrac{e^8 + 5}{2}$

8 b $e^{4x} = 5 \quad$ (inverse of e)

$4x = \ln 5 \quad (\div 4)$

$x = \dfrac{\ln 5}{4}$

c $24 - e^{-2x} = 10 \quad (+e^{-2x})$

$24 = 10 + e^{-2x} \quad (-10)$

$14 = e^{-2x} \quad$ (inverse of e)

$\ln(14) = -2x \quad (\div -2)$

$-\frac{1}{2}\ln(14) = x$

$x = -\frac{1}{2}\ln(14)$

d $\ln(x) + \ln(x-3) = 0$

$\ln(x(x-3)) = 0$

$x(x-3) = e^0$

$x(x-3) = 1$

$x^2 - 3x - 1 = 0$

$x = \dfrac{3 \pm \sqrt{9+4}}{2}$

$= \dfrac{3 \pm \sqrt{13}}{2}$

$= \dfrac{3 + \sqrt{13}}{2}$

(x cannot be negative because of initial equation)

e $e^x + e^{-x} = 2$

$e^x + \dfrac{1}{e^x} = 2 \quad (\times e^x)$

$(e^x)^2 + 1 = 2e^x$

$(e^x)^2 - 2e^x + 1 = 0$

$(e^x - 1)^2 = 0$

$e^x = 1$

$x = \ln 1 = 0$

f $\ln 2 + \ln x = 4$

$\ln 2x = 4$

$2x = e^4$

$x = \dfrac{e^4}{2}$

9 $P = 100 + 850e^{-\frac{t}{2}}$

a New price is when $t = 0$

Substitute $t = 0$ into $P = 100 + 850e^{-\frac{t}{2}}$ to give:

$P = 100 + 850e^{-\frac{0}{2}} \quad (e^0 = 1)$

$= 100 + 850 = 950$

The new price is £950

b After 3 years $t = 3$.

Substitute $t = 3$ into $P = 100 + 850e^{-\frac{t}{2}}$ to give:

$P = 100 + 850e^{-\frac{3}{2}} = 289.66$

Price after 3 years is £290 (to nearest £)

c It is worth less than £200 when $P < 200$

Substitute $P = 200$ into $P = 100 + 850e^{-\frac{t}{2}}$ to give:

$$200 = 100 + 850e^{-\frac{t}{2}}$$

$$100 = 850e^{-\frac{t}{2}}$$

$$\frac{100}{850} = e^{-\frac{t}{2}}$$

$$\ln\left(\frac{100}{850}\right) = -\frac{t}{2}$$

$$t = -2\ln\left(\frac{100}{850}\right)$$

$$t = 4.28$$

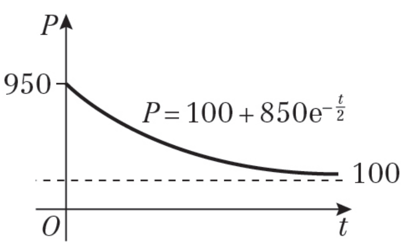
It is worth less than £200 after 4.28 years.

d As $t \to \infty$, $e^{-\frac{t}{2}} \to 0$

Hence, $P \to 100 + 850 \times 0 = 100$

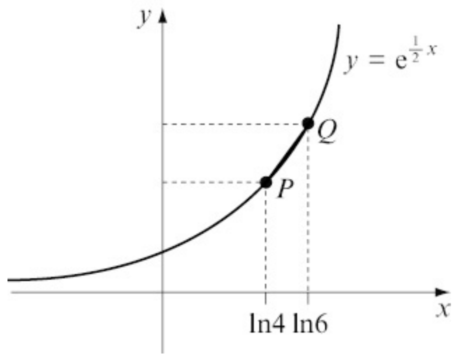
The computer will be worth £100 eventually.

e



9 f A good model. The computer will always be worth something.

10 a



Q has y-coordinate $e^{\frac{1}{2}\ln 16} = e^{\ln 16^{\frac{1}{2}}} = 16^{\frac{1}{2}} = 4$

P has y-coordinate $e^{\frac{1}{2}\ln 4} = e^{\ln 4^{\frac{1}{2}}} = 4^{\frac{1}{2}} = 2$

Gradient of the line $PQ = \dfrac{\text{change in } y}{\text{change in } x}$

$$= \frac{4-2}{\ln 16 - \ln 4}$$

$$= \frac{2}{\ln\left(\frac{16}{4}\right)}$$

$$= \frac{2}{\ln 4}$$

Using $y = mx + c$, the equation of the line PQ is:

$y = \dfrac{2}{\ln 4}x + c$

$(\ln 4, 2)$ lies on the line so

$y = \dfrac{2}{\ln 4}x + c$

$2 = 2 + c$

$c = 0$

Equation of PQ is $y = \dfrac{2x}{\ln 4}$

b The line passes through the origin as $c = 0$.

c Length from $(\ln 4, 2)$ to $(\ln 16, 4)$ is

$\sqrt{(\ln 16 - \ln 4)^2 + (4-2)^2}$

$= \sqrt{\left(\ln\dfrac{16}{4}\right)^2 + 2^2}$

$= \sqrt{(\ln 4)^2 + 4} = 2.43$

11 a $T = 55e^{-\frac{t}{8}} + 20$

t is the time in minutes and time cannot be negative as you can't go back in time.

b The starting temperature of the cup of tea is when $t = 0$

$T = 55e^{-\frac{0}{8}} + 20 = 75°\text{C}$

c When $T = 50°\text{C}$

$$55e^{-\frac{t}{8}} + 20 = 50$$

$$55e^{-\frac{t}{8}} = 30$$

$$e^{-\frac{t}{8}} = \frac{30}{55}$$

$$\ln\left(e^{-\frac{t}{8}}\right) = \ln\left(\frac{30}{55}\right)$$

$$-\frac{t}{8} = \ln\left(\frac{30}{55}\right)$$

$$t = -8\ln\left(\frac{30}{55}\right)$$

$$= 4.849...$$

$$\approx 5 \text{ minutes}$$

d The exponential term will always be positive, so the overall temperature will be greater than 20°C.

12 a As $S = aV^b$

$\log S = \log(aV^b)$

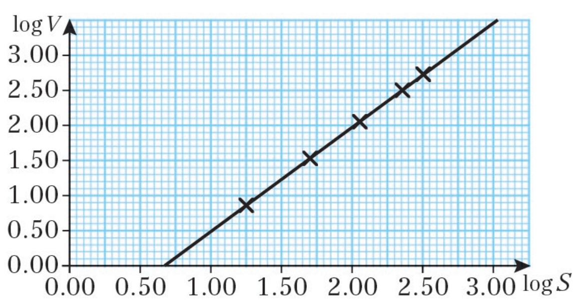
$\log S = \log a + \log(V^b)$

$\log S = \log a + b\log V$

b

$\log S$	1.26	1.70	2.05	2.35	2.50
$\log V$	0.86	1.53	2.05	2.49	2.72

c



12 d b is the gradient $= \dfrac{2.72 - 0.86}{2.5 - 1.26}$

$= \dfrac{1.86}{1.24} = 1.5$

Intercept $= \log a$

$\log a = -1.05$

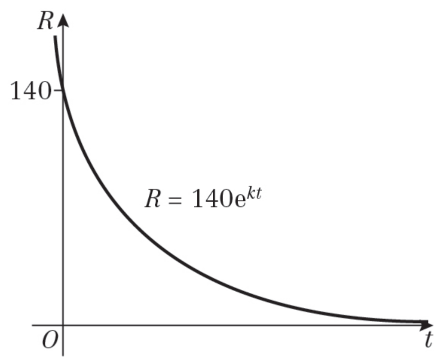
$10^{-1.05} = a$

$a = 0.0891...$

$a \approx 0.09$

13 a kt must be negative as the model is for decay, not growth. As t (for time) is always positive, k must be negative.

b



c $R = 140e^{kt}$

When $R = 70$ and $t = 30$

$$70 = 140e^{30k}$$

$$\tfrac{1}{2} = e^{30k}$$

$$\ln\tfrac{1}{2} = \ln\left(e^{30k}\right)$$

$$= 30k$$

$$k = \tfrac{1}{30}\ln\tfrac{1}{2}$$

$$= \tfrac{1}{30}\ln\left(2^{-1}\right)$$

$$= -\tfrac{1}{30}\ln 2$$

So $c = -\frac{1}{30}$

14 a $V = e^{0.4x} - 1$

When $x = 5$

$$V = e^{0.4\times5} - 1$$

$$= e^2 - 1$$

$$= 7.389... - 1$$

$$= 6.389...$$

$$\approx 6.4 \text{ million views}$$

b $V = e^{0.4x} - 1$

$$\frac{dV}{dx} = 0.4e^{0.4x}$$

14 c When $x = 100$

$$\frac{dV}{dx} = 0.4e^{0.4 \times 100}$$
$$= 0.4e^{40}$$
$$= 9.415... \times 10^{16} \text{ million}$$

So 9.4×10^{22} new views per day

d This number is greater than the population of the world, so the model is not valid after 100 days.

15 a $M = \frac{2}{3}\log_{10}(S) - 10.7$

When $S = 2.24 \times 10^{22}$

$$M = \frac{2}{3}\log_{10}(2.24 \times 10^{22}) - 10.7$$
$$= \frac{2}{3}(\log_{10} 2.24 + \log 10^{22}) - 10.7$$
$$= \frac{2}{3}(0.3502... + 22) - 10.7$$
$$= 4.2$$

b i When $M = 6$

$$6 = \frac{2}{3}\log_{10}(S) - 10.7$$
$$16.7 = \frac{2}{3}\log_{10}(S)$$
$$25.05 = \log_{10}(S)$$
$$10^{25.05} = S$$
$$S = 1.12 \times 10^{25} \text{ dyne cm}$$

ii When $M = 7$

$$7 = \frac{2}{3}\log_{10}(S) - 10.7$$
$$17.7 = \frac{2}{3}\log_{10}(S)$$
$$26.55 = \log_{10}(S)$$
$$10^{26.55} = S$$
$$S = 3.55 \times 10^{26} \text{ dyne cm}$$

c $$\frac{3.55 \times 10^{26}}{1.12 \times 10^{25}} = 31.6...$$
$$\approx 32 \text{ times}$$

16 a The student goes wrong in line 2, where the subtraction should be a division (as in line 2 below).

16 b The full working should have looked like this:

$$\log_2 x - \frac{1}{2}\log_2(x+1) = 1$$
$$\log_2 x - \log_2\left((x+1)^{\frac{1}{2}}\right) = 1$$
$$\log_2 x - \log_2(\sqrt{x+1}) = 1$$
$$\log_2 \frac{x}{\sqrt{x+1}} = 1$$
$$\frac{x}{\sqrt{x+1}} = 2^1$$
$$x = 2\sqrt{x+1} \quad \text{(square)}$$
$$x^2 = 4x + 4$$
$$x^2 - 4x - 4 = 0 \quad \text{(use quadratic formula)}$$
$$x = 2 + 2\sqrt{2}$$

($x \neq 2 - 2\sqrt{2}$ because log cannot take negative input values)

Challenge

a $y = 9^x = (3^2)^x = 3^{2x}$

So $\log_3 y = 2x$

b As $y = 9^x$

$$\log_9 y = \log_9(9^x)$$
$$\log_9 y = x\log_9 9$$
$$\log_9 y = 1, \text{ so } \log_9 y = x$$

$2x = 2\log_9 y$ and from **a**, $2x = \log_3 y$

So $\log_3 y = 2\log_9 y$

$$\log_3 y = \log_9 y^2$$

c Using $\log_3 y = \log_9 y^2$

$$\log_3(2-3x) = \log_9(2-3x)^2$$
$$= \log_9(4 - 12x + 9x^2)$$

So $\log_9(4 - 12x + 9x^2) = \log_9(6x^2 - 19x + 2)$

Therefore $4 - 12x + 9x^2 = 6x^2 - 19x + 2$

$$3x^2 + 7x + 2 = 0$$
$$(3x + 1)(x + 2) = 0$$
$$x = -\tfrac{1}{3} \text{ or } x = -2$$

17 $9^x - 11(3^x) + 18 = 0$

$(3^x)^2 - 11(3^x) + 18 = 0$

Let $u = 3^x$

$u^2 - 11u + 18 = 0$

$(u - 9)(u - 2) = 0$

$u = 2$ or $u = 9$

When $u = 9$

$3^x = 9$

$\ln(3^x) = \ln 9$

$x = \dfrac{\ln 9}{\ln 3} = 2$

When $u = 2$

$3^x = 2$

$\ln(3^x) = \ln 2$

$x = \dfrac{\ln 2}{\ln 3} = 0.631$ (3 s.f.)

So $x = 2$ or $x = 0.631$ (3 s.f.)

Review exercise 3

1 The two vectors are parallel
so $9\mathbf{i} + q\mathbf{j} = \lambda(2\mathbf{i} - \mathbf{j})$

Equating coefficients:
$9 = 2\lambda$
$\lambda = 4.5$
$q = -\lambda$
$= -4.5$

2 $|5\mathbf{i} - k\mathbf{j}| = |2k\mathbf{i} + 2\mathbf{j}|$
$\sqrt{5^2 + k^2} = \sqrt{(2k)^2 + 2^2}$
$25 + k^2 = 4k^2 + 4$
$3k^2 = 21$
$k^2 = 7$
$k = \pm\sqrt{7}$
The positive value of k is $\sqrt{7}$.

3 a $\overrightarrow{CX} = \begin{pmatrix} 1-9 \\ -3-6 \end{pmatrix} = \begin{pmatrix} -8 \\ -9 \end{pmatrix}$
$|\overrightarrow{CX}| = \sqrt{8^2 + 9^2} = \sqrt{145}$
$\overrightarrow{CY} = \begin{pmatrix} 1-13 \\ -3+2 \end{pmatrix} = \begin{pmatrix} -12 \\ -1 \end{pmatrix}$
$|\overrightarrow{CY}| = \sqrt{12^2 + 1^2} = \sqrt{145}$
$\overrightarrow{CZ} = \begin{pmatrix} 1-0 \\ -3+15 \end{pmatrix} = \begin{pmatrix} 1 \\ 12 \end{pmatrix}$
$|\overrightarrow{CZ}| = \sqrt{1^2 + 12^2} = \sqrt{145}$
Therefore, $|\overrightarrow{CX}| = |\overrightarrow{CY}| = |\overrightarrow{CZ}|$

b Centre of the circle is point $C(1, -3)$.
Radius of the circle is $\sqrt{145}$.
Equation of the circle is
$(x - 1)^2 + (y + 3)^2 = 145$

4 a $\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AC}$
$= -(9\mathbf{i} + 2\mathbf{j}) + (7\mathbf{i} - 6\mathbf{j})$
$= -2\mathbf{i} - 8\mathbf{j}$

b For triangle ABC to be isosceles two of the sides must be equal.
$AB = \sqrt{9^2 + 2^2} = \sqrt{85}$
$BC = \sqrt{2^2 + 8^2} = \sqrt{68}$
$AC = \sqrt{7^2 + 6^2} = \sqrt{85}$
$AB = AC$, therefore triangle ABC is isosceles

4 c Using the cosine rule
$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$
$$\cos B = \frac{\sqrt{68}^2 + \sqrt{85}^2 - \sqrt{85}^2}{2\sqrt{68}\sqrt{85}}$$
$$\cos B = \frac{68 + 85 - 85}{2\sqrt{5780}}$$
$$\cos B = \frac{68}{68\sqrt{5}}$$
$$\cos B = \frac{1}{\sqrt{5}}$$
So $\cos \angle ABC = \dfrac{1}{\sqrt{5}}$

5 $\mathbf{a} + \mathbf{b} = \begin{pmatrix} 8 \\ 23 \end{pmatrix} + \begin{pmatrix} -15 \\ x \end{pmatrix} = \begin{pmatrix} -7 \\ 23 + x \end{pmatrix}$

or : $-7\mathbf{i} + (23 + x)\mathbf{j}$

$\mathbf{b} - \mathbf{c} = \begin{pmatrix} -15 \\ x \end{pmatrix} - \begin{pmatrix} -13 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ x - 2 \end{pmatrix}$

or : $-2\mathbf{i} + (x - 2)\mathbf{j}$

As $\mathbf{a} + \mathbf{b}$ is parallel to $\mathbf{b} - \mathbf{c}$
$-7\mathbf{i} + (23 + x)\mathbf{j} = \lambda(-2\mathbf{i} + (x - 2)\mathbf{j})$

Equating coefficients and solving simultaneously
$-7 = -2\lambda$ and $23 + x = \lambda(x - 2)$
$\lambda = 3.5$
$23 + x = 3.5(x - 2)$
$23 + x = 3.5x - 7$
$2.5x = 30$
$x = 12$

6 a $\mathbf{R} = 2\mathbf{i} - 5\mathbf{j} + \mathbf{i} + \mathbf{j}$
$= 3\mathbf{i} - 4\mathbf{j}$
$|\mathbf{R}| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$ N

b $\mathrm{R}_{\text{new}} = 3\mathbf{i} - 4\mathbf{j} + k\mathbf{j}$
$\tan 45° = 1$, so the coefficients of $\mathbf{i}$ and $\mathbf{j}$ are equal.
So $-4\mathbf{j} + k\mathbf{j} = 3\mathbf{i}$
So $k = 7$

7 $\sin 60° = \dfrac{x}{100}$

$x = 100 \sin 60°$

$= 50\sqrt{3}$

Using Pythagoras' theorem:

$y = \sqrt{100^2 - (50\sqrt{3})^2} = \sqrt{2500} = 50$

or using $\cos 60° = \dfrac{y}{100}$ so $y = 50$

$m = 50\sqrt{3} + 30$ and $n = 50$

8 a Call the finish line F:

$\overrightarrow{AF} = -65\mathbf{i} + 180\mathbf{j} - 10\mathbf{i} = -75\mathbf{i} + 180\mathbf{j}$

$\text{AF} = \sqrt{75^2 + 180^2} = \sqrt{38025} = 195$

$\overrightarrow{BF} = 100\mathbf{i} + 120\mathbf{j} - 10\mathbf{i} = 90\mathbf{i} + 120\mathbf{j}$

$\text{BF} = \sqrt{90^2 + 120^2} = \sqrt{22500} = 150$

$150 < 195$, so boat B is closer to the finish line.

b Speed of boat $A = \sqrt{2.5^2 + 6^2}$

$= \sqrt{42.25}$

$= 6.5$ m/s

Speed of boat $B = \sqrt{3^2 + 4^2}$

$= \sqrt{25}$

$= 5$ m/s

$\text{Time} = \dfrac{\text{Distance}}{\text{Speed}}$

Time taken for boat A to reach the finish line $= \dfrac{195}{6.5} = 30$ s

Time taken for boat B to reach the finish line $= \dfrac{150}{5} = 30$ s

Both boats reach the finish line at the same time.

9 $f(x) = 5x^2$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{5(x+h)^2-5x^2}{h} \\
&= \lim_{h\to 0}\frac{5x^2+10xh+5h^2-5x^2}{h} \\
&= \lim_{h\to 0}\frac{10xh+5h^2}{h} \\
&= \lim_{h\to 0}\frac{h(10x+5h)}{h} \\
&= \lim_{h\to 0}(10x+5h)
\end{aligned}$$

As $h \to 0$, $10x + 5h \to 10x$, so $f'(x) = 10x$

10 $y = 4x^3 - 1 + 2x^{\frac{1}{2}}$

$\dfrac{dy}{dx} = \left(4\times 3x^2\right) + \left(2\times\dfrac{1}{2}x^{-\frac{1}{2}}\right)$

$\dfrac{dy}{dx} = 12x^2 + x^{-\frac{1}{2}}$

Or:

$\dfrac{dy}{dx} = 12x^2 + \dfrac{1}{x^{\frac{1}{2}}}$

Or:

$\dfrac{dy}{dx} = 12x^2 + \dfrac{1}{\sqrt{x}}$

11 a $y = 4x + 3x^{\frac{3}{2}} - 2x^2$

$\dfrac{dy}{dx} = \left(4\times 1x^0\right) + \left(3\times\tfrac{3}{2}x^{\frac{1}{2}}\right) - \left(2\times 2x^1\right)$

$\dfrac{dy}{dx} = 4 + \tfrac{9}{2}x^{\frac{1}{2}} - 4x$

b For $x = 4$,

$$\begin{aligned}
y &= (4\times 4) + \left(3\times 4^{\frac{3}{2}}\right) - \left(2\times 4^2\right) \\
&= 16 + (3\times 8) - 32 \\
&= 16 + 24 - 32 \\
&= 8
\end{aligned}$$

So P (4, 8) lies on C.

11 c For $x = 4$,

$$\frac{dy}{dx} = 4 + \left(\tfrac{9}{2} \times 4^{\frac{1}{2}}\right) - (4 \times 4)$$
$$= 4 + \left(\tfrac{9}{2} \times 2\right) - 16$$
$$= 4 + 9 - 16$$
$$= -3$$

This is the gradient of the tangent.

The gradient of the normal at P is $\frac{1}{3}$.

The normal is perpendicular to the tangent, so the gradient is $-\frac{1}{m}$.

Equation of the normal:

$$y - 8 = \frac{1}{3}(x - 4)$$
$$y - 8 = \frac{x}{3} - \frac{4}{3}$$
$$3y - 24 = x - 4$$
$$3y = x + 20$$
$$y = \frac{1}{3}x + \frac{20}{3}$$

d $y = 0$:

$0 = x + 20$

$x = -20$

Q is the point $(-20, 0)$.

$$PQ = \sqrt{(4 - -20)^2 + (8 - 0)^2}$$
$$= \sqrt{24^2 + 8^2}$$
$$= \sqrt{576 + 64}$$
$$= \sqrt{640}$$
$$= \sqrt{64} \times \sqrt{10}$$
$$= 8\sqrt{10}$$

12 a

$$y = 4x^2 + \frac{5 - x}{x}$$
$$= 4x^2 + 5x^{-1} - 1$$
$$\frac{dy}{dx} = \left(4 \times 2x^1\right) + \left(5x \times -1x^{-2}\right)$$
$$\frac{dy}{dx} = 8x - 5x^{-2}$$

At P, $x = 1$, so

$$\frac{dy}{dx} = (8 \times 1) - \left(5 \times 1^{-2}\right)$$
$$= 8 - 5 = 3$$

12 b At $x = 1$, $\frac{dy}{dx} = 3$

The value of $\frac{dy}{dx}$ is the gradient of the tangent.

At $x = 1$, $y = \left(4 \times 1^2\right) + \frac{5 - 1}{1}$

$$y = 4 + 4 = 8$$

Equation of the tangent:

$y - 8 = 3(x - 1)$

$y = 3x + 5$

c $y = 0 : 0 = 3x + 5$

$$3 = -5$$
$$x = -\tfrac{5}{3}$$

So $k = -\tfrac{5}{3}$

13 a

$$f(x) = \frac{(2x + 1)(x + 4)}{\sqrt{x}}$$
$$= \frac{2x^2 + 9x + 4}{\sqrt{x}}$$
$$= 2x^{\frac{3}{2}} + 9x^{\frac{1}{2}} + 4x^{\frac{-1}{2}}$$

$P = 2,\ Q = 9,\ R = 4$

b

$$f'(x) = \left(2 \times \tfrac{3}{2}x^{\frac{1}{2}}\right) + \left(9 \times \tfrac{1}{2}x^{-\frac{1}{2}}\right) + \left(4 \times -\tfrac{1}{2}x^{-\frac{3}{2}}\right)$$
$$f'(x) = 3x^{\frac{1}{2}} + \tfrac{9}{2}x^{-\frac{1}{2}} - 2x^{-\frac{3}{2}}$$

c At $x = 1$,

$$f'(1) = \left(3 \times 1^{\frac{1}{2}}\right) + \left(\tfrac{9}{2} \times 1^{-\frac{1}{2}}\right) - \left(2 \times 1^{-\frac{3}{2}}\right)$$
$$= 3 + \tfrac{9}{2} - 2$$
$$= \tfrac{11}{2}$$

The line $2y = 11x + 3$ is

$$y = \tfrac{11}{2}x + \tfrac{3}{2}$$

$\therefore$ The gradient is $\tfrac{11}{2}$.

The tangent to the curve where $x = 1$ is parallel to this line, since the gradients are equal.

14 $f(x) = x^3 - 12x^2 + 48x$
$f'(x) = 3x^2 - 24x + 48$
$= 3(x-4)^2$
$(x-4)^2 > 0$ for all real values of x

So $3x^2 - 24x + 48 > 0$ for all real values of x.

So $f(x)$ is increasing for all real values of x.

15 a $y = x + \frac{2}{x} - 3$

When $y = 0$, $x + \frac{2}{x} - 3 = 0$
$x^2 + 2 - 3x = 0$
$x^2 - 3x + 2 = 0$
$(x-1)(x-2) = 0$
$x = 1$ or $x = 2$

$A(1, 0)$ and $B(2, 0)$

b $y = x + 2x^{-1} - 3$

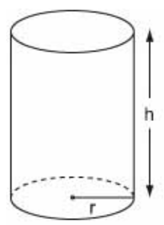
$$\frac{dy}{dx} = 1 - 2x^{-2}$$
$$= 1 - \frac{2}{x^2}$$

Let $\frac{dy}{dx} = 0$ to find the minimum

$x^2 = 2$
$x = \pm\sqrt{2}$
x is positive, so $x = \sqrt{2}$.
When $x = \sqrt{2}$,
$$y = \sqrt{2} + \frac{2}{\sqrt{2}} - 3$$
$$= \sqrt{2} + \frac{2\sqrt{2}}{2} - 3$$
$$= 2\sqrt{2} - 3$$

C has coordinates $(\sqrt{2}, 2\sqrt{2} - 3)$

16



16 Draw a diagram. Let h be the height of the cylinder.

a Surface area, $S = 2\pi rh + 2\pi r^2$
Volume $= \pi r^2 h = 128\pi$
$$h = \frac{128\pi}{\pi r^2}$$
$$= \frac{128}{r^2}$$
so $S = 2\pi r \times \frac{128}{r^2} + 2\pi r^2$
$= \frac{256\pi}{r} + 2\pi r^2$ (as required)

b $$\frac{ds}{dr} = 4\pi r - \frac{256\pi}{r^2}$$
$$4\pi r - \frac{256\pi}{r^2} = 0$$
$$4\pi r = \frac{256\pi}{r^2}$$
$$r^3 = 64$$
$$r = 4\,\text{cm}$$
When $r = 4$,
$$S = \frac{256\pi}{(4)} + 2\pi(4)^2$$
$$= 64\pi + 32\pi$$
$$= 96\pi\,\text{cm}^2$$

17 a $y = 3x^2 + 4\sqrt{x}$
$$= 3x^2 + 4x^{\frac{1}{2}}$$
$$\frac{dy}{dx} = (3 \times 2x^1) + \left(4 \times \frac{1}{2}x^{-\frac{1}{2}}\right)$$
$$\frac{dy}{dx} = 6x + 2x^{-\frac{1}{2}}$$
Or:
$$\frac{dy}{dx} = 6x + \frac{2}{x^{\frac{1}{2}}} = 6x + \frac{2}{\sqrt{x}}$$

b $$\frac{dy}{dx} = 6x + 2x^{\frac{-1}{2}}$$
$$\frac{d^2y}{dx^2} = 6 + \left(2 \times -\frac{1}{2}x^{-\frac{3}{2}}\right)$$
$$= 6 - x^{\frac{-3}{2}}$$

17 b Or:

$$\frac{d^2y}{dx^2} = 6 - \frac{1}{x^{\frac{3}{2}}}$$

Or :

$$\frac{d^2y}{dx^2} = 6 - \frac{1}{x\sqrt{x}}$$

c $\int\left(3x^2 + 4x^{\frac{1}{2}}\right)dx = \frac{3x^3}{3} + \frac{4x^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} + C$

$= x^3 + 4\left(\frac{2}{3}\right)x^{\frac{3}{2}} + C$

$= x^3 + \frac{8}{3}x^{\frac{3}{2}} + C$

$\left(\text{Or} : x^3 + \frac{8}{3}x\sqrt{x} + C\right)$

18 a $f'(x) = 6x^2 - 10x - 12$

$f(x) = \frac{6x^3}{3} - \frac{10x^2}{2} - 12x + C$

When $x = 5$, $y = 65$, so:

$65 = \frac{6\times125}{3} - \frac{10\times25}{2} - 60 + C$

$65 = 250 - 125 - 60 + C$

$C = 65 + 125 + 60 - 250$

$C = 0$

$f(x) = 2x^3 - 5x^2 - 12x$

b $f(x) = x\left(2x^2 - 5x - 12\right)$

$f(x) = x(2x+3)(x-4)$

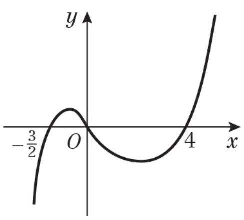
c Curve meets x-axis where $y = 0$

$x(2x+3)(x-4) = 0$

$x = 0, x = -\frac{3}{2}, x = 4$

When $x \to \infty, y \to \infty$

When $x \to -\infty, y \to -\infty$



Crosses x-axis at $\left(-\frac{3}{2}, 0\right)$, $(0,0)$ and $(4,0)$.

19 $\int_1^8 x^{\frac{1}{3}} - x^{\frac{-1}{3}}\,dx$

$= \left[\frac{3}{4}x^{\frac{4}{3}} - \frac{3}{2}x^{\frac{2}{3}}\right]_1^8$

$= \left(\frac{3}{4}(8)^{\frac{4}{3}} - \frac{3}{2}(8)^{\frac{2}{3}}\right) - \left(\frac{3}{4}(1)^{\frac{4}{3}} - \frac{3}{2}(1)^{\frac{2}{3}}\right)$

$= \left(\frac{3}{4}(16) - \frac{3}{2}(4)\right) - \left(\frac{3}{4}(1) - \frac{3}{2}(1)\right)$

$= \frac{27}{4}$

$= 6\frac{3}{4}$

20 $\int_0^6 (x^2 - kx)\,dx$

$= \left[\frac{x^3}{3} - \frac{kx^2}{2}\right]_0^6$

$= \left(\frac{6^3}{3} - \frac{k(6)^2}{2}\right) - \left(\frac{0^3}{3} - \frac{k(0)^2}{2}\right)$

$= 72 - 18k$

Given that $\int_0^6 (x^2 - kx)\,dx = 0$

$72 - 18k = 0$

$k = 4$

21 a $-x^4 + 3x^2 + 4 = 0$

$(-x^2 + 4)(x^2 + 1) = 0$

$(2 - x)(2 + x)(x^2 + 1) = 0$

$x^2 + 1 = 0$ has no real solutions.

So there are two solutions $x = -2$ or $x = 2$.

$A(-2, 0)$ and $B(2, 0)$

b $R = \int_{-2}^{2} (-x^4 + 3x^2 + 4)\,dx$

$= \left[-\frac{x^5}{5} + \frac{3x^3}{3} + 4x\right]_{-2}^{2}$

$= \left[-\frac{x^5}{5} + x^3 + 4x\right]_{-2}^{2}$

$= \left(-\frac{2^5}{5} + 2^3 + 4(2)\right) -$

$\left(-\frac{(-2)^5}{5} + (-2)^3 + 4(-2)\right)$

$= \left(-\frac{32}{5} + 8 + 8\right) - \left(\frac{32}{5} - 8 - 8\right)$

$= 19.2$ units^2

22 Area $= \int_1^4 (x-1)(x-4)\ \mathrm{d}x$

$= \int_1^4 x^2 - 5x + 4\ \mathrm{d}x$

$= \left[\frac{x^3}{3} - \frac{5x^2}{2} + 4x\right]_1^4$

$= \left(\frac{(4)^3}{3} - \frac{5(4)^2}{2} + 4(4)\right)$

$= -\left(\frac{(1)^3}{3} - \frac{5(1)^2}{2} + 4(1)\right)$

$= -4\frac{1}{2}$

$\therefore$ Area $= 4\frac{1}{2}$ units^2 (area cannot be a negative value)

23 a Solving simultaneously

$5 - x^2 = 3 - x$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$x = 2$ or $x = -1$

when $x = 2$, $y = 1$

when $x = -1$, $y = 4$

$P(-1, 4)$ and $Q(2, 1)$

b Shaded area =

area under the curve between P and Q and the x-axis − area of trapezium

Area $= \int_{-1}^{2} (5 - x^2)\ \mathrm{d}x - \frac{1}{2} \times 3(1+4)$

$= \left[5x - \frac{x^3}{3}\right]_{-1}^{2} - \frac{15}{2}$

$= \left(5(2) - \frac{2^3}{3}\right) - \left(5(-1) - \frac{(-1)^3}{3}\right) - \frac{15}{2}$

$= \left(10 - \frac{8}{3}\right) - \left(-5 + \frac{1}{3}\right) - \frac{15}{2}$

$= 4.5$ units^2

24 a $k = -1$

At point A, $x = 0$

$f(x) = 3e^0 - 1$

$= 2$

$A(0, 2)$ The y-coordinate of A is 2.

24 b At point B, $y = 0$

$3e^{-x} - 1 = 0$

$3e^{-x} = 1$

$e^{-x} = \frac{1}{3}$

$\ln(e^{-x}) = \ln\frac{1}{3}$

$-x = \ln\frac{1}{3}$

$x = -\ln\frac{1}{3}$

$= \ln\left(\frac{1}{3}\right)^{-1}$

$= \ln 3$ (which is the x-coordinate of B)

25 $T = 400e^{-0.05t} + 25, \quad t \geq 0$

a let $t = 0$

$T = 400 \times e^0 + 25 = 425°\mathrm{C}$

b let $T = 300$

$300 = 400e^{-0.05t} + 25$

$300 - 25 = 400e^{-0.05t}$

$275 = 400e^{-0.05t}$

$\frac{275}{400} = e^{-0.05t}$

Take ln of both sides:

$\ln\left(\frac{275}{400}\right) = -0.05t$

$\frac{-1}{0.05}\ln\left(\frac{275}{400}\right) = t$

$t = 7.49$ minutes

c $T = 400e^{-0.05t} + 25$

$\frac{\mathrm{d}T}{\mathrm{d}t} = 400e^{-0.05t} \times -0.05$

$= -20e^{-0.05t}$

let $t = 50$

$\frac{\mathrm{d}T}{\mathrm{d}t} = -20e^{-0.05t \times 50}$

$= -20e^{-2.5}$

$= 1.64$

The rate the temperature is decreasing is 1.64°C/min

d $T = 400e^{-0.05t} + 25, \quad t \geq 0$

$e^{-0.05t}$ tends to 0, so effectively the minimum value of T is 25°C. Therefore, 20°C is not possible.

25 e In the given model, the temperature after a long period of time is 25 °C.

Replace 25 with 15 to give:

$T = 410e^{-0.05t} + 15, t \geq 0$

26 a
$$5^x = 0.75$$
$$x \log 5 = \log 0.75$$
$$x = \frac{\log 0.75}{\log 5}$$
$$x = -0.179$$

b
$$2\log_5 x - \log_5 3x = 1$$
$$\log_5 x^2 - \log_5 3x = 1$$
$$\log_5\left(\frac{x^2}{3x}\right) = 1$$
$$5^1 = \frac{x^2}{3x} = \frac{x}{3}$$
$$x = 15$$

27 a
$$3^{2x-1} = 10$$
$$(2x-1)\log 3 = \log 10$$
$$2x - 1 = \frac{\log 10}{\log 3}$$
$$2x = \frac{1}{\log 3} + 1 \qquad (\log 10 = 1)$$
$$x = \frac{1}{2}\left(\frac{1}{\log 3} + 1\right)$$
$$= 1.55$$

b
$$\log_2 x + \log_2(9 - 2x) = 2$$
$$\log_2 x(9 - 2x) = 2$$
$$2^2 = x(9 - 2x)$$
$$4 = 9x - 2x^2$$
$$2x^2 - 9x + 4 = 0$$
$$(2x - 1)(x - 4) = 0$$
$$x = \tfrac{1}{2} \text{ or } x = 4$$

28 a
$$\log_p 12 - \left(\tfrac{1}{2}\log_p 9 + \tfrac{1}{3}\log_p 8\right)$$
$$= \log_p 12 - \left(\log_p 9^{\frac{1}{2}} + \log_p 8^{\frac{1}{3}}\right)$$
$$= \log_p 12 - \left(\log_p 3 + \log_p 2\right)$$
$$= \log_p 12 - \left(\log_p(3 \times 2)\right)$$
$$= \log_p 12 - \log_p 6$$

28 a
$$= \log_p\left(\tfrac{12}{6}\right)$$
$$= \log_p 2$$

b
$$\log_4 x = -1.5$$
$$4^{-1.5} = x$$
$$x = \tfrac{1}{8} \text{ or } 0.125$$

29 a
$$\ln x + \ln 3 = \ln 6$$
$$\ln 3x = \ln 6$$
$$3x = 6$$
$$x = 2$$

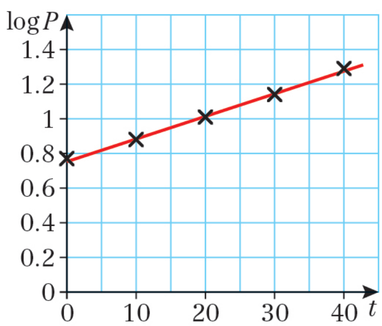
b
$$e^x + 3e^{-x} = 4$$
$$e^x + \frac{3}{e^x} = 4$$
$$e^{2x} + 3 = 4e^x$$
$$e^{2x} - 4e^x + 3 = 0$$

let $y = e^x$
$$y^2 - 4y + 3 = 0$$
$$(y-3)(y-3) = 0$$
$$y = 3 \text{ or } 1$$
$$y = e^x$$
$$e^x = 3 \text{ or } e^x = 1$$
$$x = 0$$
$$x = \ln 3 \text{ or } x = 0$$

30 a

Time in years since 1970, t	$\log P$
0	0.77
10	**0.88**
20	**1.01**
30	**1.14**
40	**1.29**

b



30 c As $P = ab^t$

$\log P = \log (ab^t)$

$\log P = \log a + \log b^t$

$\log P = \log a + t \log b$

This is a linear relationship where the gradient is $\log b$ and the intercept is $\log a$.

d Intercept = 0.77

$\log a = 0.77$

$a = 10^{0.77}$

$= 5.888...$

≈ 5.9 (2 s.f.)

$$\text{Gradient} = \frac{1.29 - 0.77}{40 - 0} = \frac{0.52}{40} = 0.013$$

$\log b = 0.013$

$b = 10^{0.013}$

$= 1.03...$

≈ 1.0

$a = 5.9$, $b = 1.0$

31 a

$$\log 2 + \log x = \log y + \log(x + y)$$
$$\log 2x = \log y + \log(x + y)$$
$$\log 2x - \log y = \log(x + y)$$
$$\log \frac{2x}{y} = \log(x + y)$$
$$\frac{2x}{y} = x + y$$
$$2x = xy + y^2$$
$$2x - xy = y^2$$
$$x(2 - y) = y^2$$
$$x = \frac{y^2}{2 - y}$$

b $0 < y < 2$

$y > 0$ given.

$x > 0$ also given, and $y^2 > 0$, so $2 - y$ must be > 0. Hence $y < 2$. Note strict inequality because denominator cannot be 0.

Challenge

1 a 090° means $\sin \theta = 0$

Therefore, $\theta = 0$

b $\cos \theta = 1$

So the vector is $1\mathbf{i}$

Magnitude $= \sqrt{1^2 + 0^2} = 1$

2 a $f'(-3) = k((-3)^2 - 3 - 6) = 0$

$f'(2) = k(2^2 + 2 - 6) = 0$

Using the factor theorem, $x + 3$ and $x - 2$ are factors of $f'(x)$.

So $f'(x) = k(x + 3)(x - 2)$

$= k(x^2 + x - 6)$

As $f(x)$ is cubic, there are no other factors of $f'(x)$.

b

$$\int k(x^2 + x - 6)\, dx = \int (kx^2 + kx - 6k)\, dx$$
$$= \frac{kx^3}{3} + \frac{kx^2}{2} - 6kx + c$$

At $(-3, 76)$

$$\frac{k(-3)^3}{3} + \frac{k(-3)^2}{2} - 6k(-3) + c = 76$$
$$-9k + \frac{9k}{2} + 18k + c = 76$$
$$\frac{27k}{2} + c = 76$$

At $(2, -49)$

$$\frac{k(2)^3}{3} + \frac{k(2)^2}{2} - 6k(2) + c = -49$$
$$\frac{8k}{3} + 2k - 12k + c = -49$$
$$-\frac{22k}{3} + c = -49$$

Solving $\frac{27k}{2} + c = 76$ and $-\frac{22k}{3} + c = -49$ simultaneously

$c = 76 - \frac{27k}{2}$ and $c = \frac{22k}{3} - 49$

So $76 - \frac{27k}{2} = \frac{22k}{3} - 49$

$456 - 81k = 44k - 294$

$125k = 750$

$k = 6$, $c = -5$

$$f(x) = \frac{kx^3}{3} + \frac{kx^2}{2} - 6kx + c$$
$$= \frac{6x^3}{3} + \frac{6x^2}{2} - 6(6)x - 5$$
$$= 2x^3 + 3x^2 - 36x - 5$$

3 $\int_0^9 \mathrm{f}(x)\,\mathrm{d}x = 24.2$

$\int_0^9 (\mathrm{f}(x)+3)\,\mathrm{d}x$

$= [\mathrm{f}'(x)+3x]_0^9$

$= (\mathrm{f}'(9)+3(9)) - (\mathrm{f}'(0)+3(0))$

$= \int_0^9 \mathrm{f}(x)\,\mathrm{d}x + 27$

$= 24.2 + 27$

$= 51.2$

4 **a** $\mathrm{f}(0) = 0^3 - k(0) + 1 = 1$

$\mathrm{g}(0) = \mathrm{e}^{2(0)} = \mathrm{e}^0 = 1$

Therefore, $\mathrm{f}(0) = \mathrm{g}(0) = 1$

$P(0, 1)$

b $\mathrm{f}'(x) = 3x^2 - k$

Gradient at $x = 0$

$\mathrm{f}'(0) = 3(0)^2 - k = -k$

Gradient of $\mathrm{g}(x)$ at $x = 0$ is $\dfrac{1}{k}$

$\mathrm{g}'(x) = 2\mathrm{e}^{2x}$

$\mathrm{g}'(0) = 2\mathrm{e}^{2(0)} = 2\mathrm{e}^0 = 2$

$\dfrac{1}{k} = 2$

$k = \dfrac{1}{2}$

Practice paper

1 a $4 = \sqrt[3]{64} = 4^{\frac{1}{3}}$

so $n = \frac{1}{3}$

b $\sqrt{50} = \sqrt{25 \times 2}$

$= \sqrt{25} \times \sqrt{2}$

$= 5\sqrt{2}$

2 $2x - 3y + 4 = 0$

$3y = 2x + 4$

$y = \frac{2}{3}x + \frac{4}{3}$

The gradient of this line is $\frac{2}{3}$.

The equation of the parallel line is:

$y - y_1 = m(x - x_1)$

$y - 6 = \frac{2}{3}(x - 5)$

$y - 6 = \frac{2}{3}x - \frac{10}{3}$

$y = \frac{2}{3}x + \frac{8}{3}$

3 a Error 1: $-\dfrac{3}{\sqrt{x}} = -3x^{-\frac{1}{2}}$, not $-3x^{\frac{1}{2}}$

Error 2: $\left[\dfrac{x^5}{5} - 2x^{\frac{3}{2}} + 2x\right]_1^2$

$= \left(\dfrac{32}{5} - 2\sqrt{8} + 4\right) - \left(\dfrac{1}{5} - 2 + 2\right)$

not $\left(\dfrac{1}{5} - 2 + 2\right) - \left(\dfrac{32}{5} - 2\sqrt{8} + 4\right)$

b $\displaystyle\int_1^2 \left(x^4 - \frac{3}{\sqrt{x}} + 2\right) dx$

$= \displaystyle\int_1^2 \left(x^4 - 3x^{-\frac{1}{2}} + 2\right) dx$

$= \left[\dfrac{x^5}{5} - \dfrac{3x^{\frac{1}{2}}}{\frac{1}{2}} + 2x\right]_1^2$

$= \left[\dfrac{x^5}{5} - 6x^{\frac{1}{2}} + 2x\right]_1^2$

$= \left(\dfrac{32}{5} - 6\sqrt{2} + 4\right) - \left(\dfrac{1}{5} - 6 + 2\right)$

$= 5.71$ (3 s.f.)

4 $2\sin^2(2x) - \cos(2x) - 1 = 0$

$2(1 - \cos^2(2x)) - \cos(2x) - 1 = 0$

$2 - 2\cos^2(2x) - \cos(2x) - 1 = 0$

$2\cos^2(2x) + \cos(2x) - 1 = 0$

$(2\cos(2x) - 1)(\cos(2x) + 1) = 0$

So $\cos(2x) = \frac{1}{2}$ or $\cos(2x) = -1$

As $0 \le x \le 180°$, $0 \le 2x \le 360°$

When $\cos(2x) = \frac{1}{2}$, $2x = 60°$

or $2x = 360° - 60° = 300°$

giving $x = 30°$ or $150°$

When $\cos(2x) = -1$, $2x = 180°$

giving $x = 90°$

$x = 30°$, $90°$ or $150°$

5 a Volume $= x \times (x + 3) \times 2x$

$= 2x^2(x + 3)$

b $2x^2(x + 3) = 980$

$2x^3 + 6x^2 = 980$

$2x^3 + 6x^2 - 980 = 0$

$x^3 + 3x^2 - 490 = 0$ (as required)

c $f(x) = x^3 + 3x^2 - 490$

$f(7) = (7)^3 + 3(7)^2 - 490$

$= 343 + 147 - 490$

$= 0$

So $x = 7$ is a solution to $x^3 + 3x^2 - 490 = 0$.

d
$$\begin{array}{r|l} & x^2 + 10x + 70 \\ \hline x - 7 & x^3 + 3x^2 + 0x - 490 \\ & \underline{x^3 - 7x^2} \\ & 10x^2 + 0x \\ & \underline{10x^2 - 70x} \\ & 70x - 490 \\ & \underline{70x - 490} \\ & 0 \end{array}$$

$x^3 + 3x^2 - 490 = (x - 7)(x^2 + 10x + 70)$

Using the discriminant for $x^2 + 10x + 70$:

$b^2 - 4ac = (10)^2 - 4(1)(70)$

$= 100 - 280$

$= -180$

As $-180 < 0$, there are no real solutions to $x^2 + 10x + 70$.

Therefore, there are no other real solutions to the equation $x^3 + 3x^2 - 490 = 0$.

6 $f(x) = x^3 - 5x^2 - 2 + x^{-2}$
$f'(x) = 3x^2 - 10x - 2x^{-3}$
When $x = -1$, gradient of curve
$= f'(-1)$
$= 3(-1)^2 - 10(-1) - \frac{2}{(-1)^3}$
$= 3 + 10 + 2$
$= 15$
So gradient of normal is $-\frac{1}{15}$.
When $x = -1$,
$y = (-1)^3 - 5(-1)^2 - 2 + \frac{1}{(-1)^2}$
$= -7$
The equation of the normal is:
$y - y_1 = m(x - x_1)$
$y + 7 = -\frac{1}{15}(x + 1)$
$15y + 105 = -x - 1$
$x + 15y + 106 = 0$

7 a $P = ab^t$
$\log_{10} P = \log_{10}(ab^t)$
$= \log_{10} a + \log_{10} b^t$
$= \log_{10} a + t\log_{10} b$
Gradient of line $= \log_{10} b$
$= \frac{2.2 - 2}{20 - 0}$
$= \frac{0.2}{20}$
$= 0.01$
Intercept $= \log_{10} a = 2$
Equation of line l is $\log_{10} P = 0.01t + 2$.

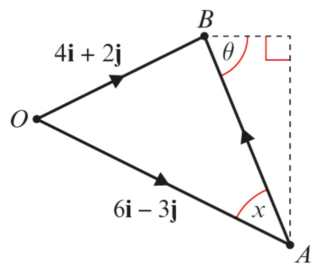
b $\log_{10} a = 2$
$a = 10^2$
$= 100$
100 is the initial population of Caledonian owlet-nightjars.

c $\log_{10} b = 0.01$
$b = 10^{0.01}$
$= 1.023$ (3 d.p.)

d $P = 100 \times 1.023^{30}$
$= 197.8\ldots$
The population when $t = 30$ is 198.

8 LHS
$= 1 + \cos^4 x - \sin^4 x$
$= 1 + (\cos^2 x + \sin^2 x)(\cos^2 x - \sin^2 x)$
$= 1 + \cos^2 x - \sin^2 x$
$= 1 - \sin^2 x + \cos^2 x$
$= \cos^2 x + \cos^2 x$
$= 2\cos^2 x$
$=$ RHS

9



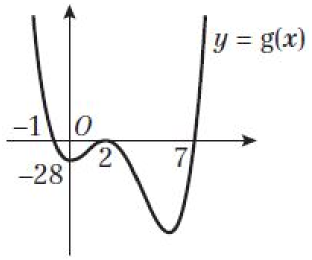
$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$
$= 4\mathbf{i} + 2\mathbf{j} - (6\mathbf{i} - 3\mathbf{j})$
$= -2\mathbf{i} + 5\mathbf{j}$
$|\overrightarrow{AB}| = \sqrt{2^2 + 5^2} = \sqrt{29}$
$\tan x = \frac{2}{5}$
$x = 21.8°$
$\theta = 21.8° + 90° = 111.8°$
$= 112°$ (3 s.f.)

10 a Using the cosine rule
$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$
$= \frac{12.7^2 + 7.5^2 - 10.6^2}{2(12.7)(7.5)}$
$= \frac{105.18}{190.5}$
$= 0.55212\ldots$
Angle $BAC = 56.5°$ (3 s.f.)

b Cost
$=$ area of lawn $\times 1.25$
$= \frac{1}{2}bc\sin A \times 1.25$
$= \frac{1}{2} \times 12.7 \times 7.5 \times \sin 56.4870... \times 1.25$
$= 49.6348...$
$= £49.63$

11 a $y = g(x) = (x - 2)^2(x + 1)(x - 7)$
$0 = (x - 2)^2(x + 1)(x - 7)$
So $x = 2$, $x = -1$ or $x = 7$

11 a The curve touches the x-axis at (2, 0) and crosses it at (−1, 0) and (7, 0).
When $x = 0$, $y = (-2)^2 \times 1 \times (-7) = -28$
So the curve crosses the y-axis at (0, −28).
$x \to \infty, y \to \infty$
$x \to -\infty, y \to \infty$



b $g(x + 3) = (x + 3 - 2)^2(x + 3 + 1)(x + 3 - 7)$
$= (x + 1)^2(x + 4)(x - 4)$
$0 = (x + 1)^2(x + 4)(x - 4)$
The roots of $g(x + 3)$ are
$x = -1$, $x = -4$ and $x = 4$.

12 $9^{2x} = 27^{x^2-5}$
$(3^2)^{2x} = (3^3)^{x^2-5}$
So $2(2x) = 3(x^2 - 5)$
$4x = 3x^2 - 15$
$3x^2 - 4x - 15 = 0$
$(3x + 5)(x - 3) = 0$
So $x = -\frac{5}{3}$ or $x = 3$

13 a $f(x)$
$= (1 - 3x)^5$
$= 1^5 + \binom{5}{1}1^4(-3x) + \binom{5}{2}1^3(-3x)^2 + \ldots$
$= 1 - 15x + 90x^2$

b $1 - 3x = 0.97$
$3x = 0.03$
$x = 0.01$
Substituting $x = 0.01$ into the expansion for $(1 - 3x)^5$:
$0.97^5 \approx 1 - 15(0.01) + 90(0.01)^2$
$= 0.859$

c This approximation is greater than the true value as the next term will be negative and the subsequent positive terms will be smaller.

14 a $$f'(x) = \frac{\sqrt{x} - x^2 - 1}{x^2}$$
$$= x^{-\frac{3}{2}} - 1 - x^{-2}$$
$$f(x) = \int (x^{-\frac{3}{2}} - 1 - x^{-2})\,dx$$
$$= \frac{x^{-\frac{1}{2}}}{-\frac{1}{2}} - x - \frac{x^{-1}}{-1} + c$$
$$= -\frac{2}{\sqrt{x}} - x + \frac{1}{x} + c$$
$$= -\frac{2\sqrt{x} + x^2 - 1}{x} + c$$
$$= -\frac{x^2 + 2\sqrt{x} - 1}{x} + c$$

b $f(3) = -1$
$$-1 = -\frac{3^2 + 2\sqrt{3} - 1}{3} + c$$
$$c = -1 + \frac{8 + 2\sqrt{3}}{3}$$
$$= \frac{5 + 2\sqrt{3}}{3}$$
$$= \frac{5}{3} + \frac{2}{3}\sqrt{3}$$
$p = \frac{5}{3}$, $q = \frac{2}{3}$, $r = 3$

15 a Substituting $x = 5$ and $y = 1$ into the equation for C:
$5^2 + 1^2 - 4(5) + 6(1) = 25 + 1 - 20 + 6$
$= 12$
Therefore, A lies on C.
Rearranging the equation:
$x^2 - 4x + y^2 + 6y = 12$
Completing the square:
$(x - 2)^2 - 4 + (y + 3)^2 - 9 = 12$
$(x - 2)^2 + (y + 3)^2 = 25$
The circle has centre (2, −3) and radius 5.

b Gradient of the radius at $A(5, 1)$
$$= \frac{y_2 - y_1}{x_2 - x_1} = \frac{1 + 3}{5 - 2} = \frac{4}{3}$$
Gradient of the tangent $= -\frac{3}{4}$
Equation of the tangent at A:
$y - y_1 = m(x - x_1)$
$y - 1 = -\frac{3}{4}(x - 5)$
$y = -\frac{3}{4}x + \frac{19}{4}$

15 c Solving the simultaneous equations

$y = x^2 - 2$ and $y = -\frac{3}{4}x + \frac{19}{4}$ to find P and Q:

$$x^2 - 2 = -\frac{3}{4}x + \frac{19}{4}$$

$$4x^2 - 8 = -3x + 19$$

$$4x^2 + 3x - 27 = 0$$

$$(4x - 9)(x + 3) = 0$$

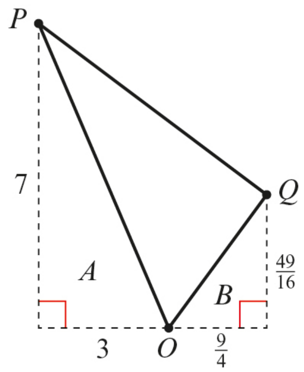
$$x = \frac{9}{4} \text{ or } x = -3$$

When $x = \frac{9}{4}$, $y = -\frac{3}{4}\left(\frac{9}{4}\right) + \frac{19}{4} = \frac{49}{16}$

When $x = -3$, $y = -\frac{3}{4}(-3) + \frac{19}{4} = 7$

The point P is $(-3, 7)$ and the point Q is $(\frac{9}{4}, \frac{49}{16})$.

Draw a diagram



Area of triangle POQ
= area of trapezium − area of triangle a
− area of triangle b

Area of trapezium $= \frac{1}{2} \times \frac{21}{4} \times \left(7 + \frac{49}{16}\right)$

$= 26\frac{53}{128}$

Area of triangle $a = \frac{1}{2} \times 3 \times 7$

$= 10\frac{1}{2}$

Area of triangle $b = \frac{1}{2} \times \frac{9}{4} \times \frac{49}{16}$

$= 3\frac{57}{128}$

Area of triangle $POQ = 26\frac{53}{128} - 10\frac{1}{2} - 3\frac{57}{128}$

$= 12\frac{15}{32}$